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Golden REAL ANALYSIS

By

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CONTENTS

LIST OF SYMBOLS AND ABBREVIATIONS

Symbol	*Meaning*
$\in$	'belongs to' or 'is an element of'
$\notin$	does not belong to
$\subset$	Is a sub-set of
$\not\subset$	Is not a sub-set of
$\supset$	Is a super-set of
$\cup$	Union of sets
$\cap$	Intersection of sets
$A \times B$	Cartesian (or cross) product of sets A and B
U (or X)	Universal set
A^c	Complement of A
$A - B$	Difference of two sets A and B (or complement of B w.r.t. A)
ϕ	Null (empty or void) set
: or | or s.t.	such that
iff	if and only if
$\forall$	for all
$\exists$	there exists
$\Rightarrow$	implies
$\Leftrightarrow$	implies and is implied by (or iff)
$A \Delta B$	Symmetric difference of A and B
N	the set of all natural numbers (or positive integers)
Z	the set of all integers
Q	the set of all rational numbers
R	the set of all real numbers
$\le$	is less than *or* equal to
$\ge$	is greater than *or* equal to
Sup. S (or l.u.b. S)	supermum (or least upper bound) of S
Inf. S (or g.l.b. s)	infimum (or greatest lower bound) of S
$\lvert x \rvert$	absolute value of x
(a, b)	open interval $a < x < b$
$[a, b]$	closed interval $a \le x \le b$
nbd	neighbourhood
A°	interior of A
A'	derived set of A
$\overline{A}$	closure of A
$f : X \to Y$	f is a function from X to Y
$\sum_{n=1}^{\infty} u_n$ (or Σu_n)	infinite series $u_1 + u_2 + u_3 + \ldots\ldots$
$\sum_{n=1}^{\infty} (-1)^{n-1} u_n, u_n > 0$	alternating series $u_1 - u_2 + u_3 - u_4 + \ldots\ldots$
$\prod_{n=1}^{\infty} u_n$ (or Πu_n)	infinite product $u_1 u_2 u_3 \ldots\ldots$

1

Sets and Functions

SECTION I—SETS

1.1. SETS

A set is a well defined collection of distinct objects.

By a **'well-defined'** collection of objects we mean that there is a rule (or rules) by means of which it is possible to say, without ambiguity, whether a particular object belongs to the collection or not. The objects in a set are **'distinct'** means we do not repeat an object over and over again in a set.

Each object belonging to a set is called an **element** (or a member) of the set. Sets are usually denoted by capital letters A, B, N, Q, R, S etc. and the elements by lower case letters a, b, c, x etc.

The symbol $\in$ is used to indicate 'belongs to' or 'is an element of'. Thus $x \in A \Rightarrow x$ is an element of the set A.

The symbol $\notin$ is used to indicate 'does not belong to' or 'is not an element of'. Thus $x \notin A \Rightarrow x$ is not an element of the set A.

Given a set S and an object p, exactly one of the following statements should be true :

(*i*) $p \in S$ (*ii*) $p \notin S$

Illustrations. (*i*) Let V be the set of vowels in English alphabet, then the elements of V are a, e, i, o, u.

$$a \in V, \quad u \in V, \quad g \notin V, \quad t \notin V.$$

(*ii*) Let E be the set of even natural numbers, then

$$6 \in E, \quad 512 \in E, \quad 3 \notin E, \quad 127 \notin E.$$

(*iii*) Let P be the set of prime numbers, then

$$2 \in P, \quad 7 \in P, \quad 6 \notin P, \quad 15 \notin P.$$

1.2. METHODS OF DESCRIBING A SET

There are two methods of describing a set.

(1) **Roster Method (or Listing Method or Tabulation Method).** In this method, a set is described by listing all its elements, separating them by commas and enclosing them within braces (curly brackets).

For example. (*i*) if A is the set of odd natural numbers less than 10, then in roster form,

$$A = \{1, 3, 5, 7, 9\}$$

(*ii*) if B is the set of letters of the word FOLLOW, then in roster form,

$$B = \{F, O, L, W\} \quad \text{(dropping the repetitions)}$$

When a set is described in roster form, the order of its elements is immaterial *i.e.*, the elements may be written down in any order.

Thus $\quad B = \{F, O, L, W\} = \{F, L, W, O\} = \{W, O, F, L\}$.

(2) **Set Builder Method (or Property Method or Rule Method).** Listing the elements of a set is sometimes difficult and sometimes impossible. We do not have a roster form of the set Q of rational numbers or the set R of real numbers. In set builder method, a set is described by means of some property which is shared by all the elements of the set.

We first write within braces a variable (x, y etc.) followed by a statement of property, in terms of the stated variable, that must be satisfied by each element of the set.

Thus if a set A is characterised by a property P, we write

$A = \{x : P(x)\}$ and read 'A is the set of all x such that x has the property P'. The symbol : stands for 'such that'.

For example. (*i*) if P is the set of all prime numbers, then

$$P = \{x : x \text{ is a prime number}\}.$$

(*ii*) if A is the set of all natural numbers between 10 and 100, then

$$A = \{x : x \in N \text{ and } 10 < x < 100\}.$$

1.3. FINITE AND INFINITE SETS

(*i*) **Finite Set.** A set is said to be finite if the number of its elements is finite *i.e.*, if its elements can be counted, one by one, with the counting coming to an end.

For example. (*a*) the set of letters in the English alphabet is finite since it has 26 (*i.e.*, finite number of) elements.

(*b*) the set of all multiples of 5 less than 1000 is a finite set.

(*ii*) **Infinite Set.** A set is said to be infinite if the number of its elements is infinite *i.e.*, if we count its elements, one by one, the counting never comes to an end.

For example. (*i*) the set of all points in a plane α is an infinite set.

(*ii*) the sets N, Z, Q, R are all infinite sets.

1.4. NULL SET

A set having no element is known as a null set or a void set or an empty set and is denoted by ϕ or { }.

For example. (*i*) $\{x : x \text{ is an integer and } x^2 = 2\} = \phi$ because there is no integer whose square is 2.

(*ii*) $\{x : x \in R \text{ and } x^2 + 1 = 0\} = \phi$ because $x^2 \geq 0$ for all $x \in R$

$\Rightarrow \quad x^2 + 1 \geq 1 \Rightarrow x^2 + 1 \neq 0.$

(*iii*) $\{x : x \neq x\} = \phi$.

1.5. SINGLETON SET

A set having only one element is called a singleton set.

For example. (*i*) {a} is a singleton set.

(*ii*) {o} or {ϕ} are not null sets but singleton set.

(*iii*) $\{x : x^3 + 1 = 0 \text{ and } x \in R\} = \{-1\}$ is a singleton set.

1.6. SUBSET AND SUPER-SET

If A and B are two sets such that every element of A is also an element of B, then A is called a subset of B and we write $A \subset B$.

Thus $\quad A \subset B$ if $x \in A \Rightarrow x \in B$.

Also if $A \subset B$, then B is called a super-set of A and we also write $B \supset A$.

For example. (*i*) if V is the set of vowels in English alphabet and S is the set of all letters in English alphabet, then $V \subset S$ and $S \supset V$.

(*ii*) if $A = \{1, 3, 5\}$ and $B = \{1, 2, 3, 4, 5\}$, then $A \subset B$ and $B \supset A$

(*iii*) if $A = \{x : x \text{ is a multiple of } 2\}$, $B = \{x : x \text{ is a multiple of } 5\}$, $C = \{x : x \text{ is a multiple of } 10\}$, then $\quad A \subset C, B \subset C$.

1.7. EQUALITY OF SETS

Two sets A and B are said to be equal iff they contain exactly the same elements and we write A = B.

Thus, A = B if every element of A is an element of B and every element of B is an element of A.

In symbols, $A = B$ iff $x \in A \Rightarrow x \in B$ and $x \in B \Rightarrow x \in A$.

Or $\quad$ A = B iff $A \subset B$ and $B \subset A$.

For example. (*i*) if $A = \{4, 5, 6, 7\}$ and $B = \{x : 4 \leq x \leq 7, x \in N\}$, then A = B.

(*ii*) if $A = \{x : x^2 = 1\}$ and $B = \{1, -1\}$, then A = B.

(*iii*) if $A = \{2\}$ and $B = \{p : p \text{ is an even prime number}\}$, then A = B.

1.8. PROPER SUBSET

If A and B are two sets such that $A \subset B$ and $A \neq B$, then A is called a proper subset of B.

Thus **A is a proper subset of B $\Rightarrow$ every element of A is an element of B but there is at least one element in B which is not in A.**

For example. (*i*) $A = \{2, 3, 4, 5, 6, ...\}$ is a proper subset of N, because $A \subset N$ and $A \neq N$, $(\because \; 1 \in N \text{ but } 1 \notin A)$.

(*ii*) N is a proper subset of Z because every natural number is an integer (*i.e.*, $N \subset Z$) but every integer need not be a natural number (*i.e.*, $N \neq Z$).

Theorem I. *Every set is a subset of itself.*

Let A be *any* set. Since $x \in A \Rightarrow x \in A \quad \therefore \quad A \subset A$.

Theorem II. *Empty set is a subset of every set.*

If possible, suppose $\phi \not\subset A$ for some set A.

$\phi \not\subset A \Rightarrow$ **there is at least one element in** ϕ which is not in A.

But this contradicts the definition of ϕ.

$\therefore$ Our supposition is wrong. Hence $\phi \subset A$ for any set A.

Theorem III. *The empty set is unique.*

If possible, let ϕ_1 and ϕ_2 be two empty sets.

Since empty set is a subset of every set.

$\therefore \quad \phi_1 \subset \phi_2$ and $\phi_2 \subset \phi_1$

$\Rightarrow \quad \phi_1 = \phi_2$. $\quad$ That proves the uniqueness of ϕ.

Theorem IV. *If a set has n elements, then the number of its subsets is* 2^n. Let a set A have n elements. The number of subsets of A having r elements is the same as the number of groups (or combinations, because the order of the elements is immaterial in a set) of r elements which can be formed out of the n elements of A and this can be done in ${}^nC_r$ ways. Thus, there are ${}^nC_r$ subsets of A having r elements.

$\therefore$ No. of subsets of A having no element $= {}^nC_0$
No. of subsets of A having one element $= {}^nC_1$
No. of subsets of A having two elements $= {}^nC_2$
....
No. of subsets of A having n elements $= {}^nC_n$

$\therefore$ Total number of subsets $= {}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n = (1+1)^n = 2^n$.

1.9. COMPARABLE SETS

If two sets A and B are such that either $A \subset B$ or $B \subset A$, then A and B are said to be comparable sets.

If neither $A \subset B$ nor $B \subset A$, then A and B are said to be *non-comparable* sets.

For example. (*i*) if $A = \{1, 3, 5\}$ and $B = \{1, 2, 3, 4, 5\}$, then A and B are comparable sets because $A \subset B$.

(*ii*) if $A = \{2, 4, 6, 8, 10\}$ and $B = \{4, 8\}$, then A and B are comparable sets because $B \subset A$.

(*iii*) if $A = B$, then A and B are comparable sets because $A \subset B$ and $B \subset A$.

(*iv*) if $A = \{1, 3, 5\}$ and $B = \{2, 4, 6\}$, then A and B are non-comparable sets because $A \not\subset B$ and $B \not\subset A$.

1.10. FAMILY OF SETS (OR SET OF SETS)

A set whose elements are also sets is called a family of sets (or a set of sets).

For example. (*i*) $\{\{1\}, \{2, 3\}, \{4, 5, 6\}\}$ is a family of sets.

(*ii*) $\{\{1\}, \{2, 3\}, 4\}$ is not a family of sets because one of its elements, namely 4, is not a set.

1.11. POWER SET OF A SET

The set of all the subsets of a set A is called the power set of A and is denoted by P(A).

Thus $P(A) = \{S : S \subset A\}$.

For example. (*i*) if $A = \{a\}$, then $P(A) = \{\phi, A\}$

(*ii*) if $B = \{2, 5\}$, then $P(B) = \{\phi, \{2\}, \{5\}, B\}$

(*iii*) if $S = \{a, b, c\}$, then $P(S) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}, S\}$.

1.12. OPERATIONS ON SETS

I. Union of Sets. The union of two sets A and B is the set of all elements which belong either to A or to B (or to both).

The union of A and B is denoted by $A \cup B$ and read as 'A union B'.

Symbolically, $A \cup B = \{x : x \in A \text{ or } x \in B\}$

For example. (*i*) If $A = \{1, 3, 6\}$ and $B = \{2, 4\}$, then $A \cup B = \{1, 2, 3, 4, 6\}$

(*ii*) If $A = \{x : x$ is an odd natural number$\}$ and $B = \{x : x$ is an even natural number$\}$

then $A \cup B = \{x : x$ is an odd or even natural number$\}$
$= \{x : x$ is a natural number$\} = N$

($\because$ every natural number is either odd or even)

On the same lines, we can define the union of more than two sets. If $A_1, A_2, \ldots, A_n$ be n given sets, then their union is denoted by $A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n$ or $\bigcup_{i=1}^{n} A_i$ and defined as the set of all elements of these sets. Thus $\bigcup_{i=1}^{n} A_i = \{x : x \in A_i$ for at least one $i, 1 \le i \le n\}$

Properties of Union of Sets

(*a*) **For any two sets A and B,** (*i*) $A \subset A \cup B$ (*ii*) $B \subset A \cup B$.

Proof. (*i*) Let x be any element of A. Then

$x \in A \Rightarrow x \in A \cup B$ | $\because$ $A \cup B$ is the set of all elements of A and B

$\therefore$ $A \subset A \cup B$

(*ii*) *Please try yourself.*

(*b*) **For any set A,** $A \cup \phi = A$

Proof. $A \cup \phi = \{x : x \in A \text{ or } x \in \phi\}$
$= \{x : x \in A\}$ [$\because$ ϕ has no element]
$= A$

(*c*) **Union of sets is idempotent *i.e.*, for any set A,** $A \cup A = A$.

Proof. $A \cup A = \{x : x \in A \text{ or } x \in A\} = \{x : x \in A\} = A$

(*d*) **Union of sets is commutative *i.e.*, for any two sets A and B,** $A \cup B = B \cup A$

Proof. $A \cup B = \{x : x \in A \text{ or } x \in B\}$
$= \{x : x \in B \text{ or } x \in A\}$
$= B \cup A$

(*e*) **Union of sets is associative *i.e.*, for any three sets A, B and C,**
$A \cup (B \cup C) = (A \cup B) \cup C$

Proof. $A \cup (B \cup C) = \{x : x \in A \text{ or } x \in B \cup C\} = \{x : x \in A \text{ or } (x \in B \text{ or } x \in C)\}$
$= \{x : (x \in A \text{ or } x \in B) \text{ or } x \in C\} = \{x : x \in A \cup B \text{ or } x \in C\}$
$= (A \cup B) \cup C$.

(*f*) $A \subset B \Rightarrow A \cup B = B$

(*g*) $x \in A \cup B \Rightarrow x \in A$ or $x \in B$,
$x \notin A \cup B \Rightarrow x \notin A$ and $x \notin B$.

II. Intersection of sets. The intersection of two sets A and B is the set of all elements which belong to both A and B.

The intersection of A and B is denoted by $A \cap B$ and read as 'A intersection B'.

Symbolically, $A \cap B = \{x : x \in A \text{ and } x \in B\}$

For example. (*i*) if $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6, 8\}$, then $A \cap B = \{2, 4\}$ (Common elements)

(*ii*) if P is the set of all prime numbers and E, the set of all even natural numbers, then $P \cap E = \{2\}$ because 2 is the only even prime number.

(*iii*) if $A = \{x : x$ is an odd natural number$\}$
and $B = \{x : x$ is an even natural number$\}$
then $A \cap B = \phi$ because no natural number is both odd and even.

On the same lines, we can define the intersection of more than two sets. If $A_1, A_2, \ldots, A_n$ be n given sets, then their intersection is denoted by

$A_1 \cap A_2 \cap A_3 \cap \ldots\ldots \cap A_n$ or $\bigcap_{i=1}^{n} A_i$ and defined as the set of all elements which belong to each A_i. Thus $\bigcap_{i=1}^{n} A_i = \{x : x \in A_i \text{ for every } i,\ 1 \le i \le n\}$

Properties of Intersection of Sets

(*a*) **For any two sets A and B,**

(*i*) $A \cap B \subset A$ (*ii*) $A \cap B \subset B$.

Proof. (*i*) Let x be any element of $A \cap B$. Then

$$x \in A \cap B \Rightarrow x \in A \text{ and } x \in B \Rightarrow x \in A \text{ (in particular)}$$

$$\therefore \quad A \cap B \subset A$$

(*ii*) *Please try yourself.*

(*b*) **For any set A,** $A \cap \phi = \phi$

Proof. Since $A \cap B \subset B$ $\therefore$ $A \cap \phi \subset \phi$ (replacing B by ϕ) ...(*i*)

Also ϕ is a subset of every set $\therefore$ $\phi \subset A \cap \phi$ ...(*ii*)

Combining (*i*) and (*ii*) $A \cap \phi = \phi$

[Here we have used the definition of equality of two sets, *i.e.*, $A = B$ iff $A \subset B$ and $B \subset A$]

(*c*) **Intersection of sets is idempotent *i.e.*, for any set A, $A \cap A = A$**

Proof. $A \cap A = \{x : x \in A \text{ and } x \in A\} = \{x : x \in A\} = A$

(*d*) **Intersection of sets is commutative *i.e.*, for any two sets A and B, $A \cap B = B \cap A$**

Proof. $A \cap B = \{x : x \in A \text{ and } x \in B\} = \{x : x \in B \text{ and } x \in A\} = B \cap A$

(*e*) **Intersection of sets is associative *i.e.*, for any three sets A, B and C,**

$$\mathbf{A \cap (B \cap C) = (A \cap B) \cap C}$$

Proof. $A \cap (B \cap C) = \{x : x \in A \text{ and } x \in B \cap C\} = \{x : x \in A \text{ and } (x \in B \text{ and } x \in C)\}$

$= \{x : (x \in A \text{ and } x \in B) \text{ and } x \in C\} = \{x : x \in A \cap B \text{ and } x \in C\}$

$= (A \cap B) \cap C.$

(*f*) $\mathbf{A \subset B \Rightarrow A \cap B = A.}$

(*g*) $\mathbf{x \in A \cap B \Rightarrow x \in A \text{ and } x \in B}$

$\mathbf{x \notin A \cap B \Rightarrow x \notin A \text{ or } x \notin B.}$

1.13. DISJOINT SETS

Two sets are said to be disjoint if they have no element in common. Thus two sets A and B are disjoint if $A \cap B = \phi$.

For example. (*i*) if $A = \{a, c, e, f\}$ and $B = \{b, d, g\}$, then A and B are disjoint sets because

$$A \cap B = \phi.$$

(*ii*) If V is the set of vowels and C, the set of consonants in English alphabet, then V and C are disjoint sets because no vowel is a consonant and *vice-versa*.

1.14. UNIVERSAL SET

In any mathematical discussion, we consider all the sets to be subsets of a given fixed set known as a universal set. It is generally denoted by U (or X).

For example. (*i*) if A is the set of B.Sc. Final year students of your college, B is the set of all boys in your college, C is the set of all cricket players in your college, G is the set of all girls in your college, then the set of all students of your college is a universal set.

(*ii*) In any study of human population, all the people in the world constitute a universal set.

1.15. DIFFERENCE OF SETS

The difference of two sets A and B is the set of all elements which are in A but not in B. The difference of sets A and B is denoted by $A - B$.

Symbolically, $A - B = \{x : x \in A \text{ and } x \notin B\}$.

For example. (*i*) if $A = \{1, 2, 3, 4, 5\}$ and $B = \{2, 4, 6, 8\}$, then $A - B = \{1, 3, 5\}$, $B - A = \{6, 8\}$

Clearly, $A - B \ne B - A$.

Thus 'the difference of sets' is not commutative.

(*ii*) if $A = \{a, b, c\}$, $B = \{c, d, e\}$ and $C = \{b, d\}$

then $B - C = \{c, e\}$, $A - (B - C) = \{a, b\}$, $A - B = \{a, b\}$, $(A - B) - C = \{a\}$

Clearly, $A - (B - C) \ne (A - B) - C$

Thus the 'difference of sets' is not associative.

Also $A - A = \phi$, $A - \phi = A$, $A - B \subset A$.

1.16. COMPLEMENT OF A SET

Let U be the universal set and $A \subset U$. Then the complement of A is the set of those elements of U which are not in A. The complement of A is denoted by A^c.

Symbolically, $A^c = U - A = \{x : x \in U \text{ and } x \notin A\} = \{x : x \notin A\}$.

For example. (*i*) if U is the set of all natural numbers and A, the set of even natural numbers, then

$A^c = U - A$

= the set of those natural numbers which are not even

= the set of odd natural numbers.

(*ii*) If U is the set of all letters of English alphabet and A is the set of consonants, then

$A^c = U - A$

= the set of letters of English alphabet which are not consonants

= the set of vowels in English alphabet

Clearly, (*i*) $\mathbf{A \cup A^c = U}$

Thus $\mathbf{x \in A \Rightarrow x \notin A^c}$ and $\mathbf{x \notin A^c \Rightarrow x \in A}$.

(*ii*) $A \cap A^c = \phi$.

(*iii*) **Complement of complement of a set is the set itself *i.e.*, $(A^c)^c = A$.**

Proof. $(A^c)^c = \{x : x \notin A^c\}$

$= \{x : x \in A\} = A$

(*iv*) $\mathbf{A - B = A \cap B^c}$

$A - B = \{x : x \in A \text{ and } x \notin B\} = \{x : x \in A \text{ and } x \in B^c\} = A \cap B^c$.

1.17. PRINCIPLE OF DUALITY

Statement. All the laws of algebra of sets remain true if we interchange union and intersection as also the universal set and the null set.

For example. (*i*) if the law is $(A \cup B)^c = A^c \cap B^c$, then its dual is obtained by interchanging union and intersection *i.e.*, dual will be $(A \cap B)^c = A^c \cup B^c$.

(*ii*) if the law is $A \cap A^c = \phi$, then its dual is obtained on replacing $\cap$ by $\cup$ and ϕ by U. *i.e.*, dual will be $A \cup A^c = U$.

ILLUSTRATIVE EXAMPLES

Example 1. *If A, B, C are any three sets, then prove that*

(*a*) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. *(Union distributes over intersection)*

(*b*) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. *(Intersection distributes over union)*.

Sol. (*a*) Let x be any element of $A \cup (B \cap C)$. Then

$x \in A \cup (B \cap C) \Rightarrow x \in A$ or $x \in B \cap C$

$\Rightarrow x \in A$ or $(x \in B$ and $x \in C) \Rightarrow (x \in A$ or $x \in B)$ and $(x \in A$ or $x \in C)$

$\Rightarrow x \in A \cup B$ and $x \in A \cup C \Rightarrow x \in (A \cup B) \cap (A \cup C)$

$\therefore \quad A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$ ...(*i*)

Again, let x be any element of $(A \cup B) \cap (A \cup C)$. Then

$x \in (A \cup B) \cap (A \cup C) \Rightarrow x \in A \cup B$ and $x \in A \cup C$

$\Rightarrow (x \in A$ or $x \in B)$ and $(x \in A$ or $x \in C) \Rightarrow x \in A$ or $(x \in B$ and $x \in C)$

$\Rightarrow x \in A$ or $x \in B \cap C \Rightarrow x \in A \cup (B \cap C)$

$\therefore \quad (A \cup B) \cap (A \cup C) \subset A \cup (B \cap C)$ ...(*ii*)

Combining (*i*) and (*ii*), $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

(*b*) Let x be any element of $A \cap (B \cup C)$. Then

$x \in A \cap (B \cup C) \Rightarrow x \in A$ and $x \in B \cup C$

$\Rightarrow x \in A$ and $(x \in B$ or $x \in C) \Rightarrow (x \in A$ and $x \in B)$ or $(x \in A$ and $x \in C)$

$\Rightarrow x \in A \cap B$ or $x \in A \cap C \Rightarrow x \in (A \cap B) \cup (A \cap C)$

$A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$ ...(*i*)

Again, let x be any element of $(A \cap B) \cup (A \cap C)$. Then

$x \in (A \cap B) \cup (A \cap C) \Rightarrow x \in A \cap B$ or $x \in A \cap C$

$\Rightarrow (x \in A$ and $x \in B)$ or $(x \in A$ and $x \in C) \Rightarrow x \in A$ and $(x \in B$ or $x \in C)$

$\Rightarrow x \in A$ and $x \in B \cup C \Rightarrow x \in A \cap (B \cup C)$

$\therefore \quad (A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$ ...(*ii*)

Combining (*i*) and (*ii*), $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Example 2. *State and prove De Morgan's laws. Or*

For any two sets A and B, prove that

(*a*) $(A \cup B)^c = A^c \cap B^c$ (*b*) $(A \cap B)^c = A^c \cup B^c$.

Sol. (*a*) Let x be any element of $(A \cup B)^c$. Then

$x \in (A \cup B)^c \Rightarrow x \notin A \cup B$

$\Rightarrow x \notin A$ and $x \notin B \Rightarrow x \in A^c$ and $x \in B^c$

$\Rightarrow x \in A^c \cap B^c$

$\therefore \quad (A \cup B)^c \subset A^c \cap B^c$ ...(*i*)

Again, let x be any element of $A^c \cap B^c$. Then

$x \in A^c \cap B^c \Rightarrow x \in A^c$ and $x \in B^c$

$\Rightarrow x \notin A$ and $x \notin B \Rightarrow x \notin A \cup B \Rightarrow x \in (A \cup B)^c$

$\therefore \quad A^c \cap B^c \subset (A \cup B)^c$ ...(*ii*)

Combining (*i*) and (*ii*), $(A \cup B)^c = A^c \cap B^c$.

In words. Complement of the union is the intersection of the complements.

(*b*) Let x be any element of $(A \cap B)^c$. Then

$x \in (A \cap B)^c \Rightarrow x \notin A \cap B$

$\Rightarrow x \notin A$ or $x \notin B \Rightarrow x \in A^c$ or $x \in B^c \Rightarrow x \in A^c \cup B^c$

$\therefore \quad (A \cap B)^c \subset A^c \cup B^c$ ...(*i*)

Again, let x be any element of $A^c \cup B^c$. Then

$x \in A^c \cup B^c \Rightarrow x \in A^c$ or $x \in B^c$

$\Rightarrow x \notin A$ or $x \notin B \Rightarrow x \notin A \cap B \Rightarrow x \in (A \cap B)^c$

$\therefore \quad A^c \cup B^c \subset (A \cap B)^c$ ...(*ii*)

Combining (*i*) and (*ii*), $(A \cap B)^c = A^c \cup B^c$.

In words. Complement of the intersection is the union of the complements.

Example 3. *If $A \subset B$ and $B \subset C$, then $A \subset C$.*

Sol. Let x be any element of A. Then

$x \in A \Rightarrow x \in B$ $[\because A \subset B]$

$\Rightarrow x \in C$ $[\because B \subset C]$

$\therefore \quad A \subset C$.

Example 4. *If $A \subset B$, $B \subset C$ and $C \subset A$, then $A = C$.*

Sol. $A \subset B$ and $B \subset C \Rightarrow A \subset C$ (See Ex. 3)

Also $C \subset A$ (given)

$\therefore \quad A = C$.

Example 5. *If A and B are any two sets, then prove that $B \subset A \Leftrightarrow A^c \subset B^c$.*

Sol. Let $B \subset A$. Let x be any element of A^c. Then

$x \in A^c \Rightarrow x \notin A \Rightarrow x \notin B$ $[\because B \subset A]$

$\Rightarrow x \in B^c \quad \therefore A^c \subset B^c$

$\therefore \quad B \subset A \Rightarrow A^c \subset B^c$ ...(*i*)

Let $A^c \subset B^c$. Let x be any element of B. Then

$x \in B \Rightarrow x \notin B^c \Rightarrow x \notin A^c$ $[\because A^c \subset B^c]$

$\Rightarrow x \in A \quad \therefore B \subset A$

$\therefore \quad A^c \subset B^c \Rightarrow B \subset A$ ...(*ii*)

Combining (*i*) and (*ii*), $B \subset A \Leftrightarrow A^c \subset B^c$.

Example 6. *For any three sets A, B and C, prove that*

(*a*) $A - (B \cup C) = (A - B) \cap (A - C)$

(*b*) $A - (B \cap C) = (A - B) \cup (A - C)$.

Sol. (*a*) Let x be any element of $A - (B \cup C)$. Then

$x \in A - (B \cup C) \Rightarrow x \in A$ and $x \notin B \cup C$

$\Rightarrow x \in A$ and $(x \notin B$ and $x \notin C)$

$\Rightarrow (x \in A$ and $x \notin B)$ and $(x \in A$ and $x \notin C)$

$\Rightarrow x \in A - B$ and $x \in A - C \Rightarrow x \in (A - B) \cap (A - C)$

$\therefore \quad A - (B \cup C) \subset (A - B) \cap (A - C)$ ...(*i*)

Again, let x be any element of $(A - B) \cap (A - C)$. Then

$x \in (A - B) \cap (A - C) \Rightarrow x \in A - B$ and $x \in A - C$

$\Rightarrow$ ($x \in A$ and $x \notin B$) and ($x \in A$ and $x \notin C$)

$\Rightarrow$ $x \in A$ and ($x \notin B$ and $x \notin C$)

$\Rightarrow$ $x \in A$ and $x \notin B \cup C \Rightarrow x \in A - (B \cup C)$

$\therefore$ $(A - B) \cap (A - C) \subset A - (B \cup C)$ ...(*ii*)

Combining (*i*) and (*ii*), $A - (B \cup C) = (A - B) \cap (A - C)$.

Second Method

$$\begin{aligned} A - (B \cup C) &= A \cap (B \cup C)^c && [\because A - B = A \cap B^c] \\ &= A \cap (B^c \cap C^c) && [\text{De Morgan's law}] \\ &= (A \cap B^c) \cap (A \cap C^c) = (A - B) \cap (A - C). \end{aligned}$$

(*b*) Let x be any element of $A - (B \cap C)$. Then

$x \in A - (B \cap C) \Rightarrow x \in A$ and $x \notin B \cap C$

$\Rightarrow$ $x \in A$ and ($x \notin B$ or $x \notin C$) $\Rightarrow$ ($x \in A$ and $x \notin B$) or ($x \in A$ and $x \notin C$)

$\Rightarrow$ $x \in A - B$ or $x \in A - C$ $\Rightarrow$ $x \in (A - B) \cup (A - C)$

$\therefore$ $A - (B \cap C) \subset (A - B) \cup (A - C)$ ...(*i*)

Again, let x be any element of $(A - B) \cup (A - C)$. Then

$x \in (A - B) \cup (A - C)$ $\Rightarrow$ $x \in A - B$ or $x \in A - C$

$\Rightarrow$ ($x \in A$ and $x \notin B$) or ($x \in A$ and $x \notin C$) $\Rightarrow$ $x \in A$ and ($x \notin B$ or $x \notin C$)

$\Rightarrow$ $x \in A$ and $x \notin B \cap C$ $\Rightarrow$ $x \in A - (B \cap C)$ ...(*ii*)

$\therefore$ $(A - B) \cup (A - C) \subset A - (B \cap C)$

Combining (*i*) and (*ii*), $A - (B \cap C) = (A - B) \cup (A - C)$.

Second Method

$$\begin{aligned} A - (B \cap C) &= A \cap (B \cap C)^c && [\because A - B = A \cap B^c] \\ &= A \cap (B^c \cup C^c) && [\text{De Morgan's Law}] \\ &= (A \cap B^c) \cup (A \cap C^c) && [\text{Distributive Law}] \\ &= (A - B) \cup (A - C). \end{aligned}$$

Example 7. *Prove that* $A - B = A - (A \cap B)$.

Sol. Let x be any element of $A - B$. Then

$x \in A - B$ $\Rightarrow$ $x \in A$ and $x \notin B$

$\Rightarrow$ $x \in A$ and $x \notin A \cap B$ $\Rightarrow$ $x \in A - (A \cap B)$

$\therefore$ $A - B \subset A - (A \cap B)$ ...(*i*)

Again, let x be any element of $A - (A \cap B)$. Then

$x \in A - (A \cap B)$ $\Rightarrow$ $x \in A$ and $x \notin A \cap B$

$\Rightarrow$ $x \in A$ and $x \notin B$ $\Rightarrow$ $x \in A - B$

$\therefore$ $A - (A \cap B) \subset A - B$

Combining (*i*) and (*ii*), $A - B = A - (A \cap B)$ ...(*ii*)

Second Method

$$\begin{aligned} A - (A \cap B) &= A \cap (A \cap B)^c && [\because A - B = A \cap B^c] \\ &= A \cap (A^c \cup B^c) && [\text{De Morgan's Law}] \\ &= (A \cap A^c) \cup (A \cap B^c) && [\text{Distributive Law}] \\ &= \phi \cup (A - B) && [\because A \cap A^c = \phi] \\ &= A - B. \end{aligned}$$

Example 8. *Prove that* $A \cap (B - C) = (A \cap B) - C$.

Sol. Let x be any element of $A \cap (B - C)$. Then

$x \in A \cap (B - C)$ $\Rightarrow$ $x \in A$ and $x \in (B - C)$

$\Rightarrow$ $x \in A$ and ($x \in B$ and $x \notin C$) $\Rightarrow$ ($x \in A$ and $x \in B$) and $x \notin C$

$\Rightarrow$ $x \in A \cap B$ and $x \notin C$ $\Rightarrow$ $x \in (A \cap B) - C$

$\therefore$ $A \cap (B - C) \subset (A \cap B) - C$ ...(*i*)

Again, let x be any element of $(A \cap B) - C$. Then

$x \in (A \cap B) - C$ $\Rightarrow$ $x \in A \cap B$ and $x \notin C$

$\Rightarrow$ ($x \in A$ and $x \in B$) and $x \notin C$ $\Rightarrow$ $x \in A$ and ($x \in B$ and $x \notin C$)

$\Rightarrow$ $x \in A$ and $x \in B - C$ $\Rightarrow$ $x \in A \cap (B - C)$

$\therefore$ $(A \cap B) - C \subset A \cap (B - C)$ ...(*ii*)

Combining (*i*) and (*ii*), $A \cap (B - C) = (A \cap B) - C$.

Second Method

$$\begin{aligned} A \cap (B - C) &= A \cap (B \cap C^c) \\ &= (A \cap B) \cap C^c && [\text{Intersection is associative}] \\ &= (A \cap B) - C. && [\because A \cap B^c = A - B] \end{aligned}$$

Example 9. *Prove that* $A - (A - B) = A \cap B$.

Sol. Let x be any element of $A - (A - B)$. Then

$x \in A - (A - B)$ $\Rightarrow$ $x \in A$ and $x \notin A - B$

$\Rightarrow$ $x \in A$ and ($x \in A$ and $x \in B$) [Note]

$\Rightarrow$ ($x \in A$ and $x \in A$) and $x \in B$

$\Rightarrow$ $x \in A$ and $x \in B$ $\Rightarrow$ $x \in A \cap B$

$\therefore$ $A - (A - B) \subset A \cap B$ ...(*i*)

Again, let x be any element of $A \cap B$. Then

$x \in A \cap B$ $\Rightarrow$ $x \in A$ and $x \in B$

$\Rightarrow$ $x \in A$ and ($x \in A$ and $x \in B$)

$\Rightarrow$ $x \in A$ and $x \notin A - B$ $\Rightarrow$ $x \in A - (A - B)$

$\therefore$ $A \cap B \subset A - (A - B)$ ...(*ii*)

Combining (*i*) and (*ii*), $A - (A - B) = A \cap B$.

Second Method

$$\begin{aligned} A - (A - B) &= A - (A \cap B^c) = A \cap (A \cap B^c)^c \\ &= A \cap (A^c \cup B) && [\because (B^c)^c = B] \\ &= (A \cap A^c) \cup (A \cap B) && [\text{Distributive Law}] \\ &= \phi \cup (A \cap B) = A \cap B. \end{aligned}$$

Example 10. *Prove that* $A = (A \cap B) \cup (A - B)$.

Sol. Let x be any element of A. Then

$x \in A$ $\Rightarrow$ $x \in A \cap U$ [$\because$ $A \cap U = A$]

$\Rightarrow$ $x \in A \cap (B \cup B^c)$ [$\because$ $U = B \cup B^c$]

$\Rightarrow$ $x \in A$ and ($x \in B$ or $x \in B^c$) $\Rightarrow$ ($x \in A$ and $x \in B$) or ($x \in A$ and $x \notin B$)

$\Rightarrow$ $x \in A \cap B$ or $x \in A - B$ $\Rightarrow$ $x \in (A \cap B) \cup (A - B)$

$\therefore$ $A \subset (A \cap B) \cup (A - B)$ ...(*i*)

Again, let x be any element of $(A \cap B) \cup (A - B)$. Then

$x \in (A \cap B) \cup (A - B) \Rightarrow x \in A \cap B$ or $x \in A - B$

$\Rightarrow (x \in A$ and $x \in B)$ or $(x \in A$ and $x \notin B) \Rightarrow x \in A$ and $(x \in B$ or $x \notin B)$

$\Rightarrow x \in A$ and $(x \in B$ or $x \in B^c) \Rightarrow x \in A$ and $x \in B \cup B^c$

$\Rightarrow x \in A$ and $x \in U \Rightarrow x \in A \cap U \Rightarrow x \in A$

$\therefore \quad (A \cap B) \cup (A - B) \subset A$ ...(ii)

Combining (i) and (ii), $A = (A \cap B) \cup (A - B)$.

Second Method

$$(A \cap B) \cup (A - B) = (A \cap B) \cup (A \cap B^c)$$
$$= A \cap (B \cup B^c) \quad \text{[Distributive Law]}$$
$$= A \cap U \quad [\because B \cup B^c = U]$$
$$= A.$$

Example 11. *Prove that* $A - (B - C) = (A - B) \cup (A \cap C)$.

Sol. Let x be any element of $A - (B - C)$. Then

$x \in A - (B - C) \Rightarrow x \in A$ and $x \notin (B - C)$

$\Rightarrow x \in A$ and $(x \notin B$ or $x \in C)$ | Note

$\Rightarrow (x \in A$ and $x \notin B)$ or $(x \in A$ and $x \in C)$

$\Rightarrow x \in A - B$ or $x \in A \cap C \Rightarrow x \in (A - B) \cup (A \cap C)$

$\therefore \quad A - (B - C) \subset (A - B) \cup (A \cap C)$ ...(i)

Again, let x be any element of $(A - B) \cup (A \cap C)$. Then

$x \in (A - B) \cup (A \cap C) \Rightarrow x \in A - B$ or $x \in A \cap C$

$\Rightarrow (x \in A$ and $x \notin B)$ or $(x \in A$ and $x \in C) \Rightarrow x \in A$ and $(x \notin B$ or $x \in C)$

$\Rightarrow x \in A$ and $x \notin (B - C) \Rightarrow x \in A - (B - C)$

$\therefore \quad (A - B) \cup (A \cap C) \subset A - (B - C)$ ...(ii)

Combining (i) and (ii), $A - (B - C) = (A - B) \cup (A \cap C)$.

Second Method

$$A - (B - C) = A - (B \cap C^c) = A \cap (B \cap C^c)^c$$
$$= A \cap (B^c \cup C) \quad [\because (A \cap B)^c = A^c \cup B^c \text{ and } (A^c)^c = A]$$
$$= (A \cap B^c) \cup (A \cap C) \quad \text{[Distributive Law]}$$
$$= (A - B) \cup (A \cap C).$$

Example 12. *Prove that* $A \cap (B - C) = (A \cap B) - (A \cap C)$.

Sol. Let x be any element of $A \cap (B - C)$. Then

$x \in A \cap (B - C) \Rightarrow x \in A$ and $x \in B - C$

$\Rightarrow x \in A$ and $(x \in B$ and $x \notin C) \Rightarrow (x \in A$ and $x \in B)$ and $(x \in A$ and $x \notin C)$

$\Rightarrow x \in A \cap B$ and $x \notin A \cap C \Rightarrow x \in (A \cap B) - (A \cap C)$

$\therefore \quad A \cap (B - C) \subset (A \cap B) - (A \cap C)$ ...(i)

Again, let x be any element of $(A \cap B) - (A \cap C)$. Then

$x \in (A \cap B) - (A \cap C) \Rightarrow x \in A \cap B$ and $x \notin A \cap C$

$\Rightarrow (x \in A$ and $x \in B)$ and $(x \in A$ and $x \notin C) \Rightarrow x \in A$ and $(x \in B$ and $x \notin C)$

$\Rightarrow x \in A$ and $x \in B - C \Rightarrow x \in A \cap (B - C)$

$\therefore \quad (A \cap B) - (A \cap C) \subset A \cap (B - C)$ ...(ii)

Combining (i) and (ii), $A \cap (B - C) = (A \cap B) - (A \cap C)$.

Example 13. *Prove that for any two sets A and B.*

$$(A - B) \cup (B - A) = (A \cup B) - (A \cap B).$$

Sol. Let x be any element of $(A - B) \cup (B - A)$. Then

$x \in (A - B) \cup (B - A) \Rightarrow x \in A - B$ or $x \in B - A$

$\Rightarrow (x \in A$ and $x \notin B)$ or $(x \in B$ and $x \notin A) \Rightarrow (x \in A$ or $x \in B)$ and $(x \notin A$ or $x \notin B)$

$\Rightarrow x \in A \cup B$ and $x \notin A \cap B \Rightarrow x \in (A \cup B) - (A \cap B)$

$\therefore \quad (A - B) \cup (B - A) \subset (A \cup B) - (A \cap B)$ ...(i)

Again, let x be any element of $(A \cup B) - (A \cap B)$. Then

$x \in (A \cup B) - (A \cap B) \Rightarrow x \in A \cup B$ and $x \notin A \cap B$

$\Rightarrow (x \in A$ or $x \in B)$ and $(x \notin A$ or $x \notin B) \Rightarrow (x \in A$ and $x \notin B)$ or $(x \in B$ and $x \notin A)$

$\Rightarrow x \in A - B$ or $x \in B - A \Rightarrow x \in (A - B) \cup (B - A)$

$\therefore \quad (A \cup B) - (A \cap B) \subset (A - B) \cup (B - A)$ ...(ii)

Combining (i) and (ii), $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$.

Example 14. *Prove that*

(i) $A \subset B$ *iff* $A \cup B = B$ (ii) $A \subset B$ *iff* $A \cap B = A$.

Sol. (i) Let $A \subset B$. To prove that $A \cup B = B$, let x be any element of $A \cup B$. Then

$x \in A \cup B \Rightarrow x \in A$ or $x \in B$

$\Rightarrow x \in B$ or $x \in B$ $(\because A \subset B)$

$\Rightarrow x \in \bar{B}$

$\therefore \quad A \cup B \subset B$ ...(i)

Also, we know that $B \subset A \cup B$ ...(ii)

Combining (i) and (ii), $A \cup B = B$

$\therefore \quad A \subset B \Rightarrow A \cup B = \overline{B}$ ...(I)

Now, let $A \cup B = B$. To prove that $A \subset B$, let x be any element of A. Then

$x \in A \Rightarrow x \in A \cup B$ $[\because A \subset A \cup B]$

$\Rightarrow x \in B$ $[\because A \cup B = B]$

$\therefore \quad A \subset B$

i.e., $A \cup B = B \Rightarrow A \subset B$ ...(II)

Combining (I) and (II), we have $A \subset B$ iff $A \cup B = B$.

(ii) Let $A \subset B$. To prove that $A \cap B = A$, we know that

$A \cap B \subset A$ ...(i)

Now, let x be any element of A. Then

$x \in A \Rightarrow x \in A$ and $x \in A$

$\Rightarrow x \in A$, and $x \in B$ $[\because A \subset B]$

$\Rightarrow x \in A \cap B$

$\therefore \quad A \subset A \cap B$ ...(ii)

Combining (i) and (ii), $A \cap B = A$,

$\therefore \quad A \subset B \Rightarrow A \cap B = A$ ...(I)

Now let $A \cap B = A$. To prove that $A \subset B$, let x be any element of A. Then

$x \in A \Rightarrow x \in A \cap B$ $[\because A = A \cap B]$

$\Rightarrow x \in A$ and $x \in B \Rightarrow x \in B$

$\therefore$ $A \subset B$

i.e., $A \cap B = A \Rightarrow A \subset B$ ...(II)

Combining (I) and (II), we have $A \subset B$ iff $A \cap B = A$.

Example 15. *Prove that $A \cup B = A \cap B \Leftrightarrow A = B$.*

Sol. Let $A \cup B = A \cap B$. To prove that $A = B$, let x be any element of A. Then

$x \in A \Rightarrow x \in A \cup B$ [$\because$ $A \subset A \cup B$]

$\Rightarrow$ $x \in A \cap B$ [$\because$ $A \cup B = A \cap B$]

$\Rightarrow$ $x \in A$ and $x \in B \Rightarrow x \in B$

$\therefore$ $A \subset B$ ...(*i*)

Again, let x be any element of B. Then

$x \in B \Rightarrow x \in A \cup B$ [$\because$ $B \subset A \cup B$]

$\Rightarrow$ $x \in A \cap B$ [$\because$ $A \cup B = A \cap B$]

$\Rightarrow$ $x \in A$ and $x \in B \Rightarrow x \in A$

$\therefore$ $B \subset A$ ...(*ii*)

Combining (*i*) and (*ii*), $A = B$.

$\therefore$ $A \cup B = A \cap B \Rightarrow A = B$ ...(I)

Now let $A = B$. To prove that $A \cup B = A \cap B$.

$A \cup B = A \cap A$ [$\because$ $B = A$]

$\Leftrightarrow$ $= A = A \cap A$

$\Leftrightarrow$ $= A \cap B$

$\therefore$ $A = B \Rightarrow A \cup B = A \cap B$ ...(II)

From (I) and (II), $A \cup B = A \cap B \Leftrightarrow A = B$.

Example 16. *Prove that $(A - B) - C = (A - C) - (B - C)$.*

Sol. Let x be any element of $(A - B) - C$. Then

$x \in (A - B) - C \Rightarrow x \in A - B$ and $x \notin C$

$\Rightarrow$ ($x \in A$ and $x \notin B$) and $x \notin C$ $\Rightarrow$ ($x \in A$ and $x \notin C$) and ($x \notin B$ and $x \notin C$)

$\Rightarrow$ $x \in A - C$ and $x \notin B - C$ $\Rightarrow$ $x \in (A - C) - (B - C)$

$\therefore$ $(A - B) - C \subset (A - C) - (B - C)$ ...(*i*)

Again, let x be any element of $(A - C) - (B - C)$. Then

$x \in (A - C) - (B - C) \Rightarrow x \in A - C$ and $x \notin B - C$

$\Rightarrow$ ($x \in A$ and $x \notin C$) and ($x \notin B$ and $x \notin C$) $\Rightarrow$ ($x \in A$ and $x \notin B$) and $x \notin C$

$\Rightarrow$ $x \in A - B$ and $x \notin C$ $\Rightarrow$ $x \in (A - B) - C$

$\therefore$ $(A - C) - (B - C) \subset (A - B) - C$ ...(*ii*)

Combining (*i*) and (*ii*), we get $(A - B) - C = (A - C) - (B - C)$.

Example 17. *Prove that $A - B = A \cap B^c = B^c - A^c$.*

Sol. Let x be any element of $A - B$. Then

$x \in A - B \Leftrightarrow x \in A$ and $x \notin B$

$\Leftrightarrow$ $x \in A$ and $x \in B^c$ $\Leftrightarrow$ $x \in A \cap B^c$

$\therefore$ $A - B = A \cap B^c$ ...(*i*)

Again $x \in A - B \Leftrightarrow x \in A$ and $x \notin B$

$\Leftrightarrow$ $x \notin A^c$ and $x \in B^c$ $\Leftrightarrow$ $x \in B^c$ and $x \notin A^c$ $\Leftrightarrow$ $x \in B^c - A^c$

$\therefore$ $A - B = B^c - A^c$ ...(*ii*)

Combining (*i*) and (*ii*), we get $A - B = A \cap B^c = B^c - A^c$.

Example 18. *Prove that $B - A^c = B \cap A$.*

Sol. *Please try yourself.*

Example 19. *Prove that*

(*i*) $A \subset B \Rightarrow A \cup C \subset B \cup C$. (*ii*) $A \subset B \Rightarrow A \cap C \subset B \cap C$.

Sol. *Please try yourself.*

Example 20. *Prove that $(A - C) \cap (B - C) = (A \cap B) - C$.*

Sol. *Please try yourself.*

Example 21. *If for two sets A and B, $A \cup B = A$ and $A \cap B = A$, then $A = B$.*

Sol. $A \cup B = A \Rightarrow B \subset A$

$A \cap B = A \Rightarrow A \subset B$

$\therefore$ $A = B$.

Example 22. *For any three sets A, B and C, prove that if $A \cup B = A \cup C$ and $A \cap B = A \cap C$, then*

$$B = C.$$

Sol. Let x be any element of B.

$x \in B \Rightarrow x \in A \cup B$ [$\because$ $B \subset A \cup B$]

$\Rightarrow$ $x \in A \cup C$ [$\because$ $A \cup B = A \cup C$]

$\Rightarrow$ $x \in A$ or $x \in C$ ...(*i*)

If $x \in B \Rightarrow x \in A$, then $x \in A \cap B$

$\Rightarrow$ $x \in A \cap C$ [$\because$ $A \cap B = A \cap C$] (in particular)

$\Rightarrow$ $x \in C$

$\therefore$ From (*i*) $x \in B \Rightarrow x \in C$

$\therefore$ $B \subset C$

Similarly $C \subset B$

Hence $B = C$.

1.18. SYMMETRIC DIFFERENCE

If A and B are any two sets, then the set $(A - B) \cup (B - A)$ is called the symmetric difference of A and B.

The symmetric difference of A and B is denoted by $A \Delta B$ and read as 'A symmetric difference B'.

For example. If $A = \{a, b, c, d, e\}$ and $B = \{c, d, e, f, g\}$, then $A - B = \{a, b\}$, $B - A = \{f, g\}$

$\therefore$ $A \Delta B = (A - B) \cup (B - A) = \{a, b\} \cup \{f, g\} = \{a, b, f, g\}$.

Example 1. *Prove the following :*

(*i*) $A \Delta A = \phi$ (*ii*) $A \Delta \phi = A$

(*iii*) $A \Delta B = B \Delta A$ (*iv*) $A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)$.

Sol. (*i*) $A \Delta A = (A - A) \cup (A - A) = \phi \cup \phi = \phi$.

(*ii*) $A \Delta \phi = (A - \phi) \cup (\phi - A) = A \cup \phi = A$

(*iii*) $A \Delta B = (A - B) \cup (B - A) = (B - A) \cup (A - B)$ [$\because$ union of sets is commutative]

$= B \Delta A$.

(*iv*) $A \cap (B \Delta C) = A \cap [(B - C) \cup (C - B)]$

$= [A \cap (B - C)] \cup [A \cap (C - B)]$ (Distributive Law)

$= (A \cap B - A \cap C) \cup (A \cap C - A \cap B) = (A \cap B) \Delta (A \cap C).$

Example 2. *Prove that* $A \Delta B = (A \cup B) - (A \cap B) = (A \cup B) \cap (A^c \cup B^c)$.

Sol. Let x be any element of $A \Delta B$. Then

$x \in A \Delta B \Rightarrow x \in (A - B) \cup (B - A)$

$\Rightarrow x \in A - B$ or $x \in B - A \Rightarrow (x \in A$ and $x \notin B)$ or $(x \in B$ and $x \notin A)$

$\Rightarrow (x \in A$ or $x \in B)$ and $(x \notin A$ or $x \notin B) \Rightarrow x \in A \cup B$ and $x \notin A \cap B$

$\Rightarrow x \in (A \cup B) - (A \cap B)$

$\therefore A \Delta B \subset (A \cup B) - (A \cap B)$ ...(1)

Again, let x be any element of $(A \cup B) - (A \cap B)$. Then

$x \in (A \cup B) - (A \cap B) \Rightarrow x \in A \cup B$ and $x \notin A \cap B$

$\Rightarrow (x \in A$ or $x \in B)$ and $(x \notin A$ or $x \notin B) \Rightarrow (x \in A$ and $x \notin B)$ or $(x \in B$ and $x \notin A)$

$\Rightarrow x \in A - B$ or $x \in B - A \Rightarrow x \in (A - B) \cup (B - A)$

$\Rightarrow x \in A \Delta B$

$\therefore (A \cup B) - (A \cap B) \subset A \Delta B$ ...(2)

From (1) and (2), $A \Delta B = (A \cup B) - (A \cap B)$ ...(3)

Since $A - B = A \cap B^c$

$\therefore (A \cup B) - (A \cap B) = (A \cup B) \cap (A \cap B)^c$

$= (A \cup B) \cap (A^c \cup B^c)$ (By De Morgan's Law)

$\therefore$ From (3), $A \Delta B = (A \cup B) - (A \cap B) = (A \cup B) \cap (A^c \cup B^c)$.

Example 3. *Prove that* $A \Delta B = \phi \Leftrightarrow A = B$.

Sol. To prove $A \Delta B = \phi \Rightarrow A = B$

$A \Delta B = \phi \Rightarrow (A - B) \cup (B - A) = \phi$

$\Rightarrow A - B = \phi$ and $B - A = \phi \Rightarrow A \subset B$ and $B \subset A \Rightarrow A = B$

To prove $A = B \Rightarrow A \Delta B = \phi$

$A \Delta B = A \Delta A = (A - A) \cup (A - A) = \phi \cup \phi = \phi$

Hence $A \Delta B = \phi \Leftrightarrow A = B$.

1.19. INDEXED FAMILY OF SETS

Let Λ be a non-empty set. If for each $\lambda \in \Lambda$, we are given a set A_λ, then we say $\{A_\lambda\}_{\lambda \in \Lambda}$ is a family of sets indexed by the set Λ.

1.20. UNION AND INTERSECTION OF AN ARBITRARY FAMILY OF SETS

(*i*) **Union.** Let $\{A_\lambda\}_{\lambda \in \Lambda}$ be an arbitrary family of sets. Then the union of this arbitrary family of sets is denoted by $\bigcup_{\lambda \in \Lambda} A_\lambda$ (read as 'union of sets A_λ for $\lambda \in \Lambda$') and defined as

$$\bigcup_{\lambda \in \Lambda} A_\lambda = \{x : x \in A_\lambda \text{ for at least one } \lambda \in \Lambda\}.$$

(*ii*) **Intersection.** Let $\{A_\lambda\}_{\lambda \in \Lambda}$ be an arbitrary family of sets. Then the intersection of this arbitrary family of sets is denoted by $\bigcap_{\lambda \in \Lambda} A_\lambda$ (read as 'intersection of sets A_λ for $\lambda \in \Lambda$') and defined as

$$\bigcap_{\lambda \in \Lambda} A_\lambda = \{x : x \in A_\lambda \text{ for every } \lambda \in \Lambda\}$$

Generalised De Morgan's Laws

If $\{A_\lambda\}_{\lambda \in \Lambda}$ be an indexed family of sets, then

(*i*) $\left(\bigcup_{\lambda \in \Lambda} A_\lambda\right)^c = \bigcap_{\lambda \in \Lambda} A_\lambda^c$ (*ii*) $\left(\bigcap_{\lambda \in \Lambda} A_\lambda\right)^c = \bigcup_{\lambda \in \Lambda} A_\lambda^c$.

Proof. (*i*) Let x be any element of $\left(\bigcup_{\lambda \in \Lambda} A_\lambda\right)^c$

$x \in \left(\bigcup_{\lambda \in \Lambda} A_\lambda\right)^c \Rightarrow x \notin \bigcup_{\lambda \in \Lambda} A_\lambda$

$\Rightarrow x \notin A_\lambda$ for any $\lambda \in \Lambda \Rightarrow x \in A_\lambda^c$ for every $\lambda \in \Lambda \Rightarrow x \in \bigcap_{\lambda \in \Lambda} A_\lambda^c$

$\therefore \left(\bigcup_{\lambda \in \Lambda} A_\lambda\right)^c \subset \bigcap_{\lambda \in \Lambda} A_\lambda^c$ ...(1)

Again, let x be any element of $\bigcap_{\lambda \in \Lambda} A_\lambda^c$

$x \in \bigcap_{\lambda \in \Lambda} A_\lambda^c \Rightarrow x \in A_\lambda^c$ for every $\lambda \in \Lambda$

$\Rightarrow x \notin A_\lambda$ for any $\lambda \in \Lambda \Rightarrow x \notin \bigcup_{\lambda \in \Lambda} A_\lambda \Rightarrow x \in \left(\bigcup_{\lambda \in \Lambda} A_\lambda\right)^c$

$\therefore \bigcap_{\lambda \in \Lambda} A_\lambda^c \subset \left(\bigcup_{\lambda \in \Lambda} A_\lambda\right)^c$ ...(2)

From (1) and (2), $\left(\bigcup_{\lambda \in \Lambda} A_\lambda\right)^c = \bigcap_{\lambda \in \Lambda} A_\lambda^c$.

(*ii*) *Please try yourself.*

SECTION II—FUNCTIONS

1.21. ORDERED PAIR

An ordered pair is a pair of elements written according to a specified order. Thus if a and b are two elements, then the ordered pair in which a occupies the first place and b, the second place is written as (a, b), whereas the ordered pair in which b occupies the first place and a, the second place is written as (b, a).

The order in which the elements occur in an ordered pair is important

$(a, b) = (b, a)$ iff $a = b$

$(a, b) = (c, d)$ iff $a = c$ and $b = d$.

1.22. (a) CARTESIAN PRODUCT OF TWO SETS

Let A and B be two sets. The set of all ordered pairs (a, b) where $a \in A$ and $b \in B$ is called the Cartesian product of A and B. It is denoted by $A \times B$ and read as 'A cross B'.

Thus $A \times B = \{(a, b) : a \in A, b \in B\}$.

For example. If A = {1, 2, 3} and B = {0, – 1}, then

$$A \times B = \{(1, 0), (1, -1), (2, 0), (2, -1), (3, 0), (3, -1)\}$$

and

$$B \times A = \{(a, b) : a \in B, b \in A\}$$
$$= \{(0, 1), (0, 2), (0, 3), (-1, 1), (-1, 2), (-1, 3)\}$$

Clearly $A \times B \neq B \times A$.

Note 1. If A has m elements and B has n elements, then A × B has $m \times n$ elements.

Note 2. Let R be the set of all real numbers. Then $R \times R = \{(x, y) : x \in R, y \in R\}$ *i.e.*, R × R is the set of all points in the Cartesian plane. It is also denoted by R^2.

1.22. (b) CARTESIAN PRODUCT OF THREE SETS

Let A, B and C be three sets. The set of all ordered triples (a, b, c) where $a \in A, b \in B$ and $c \in C$ is called the Cartesian product of A, B and C. It is denoted by A × B × C.

Thus $A \times B \times C = \{(a, b, c) : a \in A, b \in B, c \in C\}$.

Note 1. $R \times R \times R = \{(x, y, z) : x \in R, y \in R, z \in R\}$ where R is the set of all real numbers. *i.e.*, R × R × R is the set of all points in the Euclidean space. It is also denoted by R^3.

Note 2. In general, if $A_1, A_2, A_3, \ldots\ldots\ldots, A_n$ be n sets, then

$$\prod_{i=1}^{n} A_i \text{ or } A_1 \times A_2 \times A_3 \times \ldots\ldots \times A_n = \{(a_1, a_2, a_3, \ldots\ldots, a_n) : a_i \in A_i, 1 \le i \le n\}$$

$(a_1, a_2, \ldots\ldots, a_n)$ is called an ordered n-tuple.

ILLUSTRATIVE EXAMPLES

Example 1. *If A = {1, 2}, B = {2, 3}, C = {3, 4} find* $(A \times B) \cup (A \times C)$ *and* $(A \times B) \cap (A \times C)$.

Sol.

$$A \times B = \{(1, 2), (1, 3), (2, 2) (2, 3)\}$$
$$A \times C = \{(1, 3), (1, 4), (2, 3), (2, 4)\}$$
$$\therefore \quad (A \times B) \cup (A \times C) = \{(1, 2), (1, 3), (2, 2) (2, 3), (1, 4), (2, 4)\}$$
$$(A \times B) \cap (A \times C) = \{(1, 3), (2, 3)\}.$$

Example 2. *Write down the Cartesian product of the three sets A, B, C where* $A = \{1, 2\}, B = \{\pi, e\}, C = \{1, e\}$.

Sol.

$$A \times B = \{(1, \pi), (1, e), (2, \pi), (2, e)\}$$
$$A \times B \times C = \{(1, \pi, 1), (1, \pi, e), (1, e, 1), (1, e, e), (2, \pi, 1), (2, \pi, e), (2, e, 1), (2, e, e)\}.$$

Example 3. *Let A = {a, b}, B = {1, 2, 3, 4, 5}, C = {3, 5, 7, 9} find* $(A \times B) \cap (A \times C)$.

Sol. *Please try yourself.*

Example 4. *Show that (i)* $A \subset B \Rightarrow A \times C \subset B \times C$

(ii) $A \subset B$ *and* $C \subset D \Rightarrow A \times C \subset B \times D$.

Sol. (*i*) Let (x, y) be any element of A × C. Then

$$(x, y) \in A \times C \quad \Rightarrow \quad x \in A \text{ and } y \in C$$
$$\Rightarrow \quad x \in B \text{ and } y \in C \qquad [\because A \subset B]$$
$$\Rightarrow \quad (x, y) \in B \times C$$
$$\therefore \quad A \times C \subset B \times C.$$

(*ii*) *Please try yourself.*

Example 5. *Prove that* $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

Sol. Let (x, y) be any element of A × (B ∪ C). Then

$$(x, y) \in A \times (B \cup C) \quad \Rightarrow \quad x \in A \text{ and } y \in B \cup C$$
$$\Rightarrow \quad x \in A \text{ and } (y \in B \text{ or } y \in C) \quad \Rightarrow \quad (x \in A \text{ and } y \in B) \text{ or } (x \in A \text{ and } y \in C).$$
$$\Rightarrow \quad (x, y) \in A \times B \text{ or } (x, y) \in A \times C \quad \Rightarrow \quad (x, y) \in (A \times B) \cup (A \times C)$$
$$\therefore \quad A \times (B \cup C) \subset (A \times B) \cup (A \times C) \qquad \ldots(i)$$

Again, let (x, y) be any element of (A × B) ∪ (A × C). Then

$$(x, y) \in (A \times B) \cup (A \times C) \quad \Rightarrow \quad (x, y) \in A \times B \text{ or } (x, y) \in A \times C$$
$$\Rightarrow \quad (x \in A \text{ and } y \in B) \text{ or } (x \in A \text{ and } y \in C) \quad \Rightarrow \quad x \in A \text{ and } (y \in B \text{ or } y \in C)$$
$$\Rightarrow \quad x \in A \text{ and } y \in B \cup C \quad \Rightarrow \quad (x, y) \in A \times (B \cup C)$$
$$\therefore \quad (A \times B) \cup (A \times C) \subset A \times (B \cup C) \qquad \ldots(ii)$$

Combining (*i*) and (*ii*), we get $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

Example 6. *Prove that* $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

Sol. *Please try yourself.*

Example 7. *Prove that* $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

Sol. Let (x, y) be any element of (A × B) ∩ (C × D). Then

$$(x, y) \in (A \times B) \cap (C \times D) \quad \Rightarrow \quad (x, y) \in A \times B \text{ and } (x, y) \in C \times D$$
$$\Rightarrow \quad (x \in A \text{ and } y \in B) \text{ and } (x \in C \text{ and } y \in D)$$
$$\Rightarrow \quad (x \in A \text{ and } x \in C) \text{ and } (y \in B \text{ and } y \in D)$$
$$\Rightarrow \quad x \in A \cap C \text{ and } y \in B \cap D \quad \Rightarrow \quad (x, y) \in (A \cap C) \times (B \cap D)$$
$$\therefore \quad (A \times B) \cap (C \times D) \subset (A \cap C) \times (B \cap D) \qquad \ldots(i)$$

Again, let (x, y) be any element of (A ∩ C) × (B ∩ D). Then

$$(x, y) \in (A \cap C) \times (B \cap D) \quad \Rightarrow \quad x \in A \cap C \text{ and } y \in B \cap D$$
$$\Rightarrow \quad (x \in A \text{ and } x \in C) \text{ and } (y \in B \text{ and } y \in D)$$
$$\Rightarrow \quad (x \in A \text{ and } y \in B) \text{ and } (x \in C \text{ and } y \in D)$$
$$\Rightarrow \quad (x, y) \in A \times B \text{ and } (x, y) \in C \times D \quad \Rightarrow \quad (x, y) \in (A \times B) \cap (C \times D)$$
$$\therefore \quad (A \cap C) \times (B \cap D) \subset (A \times B) \cap (C \times D) \qquad \ldots(ii)$$

Combining (*i*) and (*ii*), we get $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

Example 8. *Prove that* $(A \times C) - (B \times C) = (A - B) \times C$.

Sol. Let (x, y) be any element of (A × C) – (B × C). Then

$$(x, y) \in (A \times C) - (B \times C) \quad \Rightarrow \quad (x, y) \in A \times C \text{ and } (x, y) \notin B \times C$$
$$\Rightarrow \quad (x \in A \text{ and } y \in C) \text{ and } (x \notin B \text{ and } y \in C) \quad \Rightarrow \quad (x \in A \text{ and } x \notin B) \text{ and } y \in C$$
$$\Rightarrow \quad x \in A - B \text{ and } y \in C \quad \Rightarrow \quad (x, y) \in (A - B) \times C$$
$$\therefore \quad (A \times C) - (B \times C) \subset (A - B) \times C \qquad \ldots(i)$$

Again, let (x, y) be any element of (A – B) × C. Then

$$(x, y) \in (A - B) \times C \quad \Rightarrow \quad x \in A - B \text{ and } y \in C$$
$$\Rightarrow \quad (x \in A \text{ and } x \notin B) \text{ and } y \in C \quad \Rightarrow \quad (x \in A \text{ and } y \in C) \text{ and } (x \notin B \text{ and } y \in C)$$
$$\Rightarrow \quad (x, y) \in A \times C \text{ and } (x, y) \notin B \times C \quad \Rightarrow \quad (x, y) \in (A \times C) - (B \times C)$$
$$\therefore \quad (A - B) \times C \subset (A \times C) - (B \times C) \qquad \ldots(ii)$$

Combining (*i*) and (*ii*), we get $(A \times C) - (B \times C) = (A - B) \times C$.

Example 9. *If A and B be non-empty sets, then show that*

$$A \times B = B \times A \text{ iff } A = B.$$

Sol. I. Let A × B = B × A. To prove that A = B.

Let x be any element of A. Then

$x \in A \Rightarrow (x, b) \in A \times B$ for $b \in B$

$\Rightarrow (x, b) \in B \times A$ | ∵ $A \times B = B \times A$

$\Rightarrow x \in B$

$\therefore A \subset B$ ...(i)

Again, let $y \in B$, then

$y \in B \Rightarrow (a, y) \in A \times B$ for $a \in A$

$\Rightarrow (a, y) \in B \times A$ | ∵ $A \times B = B \times A$

$\Rightarrow y \in A$

$\therefore B \subset A$ ...(ii)

Combining (i) and (ii), A = B.

II. Let A = B. To Prove that A × B = B × A

$A \times B = A \times A$ | ∵ B = A

$B \times A = A \times A$ | ∵ B = A

$\therefore A \times B = B \times A$

Hence $A \times B = B \times A$ iff A = B.

.23. RELATIONS

(i) **Def.** If A and B are two sets, then a **relation R** from A to B is a subset of the Cartesian product A × B.

[A **binary relation** is a relation between two objects. In this book, we are only concerned with binary relations, therefore, onwards by a 'relation' we shall mean a 'binary relation'.]

If a is related to b under the relation R, then we write aRb.

Thus $R = \{(x, y) : x \in A, y \in B \text{ and } xRy\}$

The set of first entries of the ordered pairs in a relation is called the **domain of the relation.** The set of second entries of the ordered pairs in a relation is called the **range of the relation.**

For example. If A = {2, 3, 4, 5, 6}, B = {2, 4, 6, 8} and

$xRy \Rightarrow x$ divides y, then the relation R from A to B is given by :

$R = \{(2, 2), (2, 4), (2, 6), (2, 8), (3, 6), (4, 4), (4, 8), (6, 6)\}$

Domain of R = {2, 3, 4, 6} | Dropping the repetitions

Range of R = {2, 4, 6, 8} | Dropping the repetitions

(ii) **Relation on a set.** A relation R from a set A to the set A itself is called a relation on the set A, *i.e.*, $R \subset A \times A$.

Thus $R = \{(x, y) : xRy \text{ and } x, y \in A\}$ is a relation on A.

For example. If A = {1, 2, 3, 4} and R is the relation 'less than', then

$R = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$.

(iii) **Identity Relation (or Diagonal Relation).** An identity relation on a set A is the set of all ordered pairs (x, y) of A × A such that $x = y$. Identity relation on A is usually denoted by I_A.

Thus $I_A = \{(x, y) : x, y \in A \text{ and } x = y\}$.

For example. If A = {a, b, c}, then

$I_A = \{(a, a), (b, b), (c, c)\}$.

(iv) **Inverse Relation.** A relation obtained from a given relation R by reversing the order of the components in the ordered pairs is called the inverse relation of R and is denoted, by R^{-1}, read as 'R inverse.'

Thus $R^{-1} = \{(y, x) : (x, y) \in R\}$.

For example. If A = {1, 2, 3, 4} and R is the relation 'less than', then

$R = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$

and $R^{-1} = \{(2, 1), (3, 1), (4, 1), (3, 2), (4, 2), (4, 3)\}$

Clearly R^{-1} is the relation 'greater than.'

(v) **Reflexive Relation.** A relation R on a set A is said to be reflexive if

$aRa \ \forall \ a \in A$ *i.e.*, if $(a, a) \in R \ \forall \ a \in A$.

For example. (i) $a = a \ \forall \ a \in R$, the set of all real numbers

⇒ The relation 'is equal to' in R is reflexive

(ii) a divides $a \ \forall \ a \in N$

⇒ The relation 'is a divisor of' in N is reflexive.

(vi) **Symmetric Relation.** A relation R on a set A is said to be symmetric if

$aRb \Rightarrow bRa, a, b \in A$ *i.e.*, if $(a, b) \in R \Rightarrow (b, a) \in R$

OR A relation R is symmetric if $R = R^{-1}$

For example. (i) $\forall \ a, b \in R, a = b \Rightarrow b = a$

∴ The relation 'is equal to' in R is symmetric.

(ii) If l_1 and l_2 are any two lines in a plane such that $l_1 \parallel l_2$ then $l_2 \parallel l_1$.

∴ The relation 'is parallel to' in the set of all straight lines in a plane is symmetric.

(vii) **Transitive Relation.** A relation R on a set A is said to be transitive if

aRb and $bRc \Rightarrow aRc$ *i.e.*, if $(a, b) \in R, (b, c) \in R \Rightarrow (a, c) \in R$.

For example. (i) If a, b, c are three real numbers such that

$a > b$ and $b > c$ then $a > c$

⇒ The relation 'is greater than' in R is transitive.

(ii) If $a, b, c \in N$ such that a divides b and b divides c then a divides c.

⇒ The relation 'is a divisor of' in N is transitive.

(viii) **Equivalence Relation.** A relation R on a set A is said to be an equivalence relation if it is

(i) reflexive (ii) symmetric (iii) transitive.

For example. (a) If A is the set of all triangles in the Cartesian plane and R, the relation 'is similar to' (or similarity of Δs), then

(i) Every Δ is similar to itself *i.e.*, $\Delta R \Delta \Rightarrow R$ is reflexive.

(ii) If Δ_1 is similar to Δ_2, then Δ_2 is similar to Δ_1, *i.e.*, $\Delta_1 R \Delta_2 \Rightarrow \Delta_2 R \Delta_1$

∴ R is symmetric.

(iii) If Δ_1 is similar to Δ_2, Δ_2 is similar to Δ_3, then Δ_1 is similar to Δ_3 *i.e.*, $\Delta_1 R \Delta_2, \Delta_2 R \Delta_3 \Rightarrow \Delta_1 R \Delta_3$

∴ R is transitive.

Hence the relation R is an equivalence relation.

(*b*) If A is the set of all straight lines in the Cartesian plane and R, the relation 'is parallel to' (or parallelism of lines), then R is an equivalence relation. (Prove it).

ILLUSTRATIVE EXAMPLES

Example 1. *Give an example of a relation which is*

(*i*) *reflexive but not symmetric or transitive.*
(*ii*) *symmetric but not reflexive or transitive.*
(*iii*) *transitive but not reflexive or symmetric.*
(*iv*) *reflexive and symmetric but not transitive.*
(*v*) *reflexive and transitive but not symmetric*
(*vi*) *symmetric and transitive but not reflexive.*

Sol. Consider $A = \{1, 2, 3, 4\}$. A relation R on A is a subset of $A \times A$.

(*i*) The relation $R = \{(1, 1) (1, 2) (2, 2) (2, 3) (3, 3), (4, 4)\}$ is reflexive but not symmetric or transitive.

$\because \forall a \in A, (a, a) \in R \Rightarrow$ R is reflexive
$(1, 2) \in R$ but $(2, 1) \notin R \Rightarrow$ R is not symmetric
$(1, 2) \in R, (2, 3) \in R$ but $(1, 3) \notin R \Rightarrow$ R is not transitive.

(*ii*) The relation $R = \{(1, 2), (2, 1)\}$ is symmetric but not reflexive or transitive.

(*iii*) The relation $R = \{(1, 2), (2, 3) (1, 3)\}$ is transitive but not reflexive or symmetric.

(*iv*) The relation $R = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 1)\}$ is reflexive and symmetric but not transitive.

(*v*) The relation $R = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 3), (1, 3)\}$ is reflexive and transitive but not symmetric.

(*vi*) The relation $R = \{(1, 2), (2, 1), (2, 3), (3, 2), (1, 1), (1, 3), (2, 2), (3, 1), (3, 3)\}$ is symmetric and transitive but not reflexive,

$\because (4, 4) \notin R.$

Example 2. *Let $S = \{1, 2, 3, 4, 5\}$ and R, the relation*

$$R = \{(1, 3), (2, 4), (3, 5), (1, 1), (2, 2), (4, 2), (3, 1)\}.$$

Is R an equivalence relation on S ? Give reasons in support of your answer.

Sol. For R to be an equivalence relation on S, R must be

(*i*) reflexive *i.e.,* $(a, a) \in R \ \forall a \in S$
(*ii*) symmetric *i.e.,* $(a, b) \in R \Rightarrow (b, a) \in R$
(*iii*) transitive *i.e.,* $(a, b) \in R, (b, c) \in R \Rightarrow (a, c) \in R.$

Now $3 \in S$ but $(3, 3) \notin R \Rightarrow$ R is not reflexive.

$\therefore$ The relation R on S is not an equivalence relation.

Example 3. *A relation R is defined on the set $N \times N$ as follows :*

$$(a, b) R (c, d) \text{ iff } a + d = b + c.$$

Prove that R is an equivalence relation on $N \times N$.

Sol. $N \times N = \{(a, b) : a, b \in N\}$, N being the set of natural numbers.

(*i*) R will be reflexive if $\forall (a, b) \in N \times N, (a, b) R (a, b)$

i.e., if $a + b = b + a$ which is true.

(*ii*) R will be symmetric if $(a, b) R(c, d) \Rightarrow (c, d) R (a, b)$

i.e., if $a + d = b + c \Rightarrow c + b = d + a$

i.e., if $a + d = b + c \Rightarrow b + c = a + d$ which is true.

(*iii*) R will be transitive if

$(a, b) R(c, d), (c, d) R(e, f) \Rightarrow (a, b) R(e, f)$

i.e., if $a + d = b + c, c + f = d + e \Rightarrow a + f = b + e$

i.e., if (on addition) $a + c + d + f = b + c + d + e \Rightarrow a + f = b + e$

i.e., if $a + f = b + e \Rightarrow a + f = b + e$ which is true.

Since R is reflexive, symmetric and transitive

$\therefore$ R is an equivalence relation on $N \times N$.

Example 4. *A relation R is defined on the set $N \times N$ as follows :*

$$(a, b) R(c, d) \text{ iff } ad = bc.$$

Prove that R is an equivalence relation on $N \times N$.

Sol. *Please try yourself.*

1.24. FUNCTIONS (or MAPPINGS)

Let X and Y be two non-empty sets. If there exists a rule '*f*' which associates to **every** element $x \in X$, a **unique** element $y \in Y$, then such a rule '*f*' is called a function or mapping from the set X to the set Y.

We write $f : X \to Y$ or $X \xrightarrow{f} Y$ and read '*f* is a function from X to Y'.

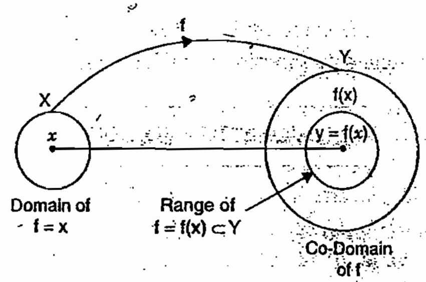
This unique element $y \in Y$ associated by *f* to an element $x \in X$ is denoted by $f(x)$ and is called **the f-image of x** or value of the function *f* at *x*. Here *x* is called the **pre-image of y** under *f*.

The set X is called the **domain** of *f* and the set Y is called the **co-domain** of *f*. The set of all *f*-images is called the **range** of *f*. Thus,

Range of $f = f(X) = \{f(x) : x \in X\}$

Clearly, $f(X) \subset Y$.

Note. From the above definition of a function, we find that $f : X \to Y$ is a particular relation from X to Y under which **each element of X** is related to a **unique element of Y**. Thus, we may also define a function as follows :



A function f from a set X to a set Y is a subset of X × Y in which each $x \in X$ appears in one and only one ordered pair belonging to f.

The set of first entries of the ordered pairs belonging to *f* is the domain of *f* and set of second entries of the ordered pairs is the range of *f*. Also, $(x, y) \in f \Rightarrow y = f(x)$.

i.e., the second entry of any ordered pair belonging to f gives the value of f at the first entry of the same ordered pair.

1.25. KINDS OF FUNCTIONS

(*a*) **Equal Functions.** Two functions f and g are said to be equal iff

(*i*) the domain of f = the domain of g

(*ii*) $f(x) = g(x)$ for all x in their common domain.

Also, then we write $f = g$.

(*b*) **Into Function.** Let $f : A \to B$ such that there is at least one element $b \in B$ which has no pre-image under f, then f is said to be a function from A *into* B.

Clearly, $f : A \to B$ is an into function if $f(A) \neq B$.

(*c*) **Onto Function (or Surjection).** Let $f : A \to B$ such that each element $b \in B$ has **at least one** pre-image under f, then f is said to be a function **from A onto B.**

Clearly, $f : A \to B$ is an onto function if $f(A) = B$.

(*d*) **One-one Function (or Injection).** Let $f : A \to B$ such that different elements of A have different f-images in B then f is said to be a one-one function.

Thus $f : A \to B$ is a one-one function if $x_1, x_2 \in A$ and

$$f(x_1) = f(x_2) \quad \Rightarrow \quad x_1 = x_2$$

or equivalently

$$x_1 \neq x_2 \quad \Rightarrow \quad f(x_1) \neq f(x_2).$$

(*e*) **Many-one Function.** Let $f : A \to B$ such that two or more different elements in A have the same f-image in B, then f is said to be a many-one function.

Thus, $f : A \to B$ is a many-one function if $\exists\, x_1, x_2 \in A$ such that

$$x_1 \neq x_2 \quad \Rightarrow \quad f(x_1) = f(x_2)$$

(*f*) **Many-one into Function.** A function $f : A \to B$ is said to be many-one into if

(*i*) $\exists\, x_1, x_2 \in A$ such that $x_1 \neq x_2 \Rightarrow f(x_1) = f(x_2)$

(*ii*) $f(A) \neq B$.

(*g*) **Many-one onto Function.** A function $f : A \to B$ is said to be many-one onto if

(*i*) $\exists\, x_1, x_2 \in A$ such that $x_1 \neq x_2 \Rightarrow f(x_1) = f(x_2)$.

(*ii*) $f(A) = B$.

(*h*) **Identity Function.** A function $f : A \to A$ is said to be an identity function if $f(x) = x$ $\forall\, x \in A$

Identity function on A is denoted by I_A and **is always one-one onto.**

(*i*) **Bijection.** A one-one onto function is called a bijection.

ILLUSTRATIVE EXAMPLES

Example 1. *Consider the sets* $X = \{1, 2, 3, 4\}$, $Y = \{a, b, c\}$.

Let $f_1 = \{(1, a), (2, a), (3, b), (4, c)\}$; $\quad f_2 = \{(1, a), (1, b), (2, a), (3, b), (4, b)\}$

$f_3 = \{(1, b), (2, b), (3, b), (4, c)\}$; $\quad f_4 = \{(1, a), (2, b), (3, c)\}$.

Answer the following :

(*i*) *Which of these f_i's are functions from X to Y ? What are their ranges ?*

(*ii*) *Which of these f_i's are onto functions ?*

(*iii*) *Which of these f_i's are one-one onto functions ?*

(*iv*) *If some f_i is not a function, explain why it is not so ?*

Sol. (*i*) Each element of X has an image in Y under f_1. Also all the ordered pairs in f_1 have different first entry.

$\therefore$ **f_1 is a function from X to Y.**

$$\text{Range of } f_1 = \text{set of second entries of ordered pairs} = \{a, b, c\} = Y.$$

Each element of Y has a pre-image under f_1.

$\therefore$ **f_1 is onto.**

$$f_1(1) = a, f_1(2) = a$$

$1, 2 \in X$ and $1 \neq 2$ but $f_1(1) = f_1(2)$. $\therefore$ **f_1 is not one-one.**

$\therefore$ **f_1 is not one-one onto.**

(*ii*) Each element of X has an image in Y under f_2. But the ordered pairs $(1, a)$ and $(1, b)$ have the same first entry.

i.e., f_2 does not associate a unique element of Y to $1 \in X$.

$\therefore$ **f_2 is not a function** from X to Y.

(*iii*) Each element of X has an image in Y under f_3. Also all the ordered pairs in f_3 have different first entry.

$\therefore$ **f_3 is a function** from X to Y.

Range of $f_3 = \{b, c\}$.

$a \in Y$ has no pre-image under f_3. $\therefore$ **f_3 is not onto.**

(*iv*) $4 \in X$ has no image in Y under f_4.

$\therefore$ **f_4 is not a function.**

Example 2. *Prove that the function $f : R \to R$ defined by $f(x) = \cos x$ is neither one-one nor onto.*

Sol. $\pm\dfrac{\pi}{3} \in R$ and $-\dfrac{\pi}{3} \neq \dfrac{\pi}{3}$.

But

$$f\left(-\frac{\pi}{3}\right) = \cos\left(-\frac{\pi}{3}\right) = \cos\frac{\pi}{3} = \frac{1}{2}$$

$$f\left(\frac{\pi}{3}\right) = \cos\frac{\pi}{3} = \frac{1}{2}$$

i.e., Different elements of the domain have the same f-image.

$\therefore$ **f is not one-one.**

Now $2 \in R$, the co-domain of f. Let x be its pre-image under f. Then

$f(x) = 2 \Rightarrow \cos x = 2$ which is absurd *i.e.*, $\exists$ no such $x \in R$.

Thus $\exists$ an element in the co-domain which has no pre-image under f.

$\therefore$ **f is not onto.**

Example 3. *Show that the mapping $f : Z^+ \to Z^+$ defined by $f(x) = x^2$, $x \in Z^+$, where Z^+ is the set of positive integers, is one-one into.*

Sol. Let n_1 and n_2 be any two distinct +ve integers.

$$n_1 \neq n_2 \Rightarrow n_1^2 \neq n_2^2 \Rightarrow f(n_1) \neq f(n_2).$$

$\therefore$ f is one-one. ...(*i*)

Now let $y \in Z^+$, the co-domain of f. Let x be its pre-image under f. Then

$$f(x) = y \Rightarrow x^2 = y \Rightarrow x = +\sqrt{y}$$

which may not be a +ve integer *e.g.*, if $y = 2$, $x = \sqrt{2} \notin Z^+$

$\therefore$ f is an into mapping ...(ii)

Combining (i) and (ii), f is a one-one into mapping.

Example 4. *Give an example of each of the following :*

(i) *A function which is one-one but not onto.*

(ii) *A function which is onto but not one-one.*

(iii) *A function which is both one-one and onto.*

(iv) *A function which is neither one-one nor onto.*

Sol. (i) Let $f : \mathbf{N} \to \mathbf{N}$ defined by $f(n) = 2n \ \forall \ n \in \mathbf{N}$.

Let $n_1, n_2 \in \mathbf{N}$ s.t. $f(n_1) = f(n_2) \Rightarrow 2n_1 = 2n_2 \Rightarrow n_1 = n_2$

$\therefore$ **f is one-one.**

Now $3 \in \mathbf{N}$. Let n be its pre-image under f. Then

$$f(n) = 3 \quad \Rightarrow \quad 2n = 3 \quad \Rightarrow \quad n = \tfrac{3}{2} \notin \mathbf{N}.$$

$\therefore$ **f is not onto.**

(ii) Let $f : \mathbf{R} \to \mathbf{R}^+$ defined by $f(x) = x^2 \ \forall \ x \in \mathbf{R}$.

For every $y \in \mathbf{R}^+, \exists \sqrt{y} \in \mathbf{R}$ s.t. $f(\sqrt{y}) = (\sqrt{y})^2 = y$

$\Rightarrow$ every element of $\mathbf{R}^+$ has pre-image under f.

$\therefore$ **f is onto.**

Now $\pm 2 \in \mathbf{R}$ and $-2 \neq 2$.

But $f(-2) = 4 = f(2)$ $\therefore$ **f is not one-one.**

(iii) Let $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x + 1 \ \forall \ x \in \mathbf{R}$.

Let $x_1, x_2 \in \mathbf{R}$ s.t. $f(x_1) = f(x_2) \Rightarrow x_1 + 1 = x_2 + 1 \Rightarrow x_1 = x_2$.

$\therefore$ **f is one-one.**

Now let y be any element of $\mathbf{R}$. Let x be its pre-image under f. Then

$$f(x) = y \quad \Rightarrow \quad x + 1 = y \quad \Rightarrow \quad x = y - 1 \in \mathbf{R}$$

i.e., for every $y \in \mathbf{R}, \exists \ x \in \mathbf{R}$ s.t. $f(x) = y$.

$\therefore$ **f is onto.**

(iv) Let $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x^2 \ \forall \ x \in \mathbf{R}$

$$f(-2) = 4 = f(2)$$
$$-2 \neq 2 \ \Rightarrow \ f(-2) = f(2).$$

$\therefore$ **f is not one-one.**

Now $-4 \in \mathbf{R}$. Let x be the pre-image of -4 under f. Then

$$f(x) = -4 \ \Rightarrow \ x^2 = -4 \ \Rightarrow \ x = \pm 2i \notin \mathbf{R}.$$

$\therefore$ **f is not onto.**

Example 5. *Prove that the mapping $f : \mathbf{N} \to \mathbf{N}$ defined by $f(n) = n^2 + n + 1$ is one-one but not onto.*

Sol. Let $n_1, n_2 \in \mathbf{N}$ s.t. $f(n_1) = f(n_2)$

$\Rightarrow \quad n_1^2 + n_1 + 1 = n_2^2 + n_2 + 1 \ \Rightarrow \ n_1^2 - n_2^2 + n_1 - n_2 = 0$

$\Rightarrow (n_1 - n_2)(n_1 + n_2 + 1) = 0$

$\Rightarrow \quad n_1 - n_2 = 0 \qquad | \because n_1, n_2 \in \mathbf{N} \therefore n_1 + n_2 + 1 \neq 0$

$\Rightarrow \quad n_1 = n_2$

$\therefore$ f is one-one.

Now $1 \in \mathbf{N}$. Let n be its pre-image under f. Then

$$f(n) = 1 \quad \Rightarrow \quad n^2 + n + 1 = 1$$
$$\Rightarrow \quad n(n+1) = 0 \quad \Rightarrow \quad n = 0, -1$$

Neither 0, nor -1 belongs to $\mathbf{N}$.

$\therefore$ $\exists$ no $n \in \mathbf{N}$ s.t. $f(n) = 1$

$\therefore$ **f is not onto.**

Example 6. *Let f be a function with domain X and range in Y and let A, B be subsets of X. Then prove that*

(i) $A \subset B \Rightarrow f(A) \subset f(B)$ (ii) $f(A \cup B) = f(A) \cup f(B)$.

Sol. (i) Let y be any element of $f(A)$. Then

$y \in f(A) \Rightarrow y = f(x)$ for some $x \in A$

$\Rightarrow y = f(x)$ for some $x \in B$ $\quad | \because A \subset B$

$\Rightarrow y = f(x)$ and $f(x) \in f(B) \Rightarrow y \in f(B) \therefore f(A) \subset f(B)$.

Note. The converse is not true.

Consider $A = [0, \pi], B = \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ and $f(x) = \sin x$

Then $f(A) = [0, 1], f(B) = [-1, 1]$

$f(A) \subset f(B)$ but $A \not\subset B$

(ii) Let y be any element of $f(A \cup B)$. Then

$y \in f(A \cup B) \Rightarrow y = f(x)$ for some $x \in A \cup B$

$\Rightarrow y = f(x)$ for some $x \in A$ or $x \in B \Rightarrow y = f(x)$ and $f(x) \in f(A)$ or $f(x) \in f(B)$

$\Rightarrow y \in f(A)$ or $y \in f(B) \Rightarrow y \in f(A) \cup f(B)$

$\therefore \quad f(A \cup B) \subset f(A) \cup f(B)$ ...(i)

Also $\quad A \subset A \cup B$ and $B \subset A \cup B$

$\Rightarrow \quad f(A) \subset f(A \cup B)$ and $f(B) \subset f(A \cup B)$

$\Rightarrow \quad f(A) \cup f(B) \subset f(A \cup B)$ ...(ii)

Combining (i) and (ii), we get $f(A \cup B) = f(A) \cup f(B)$

Note. $f(x) \in f(A)$ does not necessarily imply that $x \in A$.

For example. If $f : \mathbf{R} \to \mathbf{R}^+$ defined by $f(x) = x^2$ and $A = [0, 2]$, then $f(A) = [0, 4]$

Now $f(-1) = 1$ so that $f(-1) \in f(A)$ but $-1 \notin A$

However $x \in A \Rightarrow f(x) \in f(A)$ always.

Example 7. *If $f : X \to Y$ and $A \subset X, B \subset X$, show that $f(A \cap B) \subset f(A) \cap f(B)$. Give an example to show that $f(A \cap B)$ and $f(A) \cap f(B)$ may be unequal.*

Sol. Let y be any element of $f(A \cap B)$. Then

$y \in f(A \cap B) \Rightarrow y = f(x)$ for some $x \in A \cap B$

$\Rightarrow y = f(x)$ for some $x \in A$ and $x \in B \Rightarrow y = f(x), f(x) \in f(A)$ and $f(x) \in f(B)$

$\Rightarrow y \in f(A)$ and $y \in f(B) \Rightarrow y \in f(A) \cap f(B)$.

$\therefore \quad f(A \cap B) \subset f(A) \cap f(B)$

Consider $f : \mathbf{R} \to \mathbf{R}^+$ defined by $f(x) = x^2$ [Here $X = \mathbf{R}, Y = \mathbf{R}^+$]

Let $A = [-2, 0]$ and $B = [0, 2]$ so that $A \subset \mathbf{R}, B \subset \mathbf{R}$ and $A \cap B = \{0\}$

Now $\quad f(A \cap B) = f(\{0\}) = \{0\}$

$f(A) = [0, 4], f(B) = [0, 4]$

$f(A) \cap f(B) = [0, 4]$

Clearly, $f(A \cap B) \neq f(A) \cap f(B)$.

1.26. COMPOSITE OF FUNCTIONS (OR PRODUCT OF FUNCTIONS)

Let $f: A \to B$ and $g: B \to C$ be two functions such that

$$f(x) = y \quad \text{and} \quad g(y) = z, \text{ where } x \in A, y \in B, z \in C.$$

Then the function $h: A \to C$ such that

$$h(x) = z = g(y) = g[f(x)] \ \forall\ x \in A$$

is called the composite of functions f and g and is denoted by $g \circ f$

i.e., $g \circ f: A \to C$ defined by $(g \circ f)(x) = g[f(x)] \ \forall\ x \in A$.

(*a*) **The composite of functions is not commutative *i.e.*, $g \circ f \neq f \circ g$.**

Let $f: R \to R$ defined by $f(x) = x^2$ and $g: R \to R$ defined by $g(x) = x + 1$. Then

$g \circ f: R \to R$ is defined as $\quad (g \circ f)(x) = g[f(x)] = g(x^2) = x^2 + 1$

Also $f \circ g: R \to R$ is defined as $\quad (f \circ g)(x) = f[g(x)] = f(x + 1) = (x + 1)^2$

$\therefore$ $g \circ f$ and $f \circ g$ are both defined but $\quad g \circ f \neq f \circ g$.

(*b*) **The composite of functions is associative *i.e.*, $(h \circ g) \circ f = h \circ (g \circ f)$.**

Let $f: A \to B$, $g: B \to C$, $h: C \to D$ be three functions.

Also, let $x \in A, y \in B, z \in C$ such that $f(x) = y$ and $g(y) = z$

Then $[(h \circ g) \circ f](x) = (h \circ g)[f(x)] = (h \circ g)(y) = h[g(y)] = h(z)$

Also $\quad [h \circ (g \circ f)(x)] = h[(g \circ f)(x)] = h[g(f(x))] = h[g(y)] = h(z)$

$\therefore \quad [(h \circ g) \circ f](x) = [h \circ (g \circ f)](x) \ \forall\ x \in A$.

Hence the composite of functions is associative.

(*c*) **The product of any function with the identity function is the function itself, *i.e.*, if $f: A \to B$, then $f \circ I_A = f = I_B \circ f$.**

Let x be any element of A. Then $\exists\ y \in B$ s.t. $f(x) = y$

$\because \quad I_A: A \to A$ and $f: A \to B$

$\therefore \quad f \circ I_A: A \to B$ and $(f \circ I_A)(x) = f[I_A(x)] = f(x)$ [$\because$ I_A is the identity function]

Also $\quad f: A \to B$ and $I_B: B \to B$

$\therefore \quad I_B \circ f: A \to B$ and $(I_B \circ f)(x) = I_B[f(x)] = I_B(y) = y = f(x)$

$\because \quad (f \circ I_A)(x) = f(x) = (I_B \circ f)(x) \ \forall\ x \in A$

$\therefore \quad f \circ I_A = f = I_B \circ f$.

(*d*) **The composite of two one-one and onto functions is also a one-one and onto function.**

Or

The composite of two bijections is also a bijection.

Let $f: A \to B$ and $g: B \to C$ be two one-one onto functions.

Then $g \circ f$ exists such that $g \circ f: A \to C$.

We have to prove that $g \circ f$ is one-one as well as onto.

Let $a_1, a_2 \in A$ such that $(g \circ f)(a_1) = (g \circ f)(a_2)$

Then $\quad (g \circ f)(a_1) = (g \circ f)(a_2) \quad \Rightarrow \quad g[f(a_1)] = g[f(a_2)]$

$\Rightarrow \quad f(a_1) = f(a_2)$ [$\because$ g is one-one]

$\Rightarrow \quad a_1 = a_2$ [$\because$ f is one-one]

$\therefore$ **$g \circ f$ is also a one-one function.**

Now let c be any element of C.

$c \in C \Rightarrow \exists\ b \in B$ s.t. $g(b) = c$ [$\because$ g is onto]

and $\quad b \in B \Rightarrow \exists\ a \in A$ s.t. $f(a) = b$ [$\because$ f is onto]

$$(g \circ f)(a) = g[f(a)] = g(b) = c$$

$\Rightarrow$ every element $c \in C$ has pre-image under $g \circ f$.

$\therefore$ **$g \circ f$ is onto.**

Hence $g \circ f$ is also one-one and onto.

1.27. INVERSE FUNCTION

Let $f: A \to B$ be a **one-one and onto function**. Then the function $g: B \to A$ which associates to each element $b \in B$ the unique element $a \in A$ such that $f(a) = b$ is called the inverse function of f. The inverse function of f is denoted by f^{-1}.

Note. Every function does not have an inverse. A function $f: A \to B$ has inverse iff f is one-one and onto. If f has inverse, then f is said to be invertible and $f^{-1}: B \to A$. Also if $a \in A$, then $f(a) = b$ where $b \in B \Rightarrow a = f^{-1}(b)$.

Theorem I. *If $f: A \to B$ is one-one and onto and $g: B \to A$ is the inverse of f, then $f \circ g = I_B$ and $g \circ f = I_A$, where I_A and I_B are the identity functions on the sets A and B respectively.*

Proof. Let $a \in A$ such that $f(a) = b$, $b \in B$

Then $\quad a = g(b)$ [$\because$ g is the inverse of f]

Now $\quad (g \circ f)(a) = g[f(a)] = g(b) = a$

$\Rightarrow \quad (g \circ f)(x) = x \ \forall\ x \in A$ $\quad$ Also $I_A(x) = x \ \forall\ x \in A$

$\therefore \quad g \circ f = I_A$

Now let $b \in B$ such that $g(b) = a$, $a \in A$.

Then $\quad b = f(a)$

Now $\quad (f \circ g)(b) = f[g(b)] = f(a) = b$

$\Rightarrow \quad (f \circ g)(x) = x \ \forall\ x \in B$. Also $I_B(x) = x \ \forall\ x \in B$

$\therefore \quad f \circ g = I_B$.

Theorem II. *A function is invertible iff it is one-one and onto*

Proof. I. Let $f: A \to B$ be invertible. Then $\exists$ a function $g: B \to A$ such that

$$g \circ f = I_A \text{ and } f \circ g = I_B.$$

(*i*) If possible, suppose f is not one-one.

Then $\exists$ two elements $a_1, a_2 \in A$ with $a_1 \neq a_2$ such that

$$f(a_1) = f(a_2)\ ;\ f(a_1), f(a_2) \in B$$

$\Rightarrow \quad g[f(a_1)] = g[f(a_2)]$ [$\because$ $g: B \to A$]

$\Rightarrow \quad (g \circ f)(a_1) = (g \circ f)(a_2)$

$\Rightarrow \quad I_A(a_1) = I_A(a_2) \Rightarrow a_1 = a_2$

which is a contradiction. $\therefore$ **f is one-one.**

(*ii*) Let b be any element of B. Then $f[g(b)] = (f \circ g)(b) = I_B(b) = b$

$\Rightarrow$ every element $b \in$ B has pre-image under f, namely $g(b)$

$\therefore$ **f is onto**

$\therefore$ f is invertible $\Rightarrow f$ is one-one and onto.

II. Let $f : A \to B$ be one-one and onto.

Then for every $b \in$ B, $\exists$ a unique $a \in$ A s.t. $f(a) = b$

Let $g : B \to A$ defined as $g(b) = a$ where $f(a) = b$

$$(g \circ f)(a) = g[f(a)] = g(b) = a \;\forall\, a \in A \quad \Rightarrow \quad g \circ f = I_A$$

Similarly $\quad f \circ g = I_B$

$\therefore$ g is the inverse of f

$\therefore$ f is one-one and onto $\Rightarrow f$ is invertible.

Theorem III. *The inverse of a function, if it exists, is unique.*

Proof. Let $f : A \to B$ be invertible. Then f is one-one and onto.

i.e., for every $b \in$ B, $\exists$ a unique $a \in$ A s.t. $f(a) = b$

Let $g : B \to A$ and $h : B \to A$ be two inverses of f.

Then $\quad g \circ f = I_A = h \circ f$

$\therefore$ $\quad g(b) = g[f(a)] = (g \circ f)(a) = I_A(a) = (h \circ f)(a) = h[f(a)] = h(b) \;\forall\, b \in B$

$\Rightarrow$ $\quad g = h.$

Theorem IV. *If $f : A \to B$ is one-one and onto, then f^{-1} is also one-one and onto.*

Or

The inverse of a bijection is also a bijection.

Or

The inverse of an invertible function is invertible.

Proof. Let $f : A \to B$ be one-one and onto $\Rightarrow f$ is invertible.

Let $g : B \to A$ be the inverse of f.

Also, let $\quad g(b_1) = a_1$ and $g(b_2) = a_2$; $a_1, a_2 \in A$; $b_1, b_2 \in B$

Then $\quad g(b_1) = g(b_2) \quad \Rightarrow \quad a_1 = a_2$

$\Rightarrow \quad f(a_1) = f(a_2) \quad \Rightarrow \quad b_1 = b_2$

$\therefore$ **g is one-one.**

Again, if a is any element of A, then

$$a \in A \;\Rightarrow\; \exists\, b \in B \text{ s.t. } f(a) = b \;\Rightarrow\; \exists\, b \in B \text{ s.t. } a = g(b)$$

$\therefore$ every $a \in$ A has pre-image under g.

$\Rightarrow$ **g is onto.**

Hence $g = f^{-1}$ is one-one and onto.

Note. f^{-1} is one-one and onto $\Rightarrow f^{-1}$ is invertible $\Rightarrow (f^{-1})^{-1}$ exists.

Theorem V. *The inverse of the inverse of a function is the function itself i.e., $(f^{-1})^{-1} = f$*

Or

If a function g be the inverse of a function f, then f is the inverse of g.

Proof. Let $f : A \to B$ be invertible. Then $\exists$ a function $g = f^{-1} : B \to A$ s.t.

$$f(a) = b \quad \Rightarrow \quad a = g(b),\; a \in A,\; b \in B.$$

Also f is invertible $\Rightarrow$ f is one-one and onto

$\Rightarrow$ g is one-one and onto [Art. 1.31]

$\Rightarrow$ g is invertible *i.e.,* g^{-1} exists.

Now $\quad (f \circ g)(b) = f[g(b)] = f(a) = b \quad \Rightarrow \quad f \circ g = I_B$

$\Rightarrow$ f is the inverse of g $\Rightarrow$ $f = g^{-1}$ $\quad \Rightarrow \quad f = (f^{-1})^{-1}.$

Theorem VI. *If $f : A \to B$ and $g : B \to C$ are two one-one and onto functions, then the inverse of $g \circ f$ exists and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.*

Proof. Let $f : A \to B$ and $g : B \to C$ be one-one and onto

$\Rightarrow$ $g \circ f$ is one-one and onto [Prove it as in Art. 1.26 (*d*)]

$\Rightarrow$ $g \circ f$ is invertible.

$\because$ $\quad g \circ f : A \to C \quad \therefore \quad (g \circ f)^{-1} : C \to A$

Also $\quad g^{-1} : C \to B$ and $f^{-1} : B \to A \quad \therefore \quad f^{-1} \circ g^{-1} : C \to A$

$\therefore$ domain of $(g \circ f)^{-1}$ = domain of $f^{-1} \circ g^{-1}$.

Now let $a \in A$, $b \in B$, $c \in C$ such that

$$f(a) = b \text{ and } g(b) = c \quad \Rightarrow \quad a = f^{-1}(b) \text{ and } b = g^{-1}(c)$$

Now $(g \circ f)(a) = g[f(a)] = g(b) = c \quad \Rightarrow \quad (g \circ f)^{-1}(c) = a$

$$(f^{-1} \circ g^{-1})(c) = f^{-1}[g^{-1}(c)] = f^{-1}(b) = a = (g \circ f)^{-1}(c)$$

Hence $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

ILLUSTRATIVE EXAMPLES

Example 1. *Let f be a function with domain X and range in Y and E, F be the subsets of Y, then prove that*

(*i*) $E \subset F \Rightarrow f^{-1}(E) \subset f^{-1}(F)$ $\qquad$ (*ii*) $f^{-1}(E \cup F) = f^{-1}(E) \cup f^{-1}(F)$

(*iii*) $f^{-1}(E \cap F) = f^{-1}(E) \cap f^{-1}(F)$.

Sol. (*i*) Let x be any element of $f^{-1}(E)$. Then

$$x \in f^{-1}(E) \Rightarrow f(x) \in E \Rightarrow f(x) \in F \qquad [\because E \subset F]$$

$\Rightarrow \quad x \in f^{-1}(F)$

$\therefore \quad f^{-1}(E) \subset f^{-1}(F).$

(*ii*) Let x be any element of $f^{-1}(E \cup F)$. Then

$$x \in f^{-1}(E \cup F) \quad \Rightarrow \quad f(x) \in E \cup F$$

$\Rightarrow \quad f(x) \in E$ or $f(x) \in F \quad \Rightarrow \quad x \in f^{-1}(E)$ or $x \in f^{-1}(F)$

$\Rightarrow \quad x \in f^{-1}(E) \cup f^{-1}(F)$

$\therefore \quad f^{-1}(E \cup F) \subset f^{-1}(E) \cup f^{-1}(F) \qquad \ldots(a)$

Again, let x be any element of $f^{-1}(E) \cup f^{-1}(F)$. Then,

$$x \in f^{-1}(E) \cup f^{-1}(F) \quad \Rightarrow \quad x \in f^{-1}(E) \text{ or } x \in f^{-1}(F)$$

$\Rightarrow \quad f(x) \in E$ or $f(x) \in F \quad \Rightarrow \quad f(x) \in E \cup F \;\Rightarrow\; x \in f^{-1}(E \cup F)$

$\therefore \quad f^{-1}(E) \cup f^{-1}(F) \subset f^{-1}(E \cup F)$ ...(b)

Combining (a) and (b), we have $f^{-1}(E \cup F) = f^{-1}(E) \cup f^{-1}(F)$.

(iii) Please try yourself.

Example 2. *Prove that*

(i) $f^{-1}(A^c) = [f^{-1}(A)]^c$ *(ii)* $f^{-1}(A - B) = f^{-1}(A) - f^{-1}(B)$.

Sol. (i) Let x be any element of $f^{-1}(A^c)$. Then

$x \in f^{-1}(A^c) \Rightarrow f(x) \in A^c \Rightarrow f(x) \notin A$

$\Rightarrow x \notin f^{-1}(A) \Rightarrow x \in [f^{-1}(A)]^c$

$\therefore \quad f^{-1}(A^c) \subset [f^{-1}(A)]^c$ ...(a)

Again, let x be any element of $[f^{-1}(A)]^c$. Then

$x \in [f^{-1}(A)]^c \Rightarrow x \notin f^{-1}(A) \Rightarrow f(x) \notin A$

$\Rightarrow f(x) \in A^c \Rightarrow x \in f^{-1}(A^c)$

$\therefore \quad [f^{-1}(A)]^c \subset f^{-1}(A^c)$ ...(b)

Combining (a) and (b), we get $f^{-1}(A^c) = [f^{-1}(A)]^c$.

Remember. $x \in f^{-1}(A) \Leftrightarrow f(x) \in A$ always.

(ii) Let x be any element of $f^{-1}(A - B)$. Then

$x \in f^{-1}(A - B) \Rightarrow f(x) \in A - B \Rightarrow f(x) \in A$ and $f(x) \notin B$

$\Rightarrow x \in f^{-1}(A)$ and $x \notin f^{-1}(B) \Rightarrow x \in f^{-1}(A) - f^{-1}(B)$

$\therefore \quad f^{-1}(A - B) \subset f^{-1}(A) - f^{-1}(B)$ ...(a)

Again, let x be any element of $f^{-1}(A) - f^{-1}(B)$. Then

$x \in f^{-1}(A) - f^{-1}(B) \Rightarrow x \in f^{-1}(A)$ and $x \notin f^{-1}(B)$

$\Rightarrow f(x) \in A$ and $f(x) \notin B \Rightarrow f(x) \in A - B \Rightarrow x \in f^{-1}(A - B)$

$\therefore \quad f^{-1}(A) - f^{-1}(B) \subset f^{-1}(A - B)$ ...(b)

Combining (a) and (b), we have $f^{-1}(A - B) = f^{-1}(A) - f^{-1}(B)$.

Example 3. *What is meant by the statement that composite of functions f, g and h is associative ? Verify the associativity for the following three functions :*

$f : N \to Z_0$ *such that* $f(x) = 2x$ $\quad g : Z_0 \to Q$ *such that* $g(x) = \dfrac{1}{x}$

$h : Q \to R$ *such that* $h(x) = e^x$.

Sol. Composite of functions f, g and h is associative

$\Rightarrow \quad h\,o(g\,o\,f) = (h\,o\,g)\,o\,f$

Now $\quad h : Q \to R,\ g\,o\,f : N \to Q \Rightarrow h\,o(g\,o\,f) : N \to R$.

Also $\quad h\,o\,g : Z_0 \to R,\ f : N \to Z_0 \Rightarrow (h\,o\,g)\,o\,f : N \to R$.

Let $x \in N$ arbitrarily, $y \in Z_0$, $z \in Q$ s.t. $f(x) = y$ and $g(y) = z$

$$[h\,o(g\,o\,f)](x) = h[(g\,o\,f)(x)] = h[g(f(x))] = h[g(2x)] = h\left(\frac{1}{2x}\right) = e^{\frac{1}{2x}}$$

$$[(h\,o\,g)\,o\,f](x) = (h\,o\,g)[f(x)] = (h\,o\,g)(2x) = h[g(2x)] = h\left(\frac{1}{2x}\right) = e^{\frac{1}{2x}}$$

$\because \quad [h\,o(g\,o\,f)](x) = [(h\,o\,g)\,o\,f](x)\ \forall\, x \in N$

$\therefore \quad h\,o(g\,o\,f) = (h\,o\,g)\,o\,f$.

Example 4. *Let $f : R \to R$ defined by $f(x) = ax + b$, where $a, b, x \in R$ and $a \neq 0$. Prove that f is invertible.*

Sol. For any $x_1, x_2 \in R$ s.t. $f(x_1) = f(x_2)$

$ax_1 + b = ax_2 + b$

$\Rightarrow \quad ax_1 = ax_2 \Rightarrow x_1 = x_2 \quad [\because a \neq 0]$

$\therefore$ **f is one-one.**

Let y be any element of R, the co-domain of f. Let x be its pre-image under f.

Then $\quad f(x) = y$

$\Rightarrow \quad ax + b = y \Rightarrow x = \dfrac{y - b}{a} \in R$.

Since every element of the co-domain has its pre-image under f.

$\therefore$ **f is onto.**

f is one-one and onto $\Rightarrow f$ is invertible.

2

Countability of Sets and the Real Number System

2.1. INITIAL SEGMENT OF N

Let $m \in N$. Then the subset $N_m = \{n : n \in N \text{ and } n \leq m\} = \{1, 2, 3, \ldots\ldots, m\}$ of N is called an initial segment of N determined by the natural number m.

e.g. $\{1, 2, 3, 4, 5\} = N_5$.

is the initial segment of N determined by 5.

Thus an initial segment of N determined by $m \in N$ contains all natural numbers from 1 to m.

2.2. EQUIVALENT SETS

Two sets A and B are said to be equivalent if $\exists$ a bijection *i.e.* a one-one and onto function, from A to B.

A is equivalent to B is written as $A \sim B$.

For example, the sets N and $E = \{2, 4, 6, \ldots\ldots\}$ of all even natural numbers are equivalent because the function.

$f : N \to E$ defined by $f(n) = 2n$, $n \in N$ is one-one from N onto E.

2.3. THEOREM

The relation '$\sim$' is an equivalence relation.

Proof. '$\sim$' will be an equivalence relation if it is :

(*i*) reflexive (*ii*) symmetric and (*iii*) transitive.

(*i*) *'$\sim$' is reflexive.*

For any set A, the identity function $I_A : A \to A$ is one-one and onto.

$\therefore$ $A \sim A$ for any set A $\Rightarrow$ '$\sim$' is reflexive.

(*ii*) *'$\sim$' is symmetric.*

Let $A \sim B$, then $\exists$ a function $f : A \to B$ which is one-one and onto. Its inverse function $f^{-1} : B \to A$ is also one-one and onto.

$\therefore$ $B \sim A$

Since $A \sim B \Rightarrow B \sim A$

'$\sim$' is symmetric.

(*iii*) *'$\sim$' is transitive.*

Let $A \sim B$ and $B \sim C$, then $\exists$ one-one and onto functions $f : A \to B$ and $g : B \to C$.

Their composite function $g \circ f : A \to C$ is also one-one and onto.

$\therefore$ $A \sim C$

Since $A \sim B$ and $B \sim C \Rightarrow A \sim C$

'$\sim$' is transitive.

From (*i*), (*ii*) and (*iii*), '$\sim$' is an equivalence relation.

Note. For $n, m \in N$, $N_m \sim N_n \Leftrightarrow m = n$.

2.4. FINITE SET

A set which is either empty or equivalent to a subset N_m of N is said to be a finite set.

Thus A is finite if $A = \phi$ or $A \sim N_m$ for some natural number m.

Note. If $A \sim N_m$, then m is called the **cardinal number** of A.

2.5. INFINITE SET

A set which is not finite is called an infinite set.

Thus A is an infinite set if $A \neq \phi$ and A is not equivalent to N_m for any $m \in N$.

2.6. DENUMERABLE (OR ENUMERABLE) SET

A set is said to be denumerable if it is equivalent to N, the set of all natural numbers. Thus A is denumerable $\Rightarrow A \sim N$.

2.7. COUNTABLE SET

A set A is said to be countable if either A is finite or A is denumerable *i.e.*, if either A is finite or $A \sim N$, the set of all natural numbers.

2.8. UNCOUNTABLE SET

A set which is not countable is said to be an uncountable set.

Thus a set A is uncountable if A is not finite and A is not equivalent to N, the set of all natural numbers.

2.9. THEOREM

A is finite and $B \subset A \Rightarrow$ B is finite (Every subset of a finite set is finite)

(*i*) If $B = \phi$, then B is finite [By def.]

(*ii*) If $B = A$, then B is finite because A is finite.

(*iii*) Suppose $B \neq \phi$ and $B \neq A$.

Since A is finite, there exists $m \in N$ such that $A \sim N_m$.

Since $B \subset A$, B has k elements where $k < m$ *i.e.* $B \sim N_k$

$\therefore$ B is finite.

2.10. THEOREM

A is infinite and $B \supset A \Rightarrow$ B is infinite (Every super-set of an infinite set is infinite)

Proof. Let B be finite, then $A \subset B$ and B is finite.

$\Rightarrow$ A is finite [Th. 2.9] which is a contradiction. Hence B is infinite.

11. THEOREM

If A and B are finite sets, then A ∩ B is also a finite set

Proof. $A \cap B \subset A$ and A is finite.

$\therefore$ $A \cap B$ is a finite set.

12. THEOREM

If A and B are finite sets, then A ∪ B is also a finite set

Proof. (*i*) If $A = \phi = B$, then $A \cup B = \phi$ is finite.

(*ii*) If $A = \phi$ or $B = \phi$, then $A \cup B$ is either B or A, both of which are finite.

$\therefore$ $A \cup B$ is a finite.

(*iii*) If $A \neq \phi$, $B \neq \phi$, since A and B are finite, there exist natural numbers n and m such at $A \sim N_n$ and $B \sim N_m$.

If $A \cap B = \phi$, then $A \cup B$ has $m + n$ elements and $A \cup B \sim N_{n+m}$

$\Rightarrow$ $A \cup B$ is finite.

If $A \cap B \neq \phi$, then there exists a natural number $k \leq \min\{n, m\}$ such that $A \cap B \sim N_k$.

Now $A \cup B$ has $n + m - k$ elements.

$\therefore$ $A \cup B \sim N_{n+m-k}$ $\Rightarrow$ $A \cup B$ is finite.

3. THEOREM

Every subset of a countable set is countable.

Proof. Let A be a countable set. Then A is either finite or denumerable.

Case I. When A is finite.

Since every subset of a finite set is finite, every subset of A is finite and hence countable.

Case II. When A is denumerable.

Here $A \sim N$, the set of natural numbers.

Let $A = \{a_1, a_2, a_3, \ldots\}$ and let $B \subset A$.

Sub-case 1. If B is finite, then B is countable. (By def.)

Sub-case 2. If B is infinite, let n_1 be the least +ve integer s.t. $a_{n_1} \in B$.

Since B is infinite, $B \neq \{a_{n_1}\}$. Let n_2 be the least +ve integer s.t. $n_2 > n_1$ and $a_{n_2} \in B$.

Since B is infinite, $B \neq (a_{n_1}, a_{n_2}\}$.

Continuing like this, $B = \{a_{n_1}, a_{n_2}, a_{n_3}, \ldots\}$ where $n_1 < n_2 < n_3 < \ldots$

Define $f : N \to B$ by $f(k) = a_{n_k}$ $\forall\, k \in N$

Then f is one-one and onto.

$\therefore$ $B \sim N$

$\Rightarrow$ B is denumerable $\Rightarrow$ B is countable.

Cor. 1. Every infinite subset of a denumerable set is denumerable.

[See Case II, Sub-case 2].

Cor. 2. If A and B are countable sets, then A ∩ B is also a countable set.

$A \cap B \subset A$ and A is countable

$\Rightarrow$ $A \cap B$ is countable.

Cor. 3. Every super-set of an uncountable set is uncountable.

Let A be an uncountable set and let B be any super-set of A.

Suppose B is countable. Then A being a subset of a countable set must be countable, which is a contradiction.

Hence B is uncountable.

2.14. THEOREM

Every infinite set has a countable subset.

Proof. Let A be an infinite set. Let $a_1 \in A$.

Since A is infinite, $A \neq \{a_1\}$. $\Rightarrow$ $\exists\, a_2 \neq a_1$ s.t. $a_2 \in A$.

Since A is infinite, $A \neq \{a_1, a_2\}$. $\Rightarrow$ $\exists\, a_3 \neq a_2 \neq a_1$ s.t. $a_3 \in A$.

Continuing like this **as long as we please**, we can have a proper subset

$$B = \{a_1, a_2, a_3, \ldots\} \text{ of A.}$$

If B is finite, B is countable. (By def.)

If B is infinite, define $f : N \to B$ by $f(k) = a_k$ $\forall\, k \in N$

Then f is one-one and onto.

$\therefore$ $B \sim N$

$\Rightarrow$ B is denumerable $\Rightarrow$ B is countable.

2.15. THEOREM

A is countable, B is countable ⇒ A ∪ B is countable.

Proof. Case I. If A and B are both finite, then so is A ∪ B.

$\Rightarrow$ $A \cup B$ is countable.

Case II. If one of A and B is finite and the other is denumerable.

Let us assume that A is finite and B is denumerable. Then we can write

$$A = \{a_1, a_2, a_3, \ldots, a_m\}\ (A \sim N_m)$$

$$B = \{b_1, b_2, b_3, \ldots\}$$

Let $C = B - A$, then $C \subset B$

Since A is finite, C is infinite.

C being an infinite subset of a denumerable set is denumerable, so we can express C as

$$C = \{c_1, c_2, c_3 \ldots\}$$

Clearly, $A \cup B = A \cup C = \{a_1, a_2, \ldots, a_m, c_1, c_2, \ldots\}$ and $A \cap C = \phi$

Define $f : N \to A \cup C$ by $f(k) = \begin{cases} a_k, & \text{if } k = 1, 2, \ldots, m \\ c_{k-m}, & \text{if } k \geq m+1 \end{cases}$

Then f is one-one and onto.

$\therefore$ $A \cup C \sim N$ $\Rightarrow$ $A \cup B \sim N$

$\Rightarrow$ $A \cup B$ is denumerable $\Rightarrow$ $A \cup B$ is countable.

Case III. If A and B are both denumerable sets, we can write.

$$A = \{a_1, a_2, a_3, \ldots\} ; B = \{b_1, b_2, b_3, \ldots\}$$

Let $C = B - A$, then $C \subset B$ and $A \cup B = A \cup C$.

If C is finite, then $A \cup C$ is countable. (By case II)

If C is infinite, then C is denumerable and we can write

$$C = \{c_1, c_2, c_3, \ldots\ldots\}$$

Clearly, $A \cup C = \{a_1, c_1, a_2, c_2, a_3, c_3, \ldots\ldots\}$

Define $f : N \to A \cup C$ by $f(n) = \begin{cases} a_{\frac{n+1}{2}} & \text{if } n \text{ is odd} \\ c_{\frac{n}{2}} & \text{if } n \text{ is even} \end{cases}$

Then f is one-one and onto.

$\therefore \quad A \cup C \sim N \Rightarrow A \cup B \sim N$

$\Rightarrow$ $A \cup B$ is denumerable $\Rightarrow$ $A \cup B$ is countable.

2.16. THEOREM

The union of a denumerable collection of denumerable sets is denumerable

Proof. Let $\{A_i\}_{i \in N}$ be a denumerable collection of denumerable sets.

~~Since each~~ A_i is denumerable, we have

$$A_1 = \{a_{11}, a_{12}, a_{13}, \ldots\ldots a_{1n}, \ldots\ldots\}$$
$$A_2 = \{a_{21}, a_{22}, a_{23}, \ldots\ldots a_{2n}, \ldots\ldots\}$$
$$A_3 = \{a_{31}, a_{32}, a_{33}, \ldots\ldots a_{3n}, \ldots\ldots\}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Then a_{ij} is the jth element of A_i.

Let us list the elements of $\bigcup_{i \in N} A_i$, as follows :

$$\begin{array}{l} a_{11} \\ a_{21}, a_{12} \\ a_{31}, a_{22}, a_{13} \\ a_{41}, a_{32}, a_{23}, a_{14} \\ \ldots\ldots\ldots\ldots\ldots \end{array}$$

From the above scheme it is evident that a_{pq} is the qth element of $(p+q-1)$th row. Thus all the elements of $\bigcup_{i \in N} A_i$ have been arranged in an infinite sequence as

$$\{a_{11}, a_{21}, a_{12}, a_{31}, a_{22}, a_{13}, a_{41}, a_{32}, a_{23}, a_{14}, \ldots\ldots\}$$

In fact, the map $f : \bigcup_{i \in N} A_i \to N$ defined by

$$f(a_{pq}) = \frac{(p+q-2)(p+q-1)}{2} + q \quad \text{gives an enumeration of } \bigcup_{i \in N} A_i$$

$\therefore \quad \bigcup_{i \in N} A_i \sim N$

Hence $\bigcup_{i \in N} A_i$ is denumerable.

2.17. THEOREM

The set of real numbers x such that $0 \le x \le 1$ is not countable

Or

The unit interval [0, 1] is not countable.

Proof. Let us assume that [0, 1] is countable.

$\Rightarrow$ either [0, 1] is finite or denumerable.

Since every interval is an infinite set, [0, 1] is denumerable.

$\Rightarrow$ There is an enumeration $x_1, x_2, x_3, \ldots\ldots$ of real numbers in [0, 1].

Expanding each x_i in the form of an infinite decimal, we have

$$x_1 = 0 \,.\, a_{11}\, a_{12}\, a_{13}\, a_{14} \ldots\ldots a_{1n} \ldots\ldots$$
$$x_2 = 0 \,.\, a_{21}\, a_{22}\, a_{23}\, a_{24} \ldots\ldots a_{2n} \ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$x_n = 0 \,.\, a_{n1}\, a_{n2}\, a_{n3}\, a_{n4} \ldots\ldots a_{nn} \ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

where each $\quad a_{ij} \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

Consider the number b with decimal representation $b = 0 \,.\, b_1\, b_2\, b_3 \ldots\ldots b_n \ldots\ldots$

where b_1 is any integer from 1 to 8 s.t. $b_1 \ne a_{11}$

b_2 is any integer from 1 to 8 s.t. $b_2 \ne a_{22}$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

b_n is any integer from 1 to 8 s.t. $b_n \ne a_{nn}$ and so on.

Clearly, $b \in [0, 1]$ and $b \ne x_n \ \forall\, n$ since the decimal representation of b is different from the decimal representation of x_n as $b_n \ne a_{nn}$. Thus b escapes enumeration and we arrive at a contradiction.

Hence [0, 1] is not countable.

2.18. THEOREM

The set of real numbers is not countable.

Proof. We know that every subset of a countable set is countable.

If R were countable, then [0, 1] which is a subset of R must also be countable.

But the unit interval [0, 1] is not countable. [Theorem 2.17]

Hence R is not countable.

2.19. THEOREM

The set of all rational numbers is countable.

Proof. Consider the sets

$$A_1 = \left\{\frac{0}{1}, \frac{-1}{1}, \frac{1}{1}, \frac{-2}{1}, \frac{2}{1}, \ldots\ldots\right\} \quad \text{(Common denom. 1)}$$

$$A_2 = \left\{\frac{0}{2}, \frac{-1}{2}, \frac{1}{2}, \frac{-2}{2}, \frac{2}{2}, \ldots\ldots\right\} \quad \text{(Common denom. 2)}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$A_n = \left\{\frac{0}{n}, \frac{-1}{n}, \frac{1}{n}, \frac{-2}{n}, \frac{2}{n}, \ldots\ldots\right\} \quad \text{(Common denom. } n\text{)}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Clearly, the set of rational numbers $Q = \bigcup_{i \in N} A_i$

Consider a mapping $f : N \to A_n$ defined by $f(r) = \begin{cases} \dfrac{r-1}{2n} & \text{if } r \text{ is odd} \\ \dfrac{-r}{2n} & \text{if } r \text{ is even} \end{cases}$

f is one-one and onto. $\therefore$ $A_n \sim N$

i.e. A_n is denumerable. $\Rightarrow$ A_n is countable.

Since $Q = \bigcup_{i \in N} A_i$ is the union of a countable collection of countable sets.

$\therefore$ Q is countable.

2.20. THEOREM

The set of all positive rational numbers is countable.

Proof. Let Q^+ denote the set of positive rational numbers ; then $Q^+ \subset Q$.

Since every subset of a countable set is countable, and Q is countable.

$\therefore$ Q^+ is countable.

2.21. THEOREM

The set of irrational numbers is uncountable.

Proof. Suppose the set of irrational numbers is countable. We know that the set of rational numbers is countable. Since R, the set of real numbers is the union of the set of rational numbers and the set of irrational numbers, therefore, R is countable. But R is not countable. We are, thus, led to a contradiction.

Hence the set of irrational numbers is uncountable.

2.22. THEOREM

A finite set is not equivalent to any of its proper subsets.

Proof. Let A be a finite set.

If $A = \phi$, then A has no proper subset and we have nothing to prove.

If $A \neq \phi$ then $A \sim N_m$ for some $m \in N$.

Let B be a proper subset of A, then B has k elements, where $k < m$ *i.e.*, $B \sim N_k$.

Since A and B do not have same number of elements, A cannot be equivalent to B.

2.23. THEOREM

Every infinite set is equivalent to a proper subset of itself.

Proof. Let A be an infinite set. Since every infinite set contains a denumerable subset. [See Th. 2.14]

Let $B = \{a_1, a_2, a_3, \ldots\}$ be a denumerable subset of A.

Let $C = A - B$, then $A = B \cup C$

Let $P = A - \{a_1\}$ be a proper subset of A

Consider the mapping $f : A \to P$

defined by $f(a_i) = a_{i+1}$ for $a_i \in B$ and $f(a) = a$ for $a \in C$

Then f is one-one and onto. Hence $A \sim P$.

Note. If A is a denumerable set, then $A \sim N$ and we can write A as the indexed set $\{a_i : i \in N\}$, where $a_i \neq a_j$ for $i \neq j$. The process of writing a denumerable set in this form is called **enumeration.**

2.24. THEOREM

The union of a finite set and a countable set is a countable set.

Proof. Let A be a finite set and B be a countable set.

If B is finite then $A \cup B$ is a finite set and hence countable.

If B is denumerable then there are two possibilities :

(*i*) $A \cap B = \phi$ and (*ii*) $A \cap B \neq \phi$

Case (*i*) When $A \cap B = \phi$

Let $A = \{a_1, a_2, \ldots, a_p\}$ and $B = \{b_1, b_2, \ldots, b_n, \ldots\}$

Then $A \cup B = \{a_1, a_2, \ldots, a_p, b_1, b_2, \ldots, b_n, \ldots\}$

Define a function $f : N \to A \cup B$ by $f(n) = \begin{cases} a_n & \text{if } 1 \le n \le p \\ b_{n-p} & \text{if } n \ge p+1 \end{cases}$

Clearly f is one-one and onto.

$\therefore$ $A \cup B \sim N$. Hence $A \cup B$ is denumerable and so countable.

Case. (*ii*) When $A \cap B \neq \phi$

Let $C = B - A$, then $C \subset B$

Since A is finite, C is infinite.

C being an infinite subset of a denumerable set is denumerable.

Clearly $A \cup B = A \cup C = \{a_1, a_2, \ldots, a_p, c_1, c_2, \ldots\}$ and $A \cap C = \phi$

$\therefore$ By case (*i*), $A \cup C$ is countable. Hence $A \cup B$ is countable.

2.25. THEOREM

The set N × N is countable.

Proof. Consider the sets

$$A_1 = \{(1, 1), (1, 2), (1, 3), \ldots\}$$
$$A_2 = \{(2, 1), (2, 2), (2, 3), \ldots\}$$
$$A_3 = \{(3, 1), (3, 2), (3, 3), \ldots\}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$A_n = \{(n, 1), (n, 2), (n, 3), \ldots\}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Clearly $N \times N = \bigcup_{n \in N} A_n$

Also the function $f : A_n \to N$ defined by

$f(n, i) = i$ is one-one and onto.

$\therefore$ A_n is denumerable. Since $N \times N$ is a denumerable collection of denumerable sets, it is denumerable and hence countable.

Corollary 1. The set of all positive rational numbers is countable.

Proof. $Q^+ = \left\{\dfrac{p}{q} ; p, q \text{ are co-prime positive integers}\right\}$

Let $A = \{(p, q) ; p, q$ are co-prime positive integers$\}$

Clearly the elements of Q^+ and A are in one-one correspondence and therefore Q^+ is countable iff A is countable. Since $A \subset N \times N$ and $N \times N$ is countable, therefore, A is countable. Hence Q^+ is countable.

Note 1. The set Q^+ is denumerable can also be proved as under.

Consider the sets $A_1 = \left\{\frac{1}{1}, \frac{2}{1}, \frac{3}{1}, \ldots\right\}$

$A_2 = \left\{\frac{1}{2}, \frac{2}{2}, \frac{3}{2}, \ldots\right\}$

$A_3 = \left\{\frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \ldots\right\}$

....

$A_n = \left\{\frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots\right\}$

....

Clearly $Q^+ = \bigcup_{n \in N} A_n$

Also, the function $f : A_n \to N$ defined by $f\left(\frac{i}{n}\right) = i$ is one-one and onto.

$\therefore$ A_n is denumerable. Since Q^+ is a denumerable collection of denumerable sets, it is denumerable and hence countable.

Note 2. $Q = Q^- \cup \{0\} \cup Q^+$ is denumerable, since Q^+ and Q^- are in one-one correspondence.

2.26. ALGEBRAIC STRUCTURE

A non-empty set with one or more compositions (operations) defined on it is called an '**algebraic structure**' or '**algebraic system**'. If A is the given non-empty set and * is a composition defined on A, then this algebraic structure is denoted by (A, *).

For example, if R is the set of real numbers, then (R, + , ×) is an algebraic structure with two compositions.

2.27. REAL NUMBER SYSTEM AS AN ORDERED FIELD

Let R be the set of real numbers and the two binary operations addition and multiplication be denoted by '+' and '.' respectively.

Then the algebraic structure (R, +, .) satisfies the following axioms :

I. Field Axioms

(*i*) *The Addition Axioms*

A_1. (Closure Law of addition) $\forall a, b \in R, a + b \in R$

A_2. (Commutative Law of addition) $\forall a, b \in R, a + b = b + a$

A_3. (Associative Law of addition) $\forall a, b, c \in R, a + (b + c) = (a + b) + c$

A_4. (Existence of additive identity) $\forall a \in R, \exists 0 \in R$ s.t. $a + 0 = 0 + a = a$

This real number '0' is called the *additive identity* of R.

A_5. (Existence of additive inverse) $\forall a \in R, \exists b \in R$ s.t. $a + b = 0 = b + a$

This real number 'b' is called the *additive inverse* of 'a'.

[But $a + b = 0 = b + a$ if $b = -a$ *i.e.*, additive inverse of a real number 'a' is its negative. Thus $a + (-a) = 0 = (-a) + a$].

(*ii*) *The Multiplication Axioms.*

M_1. (Closure Law of multiplication) $\forall a, b \in R, a.b \in R$

M_2. (Commutative Law of multiplication) $\forall a, b, \in R, a.b = b.a$

M_3. (Associative Law of multiplication) $\forall a, b, c \in R, a.(b.c) = (a.b).c$

M_4. (Existence of multiplicative identity) $\forall a \in R, \exists 1 \in R$ s.t. $a.1 = 1.a = a$

This real number '1' is called the *multiplicative identity* of R.

M_5. (Existence of multiplicative inverse) $\forall a \in R, a \neq 0, \exists b \in R$ s.t. $a.b = b.a = 1$

The real number b is called the *multiplicative inverse* of a and is denoted by a^{-1} or $\frac{1}{a}$ which is the reciprocal of a.

(*iii*) *Distributivity.* Multiplication distributes over addition in R.

D. $\forall a, b, c \in R \quad a.(b + c) = a.b + a.c.$

[☞ A non-empty set with at least two elements in it and with two binary operation satisfying *all the* above *eleven axioms* A_1 to A_5, M_1 to M_5 and D is called a field.

Thus (Q, +, .) is also a field whereas (Z, + , .) and (N, +, .) are not so.]

II. Order Axioms (>)

O_1. $\forall a, b \in R$, exactly one of the following holds :

(*i*) $a > b$ (*ii*) $a = b$ (*iii*) $b > a$ (Trichotomy Law)

O_2. $\forall a, b, c \in R, a > b$ and $b > c \Rightarrow a > c$ (Transitivity)

O_3. $\forall a, b \in R, a > b \Rightarrow a + c > b + c \; \forall c \in R$

This is known as *monotone law of addition.*

O_4. $\forall a, b, c \in R, a > b$ and $c > 0 \Rightarrow ac > bc$

This is known as *monotone law of multiplication.*

☞ Ordered Field

If a *field* satisfies *all the four* order axioms O_1 to O_4, then it is called an ordered field.

[Thus (Q, +, .) is an ordered field.]

Hence (R, +, .) is an ordered field.

2.28. THE SET OF RATIONAL NUMBERS AS AN ORDERED FIELD

The set Q of rational numbers is an ordered field because (Q, +, .) satisfies the following axioms.

I. Field Axioms

(*i*) *The Addition Axioms*

A_1. $\forall a, b \in Q, \quad a + b \in Q$ (Closure Law of addition)

A_2. $\forall a, b \in Q, \quad a + b = b + a$ (Commutative Law of addition)

A_3. $\forall a, b, c \in Q, \quad a + (b + c) = (a + b) + c$ (Associative Law of addition)

A_4. $\forall a \in Q, \exists 0 \in Q$ s.t. $a + 0 = 0 + a = a$

This rational number '0' is called the *additive identity* of Q.

A_5. $\forall a \in Q, \exists b \in Q$ s.t. $a + b = b + a = 0$.

This rational number b is called the *additive inverse* of a or negative of a and is denoted by $-a$.

Thus $a + (-a) = (-a) + a = 0.$

(*ii*) *The Multiplication Axioms.*

M_1. $\forall\, a, b \in Q \;\; a.b \in Q$ (Closure Law of multiplication)

M_2. $\forall\, a, b \in Q \;\; a.b = b.a$ (Commutative Law of multiplication)

M_3. $\forall\, a, b, c \in Q \;\; a.(b.c) = (a.b).c$ (Associative Law of Multiplication)

M_4. $\forall\, a \in Q, \exists\, 1 \in Q$ s.t. $a.1 = 1.a = a$.

This rational number '1' is called the *multiplicative identity* of Q.

M_5. $\forall\, a \in Q, a \neq 0, \exists\, b \in Q$ s.t. $a.b = b.a = 1$

The rational number b is called the *multiplicative inverse* of a and is denoted by a^{-1} or $\frac{1}{a}$ which is the reciprocal of a.

(*iii*) *Distributivity*

Multiplication distributes over addition in Q.

D. $\forall\, a, b, c \in Q,\;\; a.(b + c) = a.b + a.c$

I. Order Axioms (>)

O_1. $\forall\, a, b \in Q$, exactly one of the following holds :

(*i*) $a > b$ (*ii*) $a = b$ (*iii*) $b > a$ (Trichotomy law)

O_2. $\forall\, a, b, c \in Q, a > b$ and $b > c \Rightarrow a > c$ (Transitivity)

O_3. $\forall\, a, b \in Q, a > b \Rightarrow a + c > b + c \;\forall\, c \in Q$

This is known as *monotone law of addition.*

O_4. $\forall\, a, b, c \in Q, a > b$ and $c > 0 \Rightarrow ac > bc$

This is known as *monotone law of multiplications*

Note. A_4 does not hold in N because $0 \notin$ N.

$\Rightarrow$ (N, +, .) is not a field.

M_5 does not hold in Z because $a \neq 0, \pm 1 \Rightarrow a^{-1} \notin Z$.

$\therefore$ (Z, +, .) is not a field.

Example 1. *Prove that in the set R of real numbers :*

(*i*) *Additive identity is unique.* (*ii*) *Multiplicative identity is unique.*

(*iii*) *Additive inverse is unique.* (*iv*) *Multiplicative inverse is unique.*

Sol. (*i*) Let, if possible, 0 and 0′ be two additive identities in R.

$\therefore \quad 0' + 0 = 0'$ ($\because$ 0 is the additive identity) $[A_4]$

Also $\quad 0 + 0' = 0$ ($\because$ 0′ is the additive identity) $[A_4]$

But $\quad 0 + 0' = 0' + 0 \quad \therefore \quad 0 = 0'$.

Thus the additive identity 0 in R is unique.

(*ii*) Let, if possible, 1 and 1′ be two multiplicative identities in R. As 1 is the multiplicative identity,

$\therefore$ We have $\quad 1.\, a = a \;\forall\, a \in R$ $[M_4]$

Putting $\quad a = 1',\; 1.1' = 1'$ ...(*i*)

Again, as 1′ is the multiplicative identity.

$\therefore \quad 1'.a = a \;\forall\, a \in R$ $[M_4]$

Replacing a by 1, we get $1'.1 = 1$ ...(*ii*)

But $\quad 1.1' = 1'.1$ $[M_2]$

$\therefore$ from (*i*) and (*ii*), $1' = 1$.

(*iii*) If possible, let there be two real numbers b and c such that

$$a + b = 0 \text{ and } a + c = 0$$

Now
$$\begin{aligned} b &= 0 + b && [A_4]\\ &= (a + c) + b && [\because\; a + c = 0]\\ &= (c + a) + b && [A_2]\\ &= c + (a + b) && [A_3]\\ &= c + 0 && [\because\; a + b = 0]\\ &= c && [A_4] \end{aligned}$$

Hence $b = c$, which proves the result.

(*iv*) Let, if possible, there be two multiplicative inverses b and c of $a\ (\neq 0) \in$ R such that

$$a.b = b.a = 1 \text{ and } a.c = c.a = 1$$

Now
$$\begin{aligned} b &= 1.b && [M_4]\\ &= (c.a)\, b && [\because\; c.a = 1]\\ &= c.(a.b) && [M_3]\\ &= c.1 && [\because\; a.b = 1]\\ &= c. && [M_4] \end{aligned}$$

$\therefore \quad b = c$ and hence the result.

Example 2. *Assuming the properties of addition and multiplication for the system of real numbers, prove that*

(*i*) *if $a + b = a + c$, then $b = c$* (*Cancellation law*)

(*ii*) *if $a.\, b = a.\, c$, then $b = c\ (a \neq 0)$* (*Cancellation law*)

(*iii*) $-(-a) = a$ (*iv*) $(a^{-1})^{-1} = a$.

Sol. (*i*) We have,
$$\begin{aligned} b &= 0 + b && [A_4]\\ &= [(-a) + a] + b && [A_5]\\ &= (-a) + (a + b) && [A_3]\\ &= (-a) + (a + c) && [\because\; a + b = a + c]\\ &= [(-a) + a] + c && [A_3]\\ &= 0 + c && [A_5]\\ &= c. && [A_4] \end{aligned}$$

Hence $\quad a + b = a + c \Rightarrow b = c$

(*ii*) $\because \quad a \neq 0, \therefore a^{-1}$ exists and is a real number.
$$\begin{aligned} b &= 1.\, b && [M_4]\\ &= (a^{-1}.\, a)\, b && [M_5]\\ &= a^{-1}(a\,.\,b) && [M_3]\\ &= a^{-1}.(a\,.\,c) && [\because\; a.b = a.c]\\ &= (a^{-1}.\,a)\,.\,c && [M_3]\\ &= 1.c = c. && [M_5, M_4] \end{aligned}$$

(*iii*) $\because \quad (-a) + a = 0$. $\therefore$ a is the additive inverse of $(-a)$, *i.e.*, $-(-a) = a$.

(*iv*) $\because \quad a^{-1}.\, a = 1$, $\therefore$ a is the multiplicative inverse of a^{-1} *i.e.*, $(a^{-1})^{-1} = a$.

Example 3. (*i*) *If $a.b = 0$, then either $a = 0$ or $b = 0$.*

(*ii*) $(a.b)^{-1} = a^{-1}.b^{-1}$ *$(a \neq 0, b \neq 0)$ where a, b are real numbers.*

Sol. (*i*) Let $a \neq 0$. Then a^{-1} exists.

$$b = 1.b = (a^{-1}.a)b = a^{-1}.(a.b) = a^{-1}.0 = 0$$

Similarly if $b \neq 0$, we can prove that $a = 0$.

(ii) $(a.\,b)(b^{-1}.\,a^{-1}) = a[(b.b^{-1})a^{-1}]$ [M_3]

$= a[1.a^{-1}] = a.a^{-1} = 1$

$\therefore$ $b^{-1}a^{-1}$ is the inverse of ab *i.e.,* $(a.b)^{-1} = b^{-1}.\,a^{-1} = a^{-1}.\,b^{-1}$. [$M_2$]

Example 4. $\forall\, x, y \in R$, *prove that*

(i) $-(x+y) = (-x) + (-y)$ *(ii)* $(-x)y = x(-y) = -(xy)$.

Sol. *(i)* $(-x) + (-y) = 0 + (-x) + (-y)$ [A_4]

$= \{-(x+y) + (x+y)\} + (-x) + (-y)$ [A_5]

$= -(x+y) + (y+x) + (-x) + (-y)$ [A_2]

$= -(x+y) + y + \{x + (-x)\} + (-y)$ [A_3]

$= -(x+y) + y + 0 + (-y)$ [A_5]

$= -(x+y) + y + (-y)$ [A_4]

$= -(x+y) + 0$ [A_5]

$= -(x+y)$ [A_4]

$-(x+y) = (-x) + (-y)$

(ii) $xy + (-x)y = \{x + (-x)\}y$ (Distributive Law)

$= 0.y$ [A_5]

$= 0$ (Property of 0)

It follows that $(-x)y$ is the additive inverse of xy (since its sum with xy is 0).

$\therefore$ $-(xy) = (-x)y$

Similarly $-(xy) = x(-y)$ | Prove it yourself.

Hence $(-x)y = x(-y) = -(xy)$.

Example 5. $\forall\, x, y \in R$, *prove that*

(i) $x > 0 \Leftrightarrow -x < 0$ *(ii)* $x > 0 \Leftrightarrow x^{-1} > 0$ *(iii)* $x^2 \geq 0$

(iv) $xy > 0 \Leftrightarrow$ *either* $(x > 0$ *and* $y > 0)$ *or* $(x < 0$ *and* $y < 0)$

(v) $xy < 0 \Leftrightarrow$ *either* $(x > 0$ *and* $y < 0)$ *or* $(x < 0$ *and* $y > 0)$.

Sol. *(i)* $x > 0 \Leftrightarrow x + (-x) > 0 + (-x)$ [O_2]

$\Leftrightarrow 0 > -x$ [A_5, A_4]

$\Leftrightarrow -x < 0$

(ii) Since $x > 0$, $x \neq 0$ $\therefore$ x^{-1} exists

Suppose $x^{-1} \leq 0$ then $x.\,x^{-1} \leq x.0$ [O_2]

$\Rightarrow$ $1 \leq 0$ [M_5]

which is false. $\therefore$ our supposition that $x^{-1} \leq 0$ is false.

Hence $x^{-1} > 0$ [Trichotomy Law]

Conversely, suppose $x^{-1} > 0$, then by what we have just proved,

$(x^{-1})^{-1} > 0 \Rightarrow x > 0$ [$\because$ $(x^{-1})^{-1} = x$]

Hence $x > 0 \Leftrightarrow x^{-1} > 0$.

(iii) For any $x \in R$, there are three possibilities $x = 0, x > 0, x < 0$

If $x = 0$ then $x.x = x.0 \Rightarrow x^2 = 0$

If $x > 0$ then $x.x > x.\,0$ [O_2]

$\Rightarrow$ $x^2 > 0$

If $x < 0$ then $x.x > x.0$ ($\because$ $x < 0$, inequality is reversed)

$\Rightarrow$ $x^2 > 0$

Thus in all cases $x^2 \geq 0$.

(iv) Suppose $xy > 0$. Also suppose that $x > 0$ so that x^{-1} exists and $x^{-1} > 0$

$xy > 0 \Rightarrow x^{-1}(xy) > x^{-1}\,0$ [O_3]

$\Rightarrow (x^{-1}.\,x)\,y > 0$ [M_3]

$\Rightarrow 1.y > 0$ [M_5]

$\Rightarrow y > 0$ [M_4]

Now suppose $x < 0$. Then x^{-1} exists and $x^{-1} < 0$

$xy > 0 \Rightarrow x^{-1}(xy) < x^{-1}.0$ [O_3]

$\Rightarrow (x^{-1}.x)y < 0$ [M_3]

$\Rightarrow 1.y < 0$ [M_5]

$\Rightarrow y < 0$ [M_4]

Thus $xy > 0 \Rightarrow$ either $(x > 0$ and $y > 0)$ or $(x < 0$ and $y < 0)$

Conversely, suppose $x > 0, y > 0$

$x > 0 \Rightarrow xy > 0.y \Rightarrow xy > 0$ [O_3]

Again if $x < 0, y < 0$ then $-x > 0$ and $-y > 0$

$\therefore$ By what we have proved above $(-x)(-y) > 0 \Rightarrow xy > 0$

Hence the result.

(v) Please try yourself.

Example 6. *Solve* $ax + b = 0$, $a, b \in R$, $(a \neq 0)$ *for the unknown* x, *belonging to* R *indicating at each step the property of* R *being used. Is the solution unique ? Would you always get a unique solution of the above equation in* Q, *if* $a, b \in Q$, $a \neq 0$ *? Justify your answer.*

Sol. The given equation is $ax + b = 0$. ...(1)

Since $b \in R$, $\therefore$ $\exists\,(-b) \in R$ such that $b + (-b) = 0$ [A_5]

Adding $-b$ to both sides of (1), we have

$(ax + b) + (-b) = 0 + (-b)$

or $ax + [b + (-b)] = -b$ [A_3 and A_4]

or $ax + 0 = -b$ [A_5]

or $ax = -b$ ...(2) [A_4]

Since $a\,(\neq 0) \in R$, $\therefore$ $\exists$ multiplicative inverse of a *i.e.,* $a^{-1} \in R$ such that

$a.\,a^{-1} = a^{-1}.\,a = 1$ [M_5]

Multiplying from the left, both sides of (2) by a^{-1}, we have

$a^{-1}(ax) = a^{-1}(-b)$

or $(a^{-1}.\,a)x = -ba^{-1}$ [M_3, M_2]

or $1.\,x = -ba^{-1}$ [M_5]

or $x = -ba^{-1}$ [M_4]

The solution $x = -ba^{-1}$ is unique in R, because the additive and multiplicative inverses are unique in R. If $a, b \in Q$, $a \neq 0$, then the solution is still unique, because the laws of addition and multiplication are same for R and Q.

Example 7. *Define a rational number and show that $\sqrt{2}$ is not a rational number.*

Sol. Any number which can be put in the form $\frac{p}{q}$ where p and q are integers and $q \neq 0$ is called a *rational number*.

For example, $\frac{3}{7}, \frac{-5}{2}, \frac{0}{-3}, \frac{6}{1}$ etc. are all rational numbers. The set of rational numbers is denoted by Q.

Let, if possible, $\sqrt{2}$ be a rational number. Then let $\sqrt{2} = \frac{p}{q}$ where p and q are integers prime to each other (*i.e.*, have no common factor other than 1) and $q \neq 0$.

Squaring, $\quad 2 = \frac{p^2}{q^2}$ or $p^2 = 2q^2$. ...(*i*)

Thus p^2 is even and hence p is even ($\because$ square of only an even number is even).

Let $p = 2m$, where m is an integer.

From (*i*), $\quad 4m^2 = 2q^2$ or $q^2 = 2m^2$

$\therefore$ q^2 is even and hence q is even.

Thus p and q are both even *i.e.*, p and q have a common factor 2, which contradicts our supposition that p and q have no common factor. Hence $\sqrt{2}$ cannot be expressed in the form $\frac{p}{q}$ and, therefore, $\sqrt{2}$ is not a rational number.

Example 8. *Prove that the following numbers are not rational numbers :*

(*a*) $\sqrt{3}$ (*b*) $\sqrt{5}$ (*c*) $\sqrt{8}$ (*d*) $\sqrt{6}$

Sol. (*a*) If possible, suppose that $\sqrt{3}$ is a rational number. Then

let $\sqrt{3} = \frac{p}{q}$ where p and q are integers prime to each other and $q \neq 0$

Squaring, $\quad \frac{p^2}{q^2} = 3$ or $p^2 = 3q^2$ ...(1)

Thus p^2 is divisible by 3. As such p must be divisible by 3, for otherwise p^2 would not be divisible by 3.

Let $\quad p = 3m$ where $m \in Z$

From (1), $\quad 9m^2 = 3q^2$ or $q^2 = 3m^2$

Thus q^2 is divisible by 3. As such q must be divisible by 3.

Hence p and q are both divisible by 3 which contradicts the hypothesis that p and q have no common factor other than 1. Thus $\sqrt{3}$ cannot be expressed in the form $\frac{p}{q}$ and therefore $\sqrt{3}$ is not a rational number.

(*b*) *Please try yourself.*

(*c*) We know that $\sqrt{4} < \sqrt{8} < \sqrt{9}$ *i.e.*, $2 < \sqrt{8} < 3$

If possible, let $\sqrt{8}$ be a rational number, then $\sqrt{8} = \frac{p}{q}$ where p and q are positive integers prime to each other.

Now $\quad 2 < \sqrt{8} < 3 \Rightarrow 2 < \frac{p}{q} < 3 \Rightarrow 2q < p < 3q \quad [\because q > 0]$

$\Rightarrow 0 < p - 2q < q$

Thus $p - 2q$ is a positive integer less than q

$\Rightarrow$ $p - 2q$ and q are co-prime.

Also p and q are co-prime.

$\therefore \quad \sqrt{8}(p - 2q) = \frac{p}{q}(p - 2q)$ is not an integer.

But $\quad \sqrt{8}(p - 2q) = \frac{p}{q}(p - 2q) = \frac{p^2}{q} - 2p = \frac{p^2}{q^2} \cdot q - 2p = 8q - 2p \quad \left[\because \frac{p}{q} = \sqrt{8}\right]$

which is an integer.

Thus we arrive at a contradiction. Hence $\sqrt{8}$ is not a rational number.

(*d*) *Please try yourself.*

Remark. In the above example, we have considered $\sqrt{n}$, where n is not a perfect square. If n is prime, use the first method. If n is a composite number, use the second method.

Note. Irrational Number. (Def.). A real number which is not rational is called an irrational number. For example, $\sqrt{2}, \sqrt{3}, \sqrt{5}, \pi$ are irrational numbers. An irrational number cannot be expressed as a terminating or recurring decimal.

Example 9. *Prove that between two different rational numbers, there lie an infinite number of rational numbers.*

Or

Prove that the set of rational numbers is dense.

Sol. Let a and b be two different rational numbers with $a < b$.

Now $\quad a < b \Rightarrow a + a < a + b$

$\Rightarrow 2a < a + b \Rightarrow a < \frac{a+b}{2}$ ...(*i*)

Also $\quad a < b \Rightarrow a + b < b + b$

$\Rightarrow a + b < 2b \Rightarrow \frac{a+b}{2} < b$ ...(*ii*)

Combining (*i*) and (*ii*), $\quad a < \frac{a+b}{2} < b$.

Since a, b and 2 are rational numbers, so is $\frac{a+b}{2} = r_1$ (say)

Thus $\quad a < r_1 < b$

$\Rightarrow$ $\exists$ a rational number $r_1 \left(= \frac{a+b}{2}\right)$ between a and b.

Now a and r_1 are two different rational numbers.

$\therefore \quad r_2 = \dfrac{a + r_1}{2}$ lies between them, *i.e.*, $a < r_2 < r_1$

r_1 and b are two different rational numbers.

$\therefore \quad r_3 = \dfrac{r_1 + b}{2}$ lies between them, *i.e.*, $r_1 < r_3 < b$.

Thus $\quad a < r_2 < r_1 < r_3 < b$.

Continuing like this, there lie infinitely many rational numbers between two different rational numbers.

Hence the set of rational numbers is dense.

Theorem I. *Define absolute value or modulus of a real number and prove that*

(i) $|x| = \max.\{x, -x\}$, *(ii)* $|x|^2 = x^2$

(iii) $x \le |x|$ *and* $-x \le |x|$ *(iv)* $|x| = |-x|$.

Proof. Def. If x be a real number, then the modulus (or absolute value or numerical value) of x is denoted by $|x|$ and defined as

$$|x| = x \quad \text{if} \quad x \ge 0$$
$$= -x \quad \text{if} \quad x < 0.$$

(i) Since x is a real number, either $x \ge 0$ or $x < 0$.

If $x \ge 0$, then $\quad |x| = x \quad$ and $\quad x \ge -x$

When $x < 0 \quad |x| = -x \quad$ and $\quad -x > x$.

Thus in each case $|x|$ is the greater of two numbers x and $-x$, *i.e.*,

$$|x| = \max.\{x, -x\}.$$

(ii) Since $|x| = x$ or $-x$ according as $x \ge 0$ or $x < 0$

$\therefore \quad |x|^2 = x^2$ or $(-x)^2$ *i.e.*, $|x|^2 = x^2$

which proves the result.

Note. $|x| = \pm\sqrt{x^2}$. Since $|x| \ge 0$, $\therefore$ rejecting the – ve sign, we have

$|x| = \sqrt{x^2}$ and this is used as definition of $|x|$.

(iii) Since $\quad |x| = \max.\{x, -x\} \ge x$ or $-x$.

$\therefore \quad x \le |x|$ and $-x \le |x|$.

(iv) $\quad |x| = \max.\{x, -x\}$

$|-x| = \max.\{-x, x\} = \max.\{x, -x\}$

$\therefore \quad |x| = |-x|$.

Aliter. We have $\quad |x| = \sqrt{x^2}$.

$\therefore \quad |-x| = \sqrt{(-x)^2} = \sqrt{x^2} = |x|$.

Theorem II. *If* $|x|$ *denotes the absolute value of x, then prove that* $|x| = \sqrt{x^2}$.

If x and y are any two real numbers, then

(a) $|x + y| \le |x| + |y|$ *(Triangle Inequality)*

(b) $|x - y| \ge ||x| - |y||$

(c) $|xy| = |x|\,|y|$ *(d)* $\left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|}$ *if* $y \ne 0$.

Proof. By definition, we have

$$|x| = x \quad \text{if} \quad x \ge 0$$
$$= -x \quad \text{if} \quad x < 0$$

so that $\quad |x|^2 = x^2$ for all x, +ve, –ve or zero.

$\therefore \quad |x| = \pm\sqrt{x^2}$.

Since $|x| \ge 0$ $\therefore$ rejecting the –ve sign, we have $|x| = \sqrt{x^2}$.

(a) Now $\quad |x + y| = \sqrt{(x+y)^2} = \sqrt{x^2 + y^2 + 2xy}$

$\le \sqrt{x^2 + y^2 + 2|x||y|} \quad | \because x \le |x|$ and $y \le |y|$

$= \sqrt{|x|^2 + |y|^2 + 2|x|.|y|} \quad | \because x^2 = |x|^2$

$= \sqrt{(|x| + |y|)^2}$

$= ||x| + |y|| \quad | \because \sqrt{a^2} = |a|$

$= |x| + |y| \quad | |a| = a$ if $a \ge 0$ and $|x| + |y| \ge 0$

Hence $\quad |x + y| \le |x| + |y|$.

(b) $\quad |x - y| = \sqrt{(x-y)^2} = \sqrt{x^2 + y^2 - 2xy}$

$\ge \sqrt{x^2 + y^2 - 2|x|.|y|} \quad | x \le |x|, y \le |y|$ so that $xy \le |x||y|$ or $-xy \ge -|x||y|$

$= \sqrt{|x|^2 + |y|^2 - 2|x||y|} = \sqrt{(|x| - |y|)^2}$

$= ||x| - |y|| \quad | \because \sqrt{a^2} = |a|$

Hence $\quad |x - y| \ge ||x| - |y||$.

(c) $\quad |xy| = \sqrt{(xy)^2} = \sqrt{x^2 . y^2} = \sqrt{x^2}\sqrt{y^2}$

$= |x|\,|y|$.

(d) $\quad \left|\dfrac{x}{y}\right| = \sqrt{\left(\dfrac{x}{y}\right)^2} = \sqrt{\dfrac{x^2}{y^2}} = \dfrac{\sqrt{x^2}}{\sqrt{y^2}} = \dfrac{|x|}{|y|}$ if $y \ne 0$.

Theorem III. *If* $r > 0$, *then prove that*

(a) $|x| < r$ *iff* $-r < x < r$ *(b)* $|x - a| < r \Leftrightarrow a - r < x < a + r$.

Proof. *(a)* $\quad |x| = \max.\{x, -x\}$

$\therefore \quad |x| < r \quad \Leftrightarrow \quad -x < r$ and $x < r$

$\Leftrightarrow \quad x > -r$ and $x < r \quad \Leftrightarrow \quad -r < x < r$.

(b) $\quad |x - a| = \max.\{x - a, a - x\}$

$\therefore \quad |x - a| < r \quad \Leftrightarrow \quad x - a < r$ and $a - x < r$

$\Leftrightarrow \quad x < a + r$ and $x > a - r \quad \Leftrightarrow \quad a - r < x < a + r$.

Example 1. *Prove that (i)* $|x-y| \le |x| + |y|$

(ii) $a < x < b \iff \left|x - \frac{a+b}{2}\right| < \frac{b-a}{2}$.

Sol. *(i)* $|x-y| = |x+(-y)|$

$\le |x| + |-y| = |x| + |y|$ $\qquad [\because \ |-y| = |y|]$

$\therefore \quad |x-y| \le |x| + |y|$.

(ii) $a < x < b$

Adding $-\frac{a+b}{2}$ throughout

$\iff \quad a - \frac{a+b}{2} < x - \frac{a+b}{2} < b - \frac{a+b}{2}$

$\iff \quad \frac{a-b}{2} < x - \frac{a+b}{2} < \frac{b-a}{2} \iff -\frac{b-a}{2} < x - \frac{a+b}{2} < \frac{b-a}{2}$

$\iff \quad \left|x - \frac{a+b}{2}\right| < \frac{b-a}{2}$. $\qquad [\because \ -r < x < r \iff |x| < r]$

Example 2. *For any real numbers x, y ; show that*

$$|x+y|^2 + |x-y|^2 = 2|x|^2 + 2|y|^2.$$

Sol. We know that for all $x \in \mathbf{R}$, $|x|^2 = x^2$

$\therefore \quad |x+y|^2 = (x+y)^2 = x^2 + y^2 + 2xy = |x|^2 + |y|^2 + 2xy$

Similarly $\quad |x-y|^2 = |x|^2 + |y|^2 - 2xy$

Adding, $\quad |x+y|^2 + |x-y|^2 = 2|x|^2 + 2|y|^2$.

Example 3. *Show that for any two real numbers a and b,*

(i) $max.\ \{a, b\} = \frac{1}{2}(a + b + |a-b|)$ $\quad$ *(ii)* $min.\ \{a, b\} = \frac{1}{2}(a + b - |a-b|)$.

Sol. *(i)* **When a > b**

$a - b > 0$ so that $|a-b| = a-b$

$\therefore \quad \frac{1}{2}(a + b + |a-b|) = \frac{1}{2}(a + b + a - b) = a = \text{max. } \{a, b\}$

When a < b

$a - b < 0$ so that $|a-b| = -(a-b) = b-a$

$\therefore \quad \frac{1}{2}(a + b + |a-b|) = \frac{1}{2}(a + b + b - a) = b = \text{max. } \{a, b\}$

(ii) **When a > b**

$a - b > 0$ so that $|a-b| = a-b$

$\therefore \quad \frac{1}{2}(a + b - |a-b|) = \frac{1}{2}\{a + b - (a-b)\} = b = \text{min. } \{a, b\}$

When a < b

$a - b < 0$ so that $|a-b| = -(a-b) = b-a$

$\therefore \quad \frac{1}{2}(a + b - |a-b|) = \frac{1}{2}\{a + b - (b-a)\} = a = \text{min. } \{a, b\}$.

Theorem IV. *(a) When is a set said to be (i) bounded above (ii) bounded below (iii) bounded ?*

(b) Give various examples of sets which are

(i) bounded below and not above. $\quad$ *(ii) bounded above but not below.*

(iii) bounded. $\quad$ *(iv) bounded neither above nor below.*

Proof. (*a*) Let S be a subset of real numbers.

(*i*) The set S is said to be *bounded above* if $\exists$ a real number K such that $x \le K \ \forall x \in S$. Then K is called an *upper bound* of S.

(*ii*) The set S is said to be *bounded below* if $\exists$ a real number k such that $x \ge k \ \forall x \in S$. Then k is called a *lower bound* of S.

(*iii*) The set S is said to be *bounded* if it is bounded above as well as below. Hence when a set S is bounded, there exist two real numbers k and K such that $k \le x \le K \ \forall x \in S$.

☞*Equality holds when S is a singleton.*

(b) (i) 1. The set N = {1, 2, 3, ...} is bounded below and not above, 1 being a lower bound.

2. The set $\{1, 2, 2^2, 2^3, ...\}$ *is bounded below and not above.*

(ii) 1. The set of negative integers = {..., – 3, – 2, – 1} is bounded above but not below, – 1 being an upper bound.

2. The set $\left\{\log \frac{1}{n} : n \in N\right\}$ *is bounded above and not below.*

(iii) 1. Every finite set is bounded above as well as below.

$\therefore$ *it is bounded.*

2. Each of the intervals [a, b], (a, b), [a, b) (a, b] is a bounded set.

3. The set $\left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}\right)$ *is a bounded set.*

(iv) Each of the sets Z = {..., – 3, – 2, – 1, 0, 1, 2, 3, ...} of integers

Q = set of rationals ; R = set of reals

is bounded neither below nor above.

Theorem V. *Show that for a bounded set S, there exists a positive number A such that* $|x| \le A \ \forall x \in S$. *Prove that the converse is also true.*

Proof. Since the set S is bounded,

$\therefore \exists$ real numbers k and K such that $k \le x \le K \ \forall \ x \in S$ $\qquad$ ...(*i*)

Let $A \ge \max.(|k|, |K|)$. Then from (*i*)

$-A \le k \le x \le K \le A$ for all $x \in S$.

or $\quad -A \le x \le A \quad \forall x \in S$

or $\quad |x| \le A \quad \forall x \in S$

Conversely $\quad |x| \le A \quad \forall x \in S$

$\Rightarrow \quad -A \le x \le A \quad \forall x \in S$

$\Rightarrow$ S is bounded.

Theorem VI. *(a) What are supremum and infimum of a set ? Prove that the supremum or infimum if they exist are unique. Do they always belong to the set ?*

(b) Find the supremum and infimum of the following sets. Find also whether they belong to the set or not.

(*i*) {3, 4, 7} (*ii*) $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\ldots\right\}$

(*iii*) $\left\{-2, -\frac{3}{2}, -\frac{4}{3}, -\frac{5}{4}, \ldots\ldots\right\}$ (*iv*) *open interval (2, 3).*

Proof. Let S be a set which is bounded above. Then there exist an infinite number of upper bounds of S, because every number greater than an upper bound is also an upper bound.

If the set of all upper bounds of a set S has a smallest member say s, than s is called the **least upper bound** (*l.u.b.*) or **supremum** (sup.) of S

The supremum s has the following properties :

(*i*) $x \le s \ \forall x \in S$ *i.e.,* s is an upper bound of S.

(*ii*) given $\varepsilon > 0$, however small, $\exists\, y \in S$, such that

$$y > s - \varepsilon.$$

Uniqueness. (*K.U. 1993*) Let, if possible, there be two supremums of S, say s and s'.

Now s and s' are both upper bounds of S.

Since s is the *l.u.b.* and s' is an upper bound of S,

$$s \le s' \qquad \ldots(i)$$

Again s' is the *l.u.b.* and s is an upper bound of S,

$$\therefore \qquad s' \le s \qquad \ldots(ii)$$

From (*i*) and (*ii*) it follows that $s = s'$.

The l.u.b (if it exists) of a set S, may or may not belong to S.

(*a*) If the set S is bounded below, it admits of an infinite number of lower bounds, because every number less than a lower bound is also a lower bound.

If the set of all lower bounds of a set S has a greatest member, say t, then t is called the **greatest lower bound** (*g.l.b.*) or **infimum** (Inf.) of S.

The infimum t has the following properties :

(*i*) $t \le x \ \ \forall x \in S$, *i.e.,* t is a lower bound of S.

(*ii*) Given $\varepsilon > 0$, however small, $\exists\, y \in S$ such that $y < t + \varepsilon$.

Uniqueness. Let, if possible, there be two *g.l.b.*'s of S, say t and t'.

Now t and t' are both lower bounds of S.

Since t is the *g.l.b.* and t' is a lower bound,

$$\therefore \qquad t' \le t \qquad \ldots(iii)$$

Again t' is the *g.l.b.* and t is a lower bound,

$$\therefore \qquad t \le t' \qquad \ldots(iv)$$

From (*iii*) and (*iv*) it follows that $t = t'$.

The g.l.b. (if it exists) of a set S, may or may not belong to the set S.

(*b*) (*i*) The set is {3, 4, 7}.

Supremum = 7, which belongs to the set and infimum is 3 which also belongs to the set.

(*ii*) Let $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \ldots\right\} = \left\{\frac{1}{n} : n \in N\right\}$

Then sup. $A = 1 \in A$ and Inf. $A = 0 \notin A$.

(*iii*) Let $B = \left\{-2, -\frac{3}{2}, -\frac{4}{3}, -\frac{5}{4} \ldots\right\} = \left\{-\frac{n+1}{n} : n \in N\right\}$

Then sup. $B = -1 \notin B$ and Inf. $B = -2 \in B$.

(*iv*) Let $C = (2, 3)$, an open interval.

$\therefore$ Sup. $C = 3 \notin C$ and Inf. $C = 2 \notin C$.

Theorem VII. *Prove that (i) Every finite set is bounded.*

(ii) Every subset of a bounded set is bounded.

Proof. (*i*) Let $A = \{a_1, a_2, a_3, \ldots\ldots, a_n\}$ be a finite set.

Let $h = \text{Min.}\ \{a_1, a_2, a_3, \ldots\ldots, a_n\}$ and $k = \text{Max.}\ \{a_1, a_2, a_3, \ldots\ldots, a_n\}$, then $h \le x \le k\ \forall x \in A$

$\therefore$ A is bounded.

(*ii*) Let A be a bounded set. Then $\exists$ two real numbers h and k s.t. $h \le x \le k\ \forall x \in A$

Let $B \subseteq A$. Then $y \in B \Rightarrow y \in A$

$$\therefore \qquad h \le y \le k\ \forall y \in B$$

$\Rightarrow$ B is bounded.

Example. *Show that (i) every subset of an unbounded set is not necessarily unbounded, (ii) every infinite set need not be unbounded.*

Sol. (*i*) Z, the set of integers is unbounded.

But $A = \{1, 2\} \subset Z$ is bounded (being finite)

(*ii*) $A = \{x : 1 < x < 3\}$ is an infinite set. Also it is bounded.

Theorem VII. (*i*) *If u is an upper bound of a set A and $u \in A$, then show that u = Sup. A.*

(ii) If l is a lower bound of a set A and $l \in A$, then show that l = Inf. A.

Proof. (*i*) Suppose u is not Sup. A but u is an upper bound of A.

Let u' be Sup. A, then $u' < u$

Now $\qquad u \in A$ and u' (= Sup. A) $< u$

$\Rightarrow$ u' is not even an upper bound of A

$\Rightarrow$ Our supposition is wrong. Hence u = Sup. A

(*ii*) *Please try yourself.*

Example 1. *Find the l.u.b. and g.l.b. if they exist, of the following sets :*

(*i*) $\left\{1 - \frac{1}{n} : n \in N\right\}$ (*ii*) $\left\{\frac{1}{5n} : n \in Z, n \ne 0\right\}$

(*iii*) $\left\{\frac{3n+2}{2n+1} : n \in N\right\}$ (*iv*) $\left\{x : x = 1 + \frac{1}{n}\ n \in N\right\}$

(*v*) $\{x : -5 < x < 3\}$ (*vi*) $\{x : x = 2^n,\ n \in N\}$

(*vii*) $\{x : x = (-1)^n \cdot n\ \forall\ \ n \in N\}$.

Sol. (*i*) $\left\{1 - \frac{1}{n} : n \in N\right\} = \left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\right\}$

l.u.b. = 1, *g.l.b.* = 0

(ii) $\left\{\frac{1}{5n} : n \in Z, n \neq 0\right\} = \left\{\ldots, -\frac{1}{15}, -\frac{1}{10}, -\frac{1}{5}, \frac{1}{5}, \frac{1}{10}, \frac{1}{15} \ldots\right\}$

$l.u.b = \frac{1}{5}$, $g.l.b. = -\frac{1}{5}$

(iii) $\left\{\frac{3n+2}{2n+1} : n \in N\right\} = \left\{\frac{5}{3}, \frac{8}{5}, \frac{11}{7}, \ldots\right\}$

$l.u.b = \frac{5}{3}$, $g.l.b = \frac{3}{2}$,

(iv) $\left\{x : x = 1 + \frac{1}{n}, n \in N\right\} = \left\{2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots\right\}$

$l.u.b. = 2$, $g.l.b. = 1$.

(v) $\{x : -5 < x < 3\}$

$l.u.b. = 3$, $g.l.b. = -5$.

(vi) $\{x : x = 2^n, n \in N\} = \{2, 2^2, 2^3, 2^4, \ldots\}$

Not bounded above, $g.l.b. = 2$.

(vii) $\{x : x = (-1)^n . n \;\forall\; n \in N\} = \{-1, 2, -3, 4, -5, 6, \ldots\} = \{\ldots, -5, -3, -1, 2, 4, 6, \ldots\}$

It is not a bounded set.

Example 2. *(i) Give an example of a bounded set which contains its l.u.b. but does not contain the g.l.b.*

(ii) Give an example of a bounded set which contains its g.l.b. but does not contain the l.u.b.

Sol. Consider (i) $A = \{x : 2 < x \leq 5\} = (2, 5]$

l.u.b. of $A = 5 \in A$, g.l.b. of $A = 2 \notin A$

(ii) Consider $B = \{x : 2 \leq x < 5\} = [2, 5)$

l.u.b. of $B = 5 \notin B$, g.l.b. of $B = 2 \in B$.

2.29. THE ORDER COMPLETENESS AXIOM

The least upper bound axiom. Every non-empty subset of R which is bounded above has the least upper bound (or supremum).

Equivalently, we have

The greatest lower bound axiom. Every non-empty subset of R which is bounded below has the greatest lower bound (or infimum).

l.u.b. axiom $\Leftrightarrow$ g.l.b. axiom

So far, we have seen that the set R of real numbers and Q of rational numbers are both ordered fields *i.e.,* so far, we are not in a position to distinguish between R and Q.

Order completeness property serves to distinguish between R and Q in as much as it is possessed by R but not by Q.

2.30. COMPLETE ORDERED FIELD

An ordered field which satisfies the completeness axiom is called a complete ordered field. R is a complete ordered field whereas Q is only an ordered field.

2.31. EXPLICIT STATEMENT OF THE PROPERTIES OF REAL NUMBERS WHICH CHARACTERISE THE REAL NUMBER SYSTEM AS A COMPLETE ORDERED FIELD

I. The field axioms

(i) Properties of addition (+)

A_1. $\forall\; a, b \in R$ $\quad a + b \in R$ (Closure Law of addition)

A_2. $\forall\; a, b \in R$ $\quad a + b = b + a$ (Commutative Law of addition)

A_3. $\forall\; a, b, c \in R$ $\quad a + (b + c) = (a + b) + c$ (Associative Law of addition)

A_4. $\forall\; a \in R, \exists\; 0 \in R$ s.t. $\quad a + 0 = 0 + a = a$ (Existence of additive identity)

A_5. $\forall\; a \in R, \exists - a \in R$ s.t $\quad a + (-a) = (-a) + a = 0$ (Existence of additive inverse)

(ii) Properties of multiplication (.)

M_1. $\forall\; a, b \in R$ $\quad a.b \in R$ (Closure Law of multiplication)

M_2. $\forall\; a, b \in R$ $\quad a.b = b.a$ (Commutative Law of multiplication)

M_3. $\forall\; a, b, c \in R$ $\quad a.(b.c) = (a.b)\, c$ (Associative Law of multiplication)

M_4. $\forall\; a \in R, \exists\; 1 \in R$ s.t. $\quad a.1 = 1.a = a$ (Existence of multiplicative identity)

M_5. $\forall\; a \in R, a \neq 0, \exists\; a^{-1} \in R$ s.t. $\quad a.a^{-1} = a^{-1}.a = 1$

(Existence of multiplicative inverse)

(iii) Distributive Laws

D_L. $\forall\; a, b, c \in R$ $\quad a.(b + c) = a.b + a.c$ [Left]

D_R. $\forall\; a, b, c \in R$ $\quad (a + b).c = a.c + b.c$ [Right]

II. The order axioms (>)

O_1. $\forall\; a, b \in R$, exactly one of the following holds

(i) $a > b$ (ii) $a = b$

(iii) $b > a$ (Trichotomy Law)

O_2. $\forall\; a, b, c \in R$, $\quad a > b$ and $b > c \;\Rightarrow\; a > c$ (Transitivity)

O_3. $\forall\; a, b \in R$, $a > b \;\Rightarrow\; a + c > b + c \;\forall\; c \in R$ (Monotone Law of addition)

O_4. $\forall\; a, b, c \in R$, $\quad a > b$ and $c > 0 \;\Rightarrow\; ac > bc$

(Monotone Law of multiplication)

III. The completeness axiom

The least upper bound axiom. Every non-empty subset of R which is bounded above has the least upper bound.

2.32. THE SET Q OF RATIONAL NUMBERS IS NOT A COMPLETE ORDERED FIELD

To establish the above result, it is enough to show that $\exists$ a non-empty subset of Q which is bounded above but does not have supremum *i.e.,* $\exists$ no rational number which is the least of all its upper bounds.

Let S be the set of all those positive rational numbers whose squares are less than 2 *i.e.,*

Let $\quad S = \{x : x \in Q^+ \text{ and } 0 < x^2 < 2\}$

Then $\quad S \neq \phi \quad [\because\; 1 \in S]$

S is bounded above, 2 being *an* upper bound.

$\Rightarrow$ S is a non-empty subset of Q and is bounded above.

Consider *any* rational number k. The following cases arise :

Case I. $k \le 0$. Since every element of S is > 0, k cannot be an upper bound of S and $\therefore$ k can't be l.u.b. of S.

Case II. $k > 0$ and $0 < k^2 < 2$.

Consider $$y = \frac{4+3k}{3+2k} \quad \ldots(i)$$

$$y^2 - 2 = \frac{16+24k+9k^2}{9+12k+4k^2} - 2 = \frac{k^2-2}{(3+2k)^2} \quad \ldots(ii)$$

$$y - k = \frac{4+3k}{3+2k} - k = \frac{2(2-k^2)}{(3+2k)} \quad \ldots(iii)$$

Since $k \in Q^+$, from (i) $\quad y \in Q^+$

$\because$ $k^2 - 2 < 0$, from (ii) $\quad y^2 - 2 < 0 \Rightarrow y^2 < 2$

from (iii) $\quad y - k > 0 \Rightarrow y > k$

$\because$ $\quad y \in Q^+$ and $y^2 < 2$ $\therefore$ $y \in S$.

Also, since $k < y$ and $y \in S$, k cannot be an upper bound of S.

Case III. *$k > 0$ and $k^2 = 2$.* Since $\exists$ no rational number whose square is 2, this possibility is ruled out.

Case IV. *$k > 0$ and $k^2 > 2$.*

Defining y as in Case II above, $y \in Q^+$, $y^2 > 2$, $y < k$ $\Rightarrow$ $2 < y^2 < k^2$

If x is *any* member of S, then $0 < x^2 < 2 < y^2 < k^2$ $\Rightarrow$ $0 < x < y < k$

which shows k and y are both upper bounds of S.

But $y < k$, $\therefore$ k cannot be the supremum.

Since k is any rational number, we conclude that no rational number can be the supremum of S.

Example 1. *Give an example of an ordered field which is not complete. Justify your answer.*

Sol. The set Q of rational number is an ordered field which is not complete. For justification, see 2.40.

Example 2. *Is every infinite subset of R which is an ordered field, complete also ? Justify your answer.*

Sol. Q, the set of rational numbers, is an infinite subset of R which is an ordered field but not complete. For justification, see 2.40.

Example 3. *Prove that the set $S = \{x : x \in Q^+ \text{ and } 0 < x^2 < 3\}$ does not have any l.u.b. in Q.*

Sol. $\quad S \neq \phi \quad [\because 1 \in S]$

S is bounded above, 3 being an upper bound.

$\Rightarrow$ S is a non-empty subset of Q and is bounded above.

Consider any rational number k. The following cases arise :

Case I. $k \le 0$. Since every element of S is positive, k cannot be an upper bound of S.

$\Rightarrow$ k cannot be the l.u.b. of S

Case II. $k > 0$ and $0 < k^2 < 3$

Consider $$y = \frac{3(k+1)}{k+3} \quad \ldots(i)$$

$$y^2 - 3 = \frac{9(k^2+2k+1)}{k^2+6k+9} - 3 = \frac{6(k^2-3)}{(k+3)^2} \quad \ldots(ii)$$

$$y - k = \frac{3(k+1)}{k+3} - k = \frac{3-k^2}{k+3} \quad \ldots(iii)$$

Since $k \in Q^+$ $\quad$ from (i) $\quad y \in Q^+$

$\because$ $k^2 - 3 < 0$ $\quad$ from (ii) $\quad y^2 - 3 < 0 \Rightarrow y^2 < 3$

from (iii) $\quad y - k > 0 \Rightarrow y > k$

Now $y \in Q^+$ and $y^2 < 3 \Rightarrow y \in S$

Also $k < y$ and $y \in S \Rightarrow k$ cannot be an upper bound of S.

Case III. $k > 0$ and $k^2 = 3$.

Since $\exists$ no rational number whose square is 3 (*i.e.*, $\sqrt{3} \notin Q$), this possibility is ruled out.

Case IV. $k > 0$ and $k^2 > 3$

Defining y as in Case II above,

$$y \in Q^+,\ y^2 > 3 \text{ and } y < k \Rightarrow 3 < y^2 < k^2$$

If x is any member of S, then

$$0 < x^2 < 3 < y^2 < k^2 \Rightarrow 0 < x < y < k$$

which shows k and y are both upper bounds of S.

But $\quad y < k$.

$\therefore$ k cannot be the l.u.b.

Since k is any rational number, we conclude that no rational number can be the l.u.b. of S.

Example 4. *Prove that the set $\{x : x \in Q^+ \text{ and } x^2 < 5\}$ does not have any l.u.b. in Q.*

Sol. *Please try yourself.*

$$\left[\textbf{Hint. } \text{Consider } y = \frac{5(k+1)}{k+5}, \text{ then } y^2 - 5 = \frac{20(k^2-5)}{(k+5)^2} \text{ and } y - k \frac{5-k^2}{k+5}\right]$$

2.33. THEOREM

Every non-empty set of real numbers which is bounded above has the least upper bound.

Proof. Let S be a non-empty set of real numbers which is bounded above. Let k be an upper bound of S.

Let $\quad T = \{-x : x \in S\}$.

Since S is a non-empty set of real numbers, so is T. Since k is an upper bound of S

$\therefore$ $\quad x \le k \quad \forall x \in S \Rightarrow -x \ge -k \quad \forall x \in S$

$\Rightarrow$ $\quad -k \le -x \quad \forall x \in S \Rightarrow -k \le y \quad \forall y \in T$

$\Rightarrow$ T is bounded below and $-k$ is a lower bound of T.

Since T is a non-empty set of real numbers which is bounded below, by completeness axiom, the set T has the greatest lower bound.

Let $\quad l = \text{g.l.b. } T$ and $u = -l$

then $\quad l \le y \quad \forall y \in T \Rightarrow -l \ge -y \quad \forall y \in T$

$\Rightarrow$ $\quad u \ge x \quad \forall x \in S \Rightarrow x \le u \quad \forall x \in S$

$\Rightarrow$ u is an upper bound of S.

Let u' be another upper bound of S, then

$$x \le u' \quad \forall\, x \in S \quad \Rightarrow \quad -x \ge -u' \quad \forall\, x \in S$$

$$\Rightarrow \quad y \ge -u' \;\forall\, y \in T \quad \Rightarrow \quad -u' \le y \quad \forall\, y \in T$$

$\Rightarrow$ $-u'$ is a lower bound of T.

Since l is the g.l.b. of T

$$\therefore \quad -u' \le l \quad \Rightarrow \quad u' \ge -l$$

$$\Rightarrow \quad u' \ge u \quad \Rightarrow \quad u \le u'$$

$\Rightarrow$ every upper bound of S, other than u, is greater than u.

$$\Rightarrow \quad u = \text{l.u.b. } S.$$

Hence S has the least upper bound.

2.34. THEOREM

Every non-empty set of real numbers which is bounded below has the greatest lower bound.

Proof. Let S be a non-empty set of real numbers which is bounded below. Let k be a lower bound of S.

Let $\quad T = \{-x : x \in S\}$

Since S is a non-empty set of real numbers, so is T.

Since k is a lower bound of S

$$\therefore \quad k \le x \quad \forall\, x \in S \quad \Rightarrow \quad -k \ge -x \quad \forall\, x \in S$$

$$\Rightarrow \quad -x \le -k \quad \forall\, x \in S \quad \Rightarrow \quad y \le -k \quad \forall\, y \in T.$$

$\Rightarrow$ T is bounded above and $-k$ is an upper bound of T.

Since T is a non-empty set of real numbers which is bounded above, by completeness axiom, the set T has the least upper bound.

Let $\quad u = \text{l.u.b. } T \quad \text{and} \quad l = -u$

then $\quad y \le u \quad \forall\, y \in T \quad \Rightarrow \quad -y \ge -u \quad \forall\, y \in T$

$$\Rightarrow \quad x \ge l \quad \forall\, x \in S \quad \Rightarrow \quad l \le x \quad \forall\, x \in S$$

$\Rightarrow$ l is a lower bound of S.

Let l' be another lower bound of S, then

$$l' \le x \quad \forall\, x \in S \quad \Rightarrow \quad -l' \ge -x \quad \forall\, x \in S$$

$$\Rightarrow \quad -l' \ge y \quad \forall\, y \in T \quad \Rightarrow \quad y \le -l' \quad \forall\, y \in T$$

$\Rightarrow$ $-l'$ is an upper bound of T.

Since u is the l.u.b. of T

$$\therefore \quad u \le -l' \quad \Rightarrow \quad -u \ge l' \quad \Rightarrow \quad l \ge l' \quad \Rightarrow \quad l' \le l$$

$\Rightarrow$ every lower bound of S, other than l, is smaller than l.

$$\Rightarrow \quad l = \text{g.l.b. } S$$

Hence S has the greatest lower bound.

2.35. ARCHIMEDEAN PROPERTY OF REAL NUMBERS

Statement. *If x is a positive real number and y is any real number, then there exists a positive integer n such that $nx > y$. i.e., $\forall\, x, y \in R$, $x > 0$, $\exists\, n \in N$ s.t. $nx > y$.*

Proof. Since y is any real number, by trichotomy law, $y > 0$ or $y = 0$ or $y < 0$

(*i*) If $y \le 0$, since $\quad x > 0, n > 0 \quad \Rightarrow \quad nx > 0$

$$\therefore \quad nx > 0 \ge y \quad \Rightarrow \quad nx > y$$

(*ii*) If $y > 0$, let us assume that the theorem is false.

Then $nx \le y$ **for every** $n \in N$ *i.e.*, the set $A = \{nx : n \in N\}$ is bounded above by y.

By the order completeness property, A has the l.u.b. say u.

Now $\quad nx \le u \quad$ **for every** $n \in N$

$$\Rightarrow \quad (n+1)x \le u \qquad [n \in N \;\Rightarrow n+1 \in N]$$

$$\Rightarrow \quad nx \le u - x \quad \text{for every } n \in N$$

$$\Rightarrow \quad nx \le u - x < u \quad \text{for every } n \in N$$

Thus $u - x$ which is strictly less than u is also an upper bound of A. *i.e.*, we have an upper bound of A which is less than the l.u.b. of A which contradicts the definition of l.u.b.

$\therefore$ Our supposition that $nx \le y$, $\forall\, n \in N$ is wrong. Hence $\exists$ an element $n \in N$ s.t. $nx > y$.

Cor. If x be any real number then there exists a positive integer n such that $n > x$.

Proof. Taking $y = x$ and $x = 1$ in the above theorem, we see that $\exists$ a positive integer n such that

$$n \,.\, 1 > x \quad i.e., \quad n > x.$$

2.36. ARCHIMEDEAN ORDERED FIELD

An **ordered field** is said to be Archimedean if it has the Archimedean Property.

2.37. THEOREM

Show that the set N of natural numbers is unbounded above.

Sol. By Archimedean Property in R, for each positive real number x, there exists $n \in N$ such that

$$n > x$$

$\Rightarrow$ There exists no positive real number x such that $\; n \le x \quad \forall\, n \in N$

$\Rightarrow$ No positive real number is an upper bound of N.

$\Rightarrow$ N is unbounded above.

(Another Proof)

Let, if possible, N be bounded above.

Also $\quad N \ne \phi \qquad [\because \; 1 \in N]$

Since N is a non-empty and bounded above subset of R, therefore, by completeness axiom of R, N has the l.u.b. u (say).

$$\Rightarrow \quad n \le u \quad \forall\, n \in N \quad \Rightarrow \quad n + 1 \le u \quad \forall\, n \in N$$

$$\Rightarrow \quad n \le u - 1 < u \quad \forall\, n \in N$$

Thus $u - 1$ which is strictly less than u is also an upper bound of N. This contradicts the fact that u is the l.u.b. of N.

Hence N is not bounded above.

Example 1. *For any $a \in R$, $a > 0$, show that there exists a natural number n such that $a > \dfrac{1}{n}$.*

Sol. Since a is a positive real number, by Archimedean property in R, $\exists\, n \in N$ s.t.

$$n\,a > 1 \quad (\text{Here } y = 1) \quad \Rightarrow \quad a > \frac{1}{n}.$$

Example 2. *If $0 \le a < \frac{1}{n}$ for every $n \in N$, show that $a = 0$.*

Sol. Suppose $a > 0$, then by Archimedean property in R, $\exists\, n \in N$ s.t. $na > 1$ i.e., $a > \frac{1}{n}$, which is a contradiction. $\left[\because\ a < \frac{1}{n}\ \forall\, n \in N\right]$

Hence $a = 0$.

Example 3. *If $0 \le a < \varepsilon$ for every $\varepsilon > 0$, show that $a = 0$.*

Sol. Suppose $a > 0$. Take $\varepsilon = \frac{a}{2} > 0$.

Now $\quad a > \frac{a}{2} \Rightarrow a > \varepsilon$, which is a contradiction. $[\because\ a < \varepsilon$ for every $\varepsilon > 0]$

Hence $\quad a = 0$.

2.38. THEOREM

For any real number x, there exists a unique integer m such that $m \le x < m + 1$.

Proof. Let $\quad A = \{n : n \in Z$ and $n \le x\}$

Then $A \ne \phi$ and A is bounded above by x.

$\therefore$ By order completeness property, A has the l.u.b. say m. x is an upper bound and m is the l.u.b. of A

$$\Rightarrow \quad m \le x \qquad \ldots(i)$$

To prove $x < m + 1$; let, if possible, $m + 1 \le x$

Also $\quad m + 1 \in Z \Rightarrow m + 1 \in A$ and $m < m + 1$

which contradicts that $\quad m = $ l.u.b. of A.

$$\therefore \quad m + 1 > x \quad i.e., \quad x < m + 1 \qquad \ldots(ii)$$

Combining (i) and (ii), $m \le x < m + 1$

Clearly m being the l.u.b. of A is unique. $[\because$ l.u.b. of a set, if it exists, is unique$]$

Cor. Putting $\quad m = n - 1,\ \ m \le x < m + 1 \Rightarrow n - 1 \le x < n$.

2.39. DENSENESS OF R

I. Between any two distinct real numbers, there is always a rational number and therefore infinitely many rational numbers.

Let the two real numbers be x and y with $x < y$ so that $y - x > 0$.

By Archimedean property in R, $\exists\, n \in N$ s.t.

$$n(y - x) > 1 \quad i.e., \quad ny > nx + 1. \qquad \ldots(i)$$

Also $\exists$ a unique integer m such that $m - 1 \le nx < m$ $\qquad \ldots(ii)$

$\therefore$ From (i), $\quad ny > nx + 1 \ge m > nx$ $\qquad$ [Using (ii)]

or $\quad nx < m < ny$ or $x < \frac{m}{n} < y$

$\therefore \frac{m}{n} (= r)$ is a rational number between x and y

Thus $\quad x < r < y$

Repeating the argument for x and r; r and y, we get rational numbers r_1 and r_2 s.t. $x < r_1 < r < r_2 < y$

Continuing like this, we get infinitely many rational numbers between x and y.

II. Between any two distinct real numbers, there is always an irrational number and therefore infinitely many irrational numbers.

Let the two real numbers be x and y with $x < y$ so that $y - x > 0$.

By Archimedean property in R, $\exists\, n \in N$ s.t.

$$n(y - x) > \sqrt{2} \quad \text{or} \quad y - x > \frac{\sqrt{2}}{n}$$

$$\therefore \quad y > x + \frac{\sqrt{2}}{n} > x + \frac{\sqrt{2}}{2n} > x \qquad \text{[Note]}$$

Since $\left(x + \frac{\sqrt{2}}{n}\right) - \left(x + \frac{\sqrt{2}}{2n}\right) = \frac{\sqrt{2}}{2n}$ is an irrational number.

$\therefore$ at least one of $x + \frac{\sqrt{2}}{n}$ and $x + \frac{\sqrt{2}}{2n}$ is irrational. Let us denote it by r. Then $x < r < y$.

Thus there is an irrational number between x and y.

Repeating the argument for x and r, r and y, we get irrational numbers r_1 and r_2 s.t. $x < r_1 < r < r_2 < y$.

Continuing like this, we get infinitely many irrational numbers between x and y.

Example 1. *Is the following statement true or false ? Justify your answer.*

Between any two distinct rational numbers, there lie infinitely many irrationals.

Sol. The given statement is true.

If x and y are two distinct rational numbers with $x < y$, then $Q \subset R \Rightarrow x$ and y are two distinct real numbers with $x < y$.

Now reproduce 2.47 II.

Example 2. *Prove that if a is a rational number and x is an irrational number, then ax (provided $a \ne 0$) and $a + x$ are irrational numbers.*

Sol. $\qquad a \in Q, x \in R - Q$

(i) $a \ne 0 \Rightarrow a^{-1}$ exists and $a^{-1} \in Q$.

Suppose $ax \in Q$. Then $a^{-1}(ax) = (a^{-1}a)x = 1 \cdot x = x \in Q$ which is a contradiction. Hence $ax \in R - Q$.

(ii) Suppose $\quad a + x \in Q$.

Since $\quad a \in Q, -a \in Q$

$\therefore \quad (-a) + (a + x) = (-a + a) + x = 0 + x = x \in Q$

which is a contradiction. Hence $a + x \in R - Q$.

Example 3. *If A and B are bounded subsets of R, then prove that the set $A + B = \{x + y : x \in A$ and $y \in B\}$ is also bounded and*

(i) Sup. $(A + B)$ = Sup. A + Sup. B

(ii) Inf. $(A + B)$ = Inf. A + Inf. B.

Sol. Since A and B are bounded, $\exists$ positive real numbers k_1 and k_2 such that

$$|x| \le k_1 \quad \forall\, x \in A \quad \text{and} \quad |y| \le k_2 \quad \forall\, y \in B$$

Let $z \in A + B$, then $z = x + y$ where $x \in A, y \in B$

$$|z| = |x+y| \le |x| + |y| \le k_1 + k_2 \quad \forall z \in A+B$$

$\Rightarrow$ A + B is bounded and so it has supremum and infimum.

(*i*) Let Sup. A $= u_1$, Sup. B $= u_2$ and Sup. (A + B) $= u$

Then $x \le u_1 \quad \forall x \in A$

$y \le u_2 \quad \forall y \in B$

$\Rightarrow \quad x + y \le u_1 + u_2 \quad \forall x \in A, y \in B$

$\Rightarrow \quad z \le u_1 + u_2 \quad \forall z \in A + B$

$\Rightarrow \quad u_1 + u_2$ is an upper bound of A + B

Since u is the l.u.b.

$\therefore \quad u \le u_1 + u_2$ ...(1)

Also, for any $\varepsilon > 0$, $\exists\, x_1 \in A$ and $y_1 \in B$ such that

$$x_1 > u_1 - \frac{\varepsilon}{2} \text{ and } y_1 > u_2 - \frac{\varepsilon}{2} \Rightarrow x_1 + y_1 > u_1 + u_2 - \varepsilon$$

$\Rightarrow \quad \exists\, z_1 = x_1 + y_1 \in A + B$ such that $z_1 > u_1 + u_2 - \varepsilon$

$\Rightarrow$ Any number $< u_1 + u_2$ cannot be an upper bound of A + B

$\Rightarrow$ Every upper bound of A + B is $\ge u_1 + u_2$

$\therefore \quad u \ge u_1 + u_2$ ...(2)

From (1) and (2), $u = u_1 + u_2$

i.e., Sup. (A + B) = Sup. A + Sup. B.

(*ii*) Let Inf. A $= l_1$, Inf. B $= l_2$ and Inf. (A + B) $= l$

Then $x \ge l_1 \quad \forall x \in A$

$y \ge l_2 \quad \forall y \in B$

$\Rightarrow \quad x + y \ge l_1 + l_2 \quad \forall x \in A, y \in B$

$\Rightarrow \quad z \ge l_1 + l_2 \quad \forall z \in A + B$

$\Rightarrow \quad l_1 + l_2$ is a lower bound of A + B

Since l is the g.l.b.

$\therefore \quad l \ge l_1 + l_2$ ...(1)

Also, for any $\varepsilon > 0$, $\exists\, x_1 \in A$ and $y_1 \in B$ such that

$$x_1 < l_1 + \frac{\varepsilon}{2} \text{ and } y_1 < l_2 + \frac{\varepsilon}{2} \Rightarrow x_1 + y_1 < l_1 + l_2 + \varepsilon$$

$\Rightarrow \quad \exists\, z_1 = x_1 + y_1 \in A + B$ such that $z_1 < l_1 + l_2 + \varepsilon$.

$\Rightarrow$ Any number $> l_1 + l_2$ cannot be a lower bound of A + B

$\Rightarrow$ Every lower bound of A + B is $\le l_1 + l_2$

$\therefore \quad l \le l_1 + l_2$ ...(2)

From (1) and (2), $l = l_1 + l_2$

i.e., Inf. (A + B) = Inf. A + Inf. B.

Example 4. *For a real number λ, let λA denote the set $\lambda A = \{\lambda x : x \in A\}$ Prove that if A is a bounded subset of R, then λA is also bounded and*

(*i*) *Sup. $\lambda A = \lambda$ Sup. A if $\lambda > 0$* (*ii*) *Sup. $\lambda A = \lambda$ Inf. A if $\lambda < 0$*

(*iii*) *Inf. $\lambda A = \lambda$ Inf. A if $\lambda > 0$*

(*iv*) *Inf. $\lambda A = \lambda$ Sup. A if $\lambda < 0$.*

Sol. Since A is bounded, $\exists$ a positive real number k such that $|x| \le k \quad \forall x \in A$

Now $\quad |\lambda x| = |\lambda|\,|x| \le |\lambda|\,k \quad \forall \lambda x \in \lambda A$

$\Rightarrow \quad \lambda A$ is bounded and so it has supremum and infimum.

(*i*) Let Sup. A $= u$, then $x \le u \quad \forall x \in A$

$\Rightarrow \quad \lambda x \le \lambda u \quad \forall \lambda x \in \lambda A, \lambda > 0$

$\Rightarrow \quad \lambda u$ is an upper bound of λA

$\Rightarrow \quad$ Sup. $\lambda A \le \lambda u$ ...(1)

Also, for any $\varepsilon > 0$, we have $\frac{\varepsilon}{\lambda} > 0$ $\quad [\because \lambda > 0]$

$\Rightarrow \quad u - \frac{\varepsilon}{\lambda} < u$ is not an upper bound of A

$\Rightarrow \quad \exists\, x_1 \in A$ such that $x_1 > u - \frac{\varepsilon}{\lambda}$

$\Rightarrow \quad \lambda x_1 > \lambda u - \varepsilon$, where $\lambda x_1 \in \lambda A, \lambda > 0$

$\Rightarrow$ Any number $< \lambda u$ cannot be an upper bound of λA

$\Rightarrow \quad$ Sup. $\lambda A \ge \lambda u$ ...(2)

From (1) and (2), Sup. $\lambda A = \lambda u$

i.e., Sup. $\lambda A = \lambda$ Sup. A for $\lambda > 0$

(*ii*) Let Ind. A $= l$, then $x \ge l \quad \forall x \in A$

$\Rightarrow \quad \lambda x \le \lambda l \quad \forall \lambda x \in \lambda A, \lambda < 0$

$\Rightarrow \quad \lambda l$ is in upper bound of λA

$\Rightarrow \quad$ Sup. $\lambda A \le \lambda l$ ...(1)

Also, for any $\varepsilon > 0$, we have $\frac{\varepsilon}{\lambda} < 0$ $\quad [\because \lambda < 0]$

$\Rightarrow \quad l - \frac{\varepsilon}{\lambda} > l$ is not a lower bound of A

$\Rightarrow \quad \exists\, x_1 \in A$ such that $x_1 < l - \frac{\varepsilon}{\lambda}$

$\Rightarrow \quad \lambda x_1 > \lambda l - \varepsilon$, where $\lambda x_1 \in \lambda A, \lambda < 0$

$\Rightarrow$ Any number $< \lambda l$ cannot be an upper bound of λA

$\Rightarrow \quad$ Sup. $\lambda A \ge \lambda l$ ...(2)

From (1) and (2), Sup. $\lambda A = \lambda l$

i.e., Sup. $\lambda A = \lambda$ Inf. A for $\lambda < 0$.

(*iii*) Let Inf. A $= l$, then $x \ge l \quad \forall x \in A$

$\Rightarrow \quad \lambda x \ge \lambda l \quad \forall \lambda x \in \lambda A, \lambda > 0$

$\Rightarrow \quad \lambda l$ is a lower bound of λA

$\Rightarrow \quad$ Inf. $\lambda A \ge \lambda l$ ...(1)

Also, for any $\varepsilon > 0$, we have $\frac{\varepsilon}{\lambda} > 0$ $\quad [\because \lambda > 0]$

$\Rightarrow \quad l + \frac{\varepsilon}{\lambda} > l$ is not a lower bound of A

$\Rightarrow\ \exists\, x_1 \in A$ such that $x_1 < l + \dfrac{\varepsilon}{\lambda}$

$\Rightarrow\quad \lambda x_1 < \lambda l + \varepsilon,$ where $\lambda x_1 \in \lambda A, \lambda > 0$

$\Rightarrow$ Any number $> \lambda l$ cannot be a lower bound of λA

$\Rightarrow\quad \text{Inf. } \lambda A \le \lambda l$ ...(2)

From (1) and (2), $\text{Inf. } \lambda A = \lambda l$

i.e., $\text{Inf. } \lambda A = \lambda \text{ Inf. A for } \lambda > 0.$

(*iv*) Let Sup. $A = u$, then $x \le u \quad \forall\, x \in A$

$\Rightarrow\quad \lambda x \ge \lambda u \,\forall\, \lambda x \in \lambda A, \lambda < 0$

$\Rightarrow\ \lambda u$ is a lower bound of λA

$\Rightarrow\quad \text{Inf. } \lambda A \ge \lambda u$ ...(1)

Also, for any $\varepsilon > 0$, we have $\dfrac{\varepsilon}{\lambda} < 0$ $[\because\ \lambda < 0]$

$\Rightarrow\quad u + \dfrac{\varepsilon}{\lambda} < u$ is not an upper bound of A

$\Rightarrow\ \exists\, x_1 \in A$ such that $x_1 > u + \dfrac{\varepsilon}{\lambda}$

$\Rightarrow\quad \lambda x_1 < \lambda u + \varepsilon,$ where $\lambda x_1 \in \lambda A, \lambda < 0$

$\Rightarrow$ Any number $> \lambda u$ cannot be a lower bound of λA

$\Rightarrow\quad \text{Inf. } \lambda A \le \lambda u$ ...(2)

From (1) and (2), $\text{Inf. } \lambda A = \lambda u$

i.e., $\text{Inf. } \lambda A = \lambda \text{ Sup. A for } \lambda < 0.$

Example 5. *Prove that*

(*i*) *l.u.b.* $(-A) = -$ *g.l.b. A* (*ii*) *g.l.b.* $(-A) = -$ *l.u.b. A.*

Sol. (*i*) Let g.l.b. $A = l$, then $x \ge l \quad \forall\ x \in A$

$\Rightarrow\quad -x \le -l \quad \forall -x \in -A$

$\Rightarrow\ -l$ is an upper bound of $-A$

$\Rightarrow\quad \text{l.u.b. } (-A) \le -l$ ...(1)

Also, for $\varepsilon > 0$, $l + \varepsilon > l$ is not a lower bound of A

$\Rightarrow\ \exists\, x_1 \in A$ such that $x_1 < l + \varepsilon \Rightarrow -x_1 > -l - \varepsilon$, where $-x_1 \in -A$

$\Rightarrow$ Any number $< -l$ is not an upper bound of $-A$

$\Rightarrow\quad \text{l.u.b. } (-A) \ge -l$ ...(2)

From (1) and (2), l.u.b. $(-A) = -l$ *i.e.*, l.u.b. $(-A) = -$ g.l.b. A.

(*ii*) *Please try yourself.*

Example 6. *If A and B are two non-empty bounded subsets of R, and* $A - B = \{a - b : a \in A, b \in B\}$, *then*

(*i*) *Sup.* $(A - B) =$ *Sup. A – Inf. B* (*ii*) *Inf.* $(A - B) =$ *Inf. A – Sup. B.*

Sol. Since A and B are bounded, $\exists$ positive real numbers k_1 and k_2 such that

$$|\,a\,| \le k_1 \quad \forall\, a \in A$$

and

$$|\,b\,| \le k_2 \quad \forall\, b \in B$$

Let $c \in A - B$, then $c = a - b$, where $a \in A, b \in B$

$$|\,c\,| = |\,a - b\,| \le |\,a\,| + |\,b\,| \le k_1 + k_2 \quad \forall\, c \in A - B$$

$\Rightarrow$ A – B is bounded and so it has supremum and infimum.

(*i*) Let Sup. $A = u$ and Inf. $B = l$

Then $a \le u \quad \forall\, a \in A$

$b \ge l \quad \forall\, b \in B$ or $-b \le -l$

$\therefore\quad a - b \le u - l \quad \forall\, a \in A, b \in B$

$\Rightarrow\quad c \le u - l \quad \forall\, c \in A - B$

$\Rightarrow\ u - l$ is an upper bound of A – B

$\Rightarrow\quad \text{Sup. } (A - B) \le u - l$ ...(1)

Also, for any $\varepsilon > 0$, $\exists\, a_1 \in A$ and $b_1 \in B$ such that

$$a_1 > u - \frac{\varepsilon}{2} \quad \text{and} \quad b_1 < l + \frac{\varepsilon}{2} \quad \text{or} \quad -b_1 > -l - \frac{\varepsilon}{2}$$

$\therefore\quad a_1 - b_1 > (u - l) - \varepsilon$

$\Rightarrow\quad \exists\, c_1 = a_1 - b_1 \in A - B$ such that $c_1 > (u - l) - \varepsilon$

$\Rightarrow$ Any number $< u - l$ cannot be an upper bound of A – B

$\Rightarrow$ Every upper bound of A – B is $\ge u - l$

$\therefore\quad \text{Sup. } (A - B) \ge u - l$ ...(2)

From (1) and (2), Sup. $(A - B) = u - l$ *i.e.*, Sup. $(A - B) =$ Sup. A – Inf. B.

(*ii*) Let Inf. $A = l$ and Sup. $B = u$

Then $a \ge l \quad \forall\, a \in A$

$b \le u \quad \forall\, b \in B$ or $-b \ge -u$

$\therefore\quad a - b \ge l - u \quad \forall\, a \in A, b \in B$

$\Rightarrow\quad c \ge l - u \quad \forall\, c \in A - B$

$\Rightarrow\ l - u$ is a lower bound of A – B

$\Rightarrow\quad \text{Inf. } (A - B) \ge l - u$ ...(1)

Also, for any $\varepsilon > 0$, $\exists\, a_1 \in A$ and $b_1 \in B$ such that

$$a_1 < l + \frac{\varepsilon}{2} \quad \text{and} \quad b_1 > u - \frac{\varepsilon}{2} \quad \text{or} \quad -b_1 < -u + \frac{\varepsilon}{2}$$

$\therefore\quad a_1 - b_1 < (l - u) + \varepsilon$

$\Rightarrow\quad \exists\, c_1 = a_1 - b_1 \in A - B$ such that $c_1 < (l - u) + \varepsilon$

$\Rightarrow$ Any number $> l - u$ cannot be a lower bound of A – B

$\Rightarrow$ Every lower bound of A – B is $\le l - u$

$\therefore\quad \text{Inf. } (A - B) \le l - u$ ...(2)

From (1) and (2), Inf. $(A - B) = l - u$ or Inf. $(A - B) =$ Inf. A – Sup. B.

Example 7. *If A and B are two bounded non-empty subsets of R, then* $A \cup B$ *is also bounded and*

(*i*) *Sup.* $(A \cup B) =$ *max.* {*Sup. A, Sup. B*} (*ii*) *Inf.* $(A \cup B) =$ *min.* {*Inf. A, Inf. B*}.

Sol. Let Sup. $A = u_1$, Sup. $B = u_2$

Inf. $A = l_1$, Inf. $B = l_2$

Let $u = \max. \{u_1, u_2\}$ and $l = \min. \{l_1, l_2\}$

Then $x \le u_1 \le u \quad \forall\, x \in A$

and $x \le u_2 \le u \quad \forall\, x \in B$

$\therefore\quad x \le u \quad \forall\, x \in A \cup B$

$\Rightarrow\ A \cup B$ is bounded above and u is an upper bound of $A \cup B$ ...(1)

Also $\quad x \geq l_1 \geq l \quad \forall\, x \in A$

and $\quad x \geq l_2 \geq l \quad \forall\, x \in B$

$\therefore \quad x \geq l \quad \forall\, x \in A \cup B$

$\Rightarrow$ A $\cup$ B is bounded below and l is a lower bound of A $\cup$ B. ...(2)

From (1) and (2), A $\cup$ B is bounded and so it has supremum and infimum.

(*i*) For any $\varepsilon > 0$, $u - \varepsilon < u$

$\therefore$ either $u - \varepsilon < u_1$ or $u - \varepsilon < u_2$ $\quad [\because\ u = \max.\{u_1, u_2\}]$

If $u - \varepsilon < u_1$, then $\exists\, x \in$ A and hence $\in$ A $\cup$ B such that $x > u - \varepsilon$

If $u - \varepsilon < u_2$, then $\exists\, x \in$ B and hence $\in$ A $\cup$ B such that $x > u - \varepsilon$

Thus, for $\varepsilon > 0$, $\exists\, x \in A \cup B$ such that $x > u - \varepsilon$ where u is an upper bound of A $\cup$ B.

$\therefore$ Sup. (A $\cup$ B) $= u = \max.\{u_1, u_2\}$

i.e., Sup. (A $\cup$ B) = max. {Sup. A, Sup. B}

(*ii*) For any $\varepsilon > 0$, $l + \varepsilon > l$

$\therefore$ either $l + \varepsilon > l_1$ or $l + \varepsilon > l_2$ $\quad [\because\ l = \min.\{l_1, l_2\}]$

If $l + \varepsilon > l_1$, then $\exists\, x \in$ A and hence $\in$ A $\cup$ B such that $x < l + \varepsilon$

If $l + \varepsilon > l_2$, then $\exists\, x \in$ B and hence $\in$ A $\cup$ B such that $x < l + \varepsilon$

Thus, for $\varepsilon > 0$, $\exists\, x \in A \cup B$ such that $x < l + \varepsilon$ where l is a lower bound of A $\cup$ B.

$\therefore$ Inf. (A $\cup$ B) $= l = \min.\{l_1, l_2\}$

i.e., Inf. (A $\cup$ B) = min. {Inf. A, Inf. B}.

3

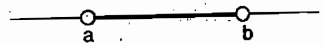
Topology of Real Numbers

3.1. INTERVALS

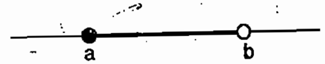
I. Finite Intervals. Let a and b be two real numbers with $a < b$.

(*i*) The set $\{x : x \in R, a \leq x \leq b\}$ is called a *closed interval* and is denoted by $[a, b]$, a and b are called the *end points* of the interval. a is called the *left end point* while b is called the *right end point*. Thus, in a closed interval, both the end points belong to the interval.

(*ii*) The set $\{x : x \in R, a < x < b\}$ is called an *open interval* and is denoted by (a, b) or $]a, b[$. In an open interval, the end points do not belong to the interval.



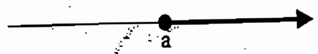
(*iii*) The set $\{x : x \in R, a \leq x < b\}$ is called left-half closed interval (or right-half open interval) and is denoted by $[a, b)$. Left end point belongs to interval while right end point does not belong to the interval.



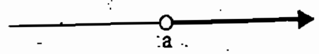
(*iv*) The set $\{x : x \in R, a < x \leq b\}$ is called right-half closed interval (or left-half open interval) and is denoted by $(a, b]$. Left end point does not belong to the interval while right end point belongs to the interval.

II. Infinite Intervals. Let a be any real number.

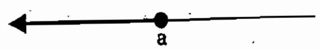
(*v*) The set $\{x : x \in R, x \geq a\}$ is called closed right ray and is denoted by $[a, \infty)$.



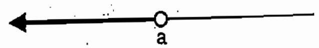
(*vi*) The set $\{x : x \in R, x > a\}$ is called open right ray and is denoted by (a, ∞).



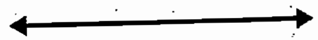
(*vii*) The set $\{x : x \in R, x \leq a\}$ is called closed left ray and is denoted by $(-\infty, a]$.



(*viii*) The set $\{x : x \in R, x < a\}$ is called open left ray and is denoted by $(-\infty, a)$.



(*ix*) The set $\{x : x \in R\}$ is also an interval and has no end points. It is denoted by $(-\infty, \infty)$.



3.2. LENGTH OF AN INTERVAL

For each interval with end points a and b $(a < b)$, $b - a$ is called the length of the interval.

Clearly, each of the intervals $[a, b]$, (a, b), $[a, b)$ and $(a, b]$ has the same length $b - a$.

3.3. FINITE AND INFINITE INTERVALS

An interval is said to be finite or infinite according as its length is finite or infinite. Thus, the intervals $[a, b]$, (a, b), $[a, b)$, $(a, b]$ are finite because $b - a$ is finite. The intervals $[a, \infty)$, (a, ∞), $(-\infty, a]$, $(-\infty, a)$, $(-\infty, \infty)$ are infinite.

Note 1. *Every interval is an infinite set* but every infinite set need not be an interval.

Note 2. A finite interval is also an infinite set because the word finite only signifies that the length of the interval is finite.

Note 3. A ray is an infinite interval.

3.4. NEIGHBOURHOOD OF A POINT

A subset N of R is called a neighbourhood of a point p if there exists an **open interval** I containing p and itself contained in N.

i.e., a subset N of R is called a neighbourhood of a point p if $\exists$ an open interval I s.t. $p \in I \subset N$.

Remark. Since an open interval containing p can be taken of the form $(p - \varepsilon, p + \varepsilon)$ where $\varepsilon > 0$, **neighbourhood of p can alternatively be defined as :**

A subset N of R is called a neighbourhood of a point p if $\exists\ \varepsilon > 0$ s.t. $(p - \varepsilon, p + \varepsilon) \subset N$.

This neighbourhood of p is called a **symmetric neighbourhood of p** with radius ε.

3.5. DELETED NEIGHBOURHOOD OF A POINT

If from the neighbourhood of a point, the point itself is excluded, we get the deleted neighbourhood of that point.

Thus, if N is a neighbourhood of a point p, then the set $N - \{p\}$ is a deleted neighbourhood of p.

Note 1. Deleted symmetric neighbourhood of a point p with radius ε is

$$(p - \varepsilon, p + \varepsilon) - \{p\} = (p - \varepsilon, p) \cup (p, p + \varepsilon) = \{x : 0 < | x - p | < \varepsilon\}$$

Note 2. **Neighbourhood is shortly written as nbd.**

Illustrations. (*i*) **An open interval is a nbd of each of its points.**

Let x be any point of the open interval (a, b). Then

$$x \in (a, b) \subset (a, b) \qquad [\because \text{ every set is a subset of itself}]$$

$\Rightarrow$ (a, b) is a *nbd* of x

But x is any point of (a, b)

$\therefore$ (a, b) is a *nbd* of each of its points.

(*ii*) **A closed interval is a nbd of each of its points except the two end points.**

Let x be any point of the open interval (a, b). Then $x \in (a, b) \subset [a, b]$

$\Rightarrow$ $[a, b]$ is *nbd* of x.

But $[a, b]$ is not a *nbd* of a. To be a *nbd* of a, it must contain an open interval $(a - \varepsilon, a + \varepsilon)$ *i.e.*, it has to contain points less than a, which it does not. Similarly $[a, b]$ is not a *nbd* of b. To be a *nbd* of b, it must contain an open interval $(b - \varepsilon, b + \varepsilon)$ *i.e.*, it has to contain points greater than b, which it does not.

Thus, $[a, b]$ is a *nbd* of each point of (a, b) but not a *nbd* of a or b.

Note. $[a, b)$ is a *nbd* of each of its points except a while $(a, b]$ is a *nbd* of each of its points except b.

(*iii*) **A non-empty finite set is not a nbd of any point.**

Let A be a non-empty finite set. A will be a *nbd* of a point x if $\exists$ an open interval I s.t. $x \in I \subset A$. But every interval is an infinite set, so is I. $\therefore$ $I \subset A$ only when A is infinite.

Since A is given to be finite, A cannot be a *nbd* of any point.

Note. Each *nbd* of a point is an infinite set but every infinite set need not be a *nbd*.

(*iv*) **The set N of natural numbers is not a nbd of any of its points.**

Let n be any natural number. For N to be a *nbd* of n, $\exists\ \varepsilon > 0$ s.t. $(n - \varepsilon, n + \varepsilon)$ is a subset of N. But $(n - \varepsilon, n + \varepsilon)$ contains infinitely many points which are not in N.

$\therefore$ N is not a *nbd* of any of its points.

Similarly the set Z of integers is not a *nbd* of any of its points. The set Q of rationals is not a *nbd* of any of its points.

(*v*) **The set R of real numbers is a nbd of each of its points.**

3.6. THEOREMS ON NEIGHBOURHOOD

Theorem I. *Any superset of the nbd of a point is also a nbd of that point.*

Proof. Let N be a *nbd* of a point x.

Then $\exists\ \varepsilon > 0$ s.t. $(x - \varepsilon, x + \varepsilon) \subset N$.

If M is a superset of N, then $M \supset N$ or $N \subset M$

$\therefore$ $(x - \varepsilon, x + \varepsilon) \subset N \subset M \Rightarrow (x - \varepsilon, x + \varepsilon) \subset M$

$\Rightarrow$ M is also a *nbd* of x.

Theorem II. *The intersection of two nbds of a point is also a nbd of that point.*

Proof. Let N_1 and N_2 be two *nbds* of a point x.

Then $\exists\ \varepsilon_1 > 0$ and $\varepsilon_2 > 0$ s.t. $(x - \varepsilon_1, x + \varepsilon_1) \subset N_1$ and $(x - \varepsilon_2, x + \varepsilon_2) \subset N_2$.

Let $\varepsilon = \min. \{\varepsilon_1, \varepsilon_2\}$. Then $(x - \varepsilon, x + \varepsilon) \subset (x - \varepsilon_1, x + \varepsilon_1) \subset N_1$

and $(x - \varepsilon, x + \varepsilon) \subset (x - \varepsilon_2, x + \varepsilon_2) \subset N_2$

$\Rightarrow$ $(x - \varepsilon, x + \varepsilon) \subset N_1 \cap N_2$

$\Rightarrow$ $N_1 \cap N_2$ is also a *nbd* of x.

Theorem III. *A set A is a nbd of x iff there exists a positive interger n such that*

$$\left(x - \frac{1}{n}, x + \frac{1}{n}\right) \subset A.$$

Proof. Part I. Let A be a *nbd* of x. Then $\exists\ \varepsilon > 0$ s.t. $(x - \varepsilon, x + \varepsilon) \subset A$ ...(*i*)

$$\because \quad \varepsilon > 0, \varepsilon^{-1} = \frac{1}{\varepsilon} \text{ exists and } \frac{1}{\varepsilon} \in R.$$

Since it is always possible to find a +ve integer greater than any given real number (Archimedean Property), we can find a +ve integer n such that

$$n > \frac{1}{\varepsilon} \quad i.e., \quad \frac{1}{n} < \varepsilon \quad \text{and} \quad -\varepsilon < -\frac{1}{n}$$

$$\therefore \quad x + \frac{1}{n} < x + \varepsilon \quad \text{and} \quad x - \varepsilon < x - \frac{1}{n}$$

$$\Rightarrow \quad \left(x - \frac{1}{n}, x + \frac{1}{n}\right) \subset (x - \varepsilon, x + \varepsilon) \subset A \qquad [\text{from } (i)]$$

$$\Rightarrow \quad \exists \text{ a +ve integer } n \text{ s.t.} \left(x - \frac{1}{n}, x + \frac{1}{n}\right) \subset A.$$

Part II. (Converse) Let $\left(x-\frac{1}{n}, x+\frac{1}{n}\right) \subset A$ ...(*ii*)

where n is a +ve integer.

Since $\left(x-\frac{1}{n}, x+\frac{1}{n}\right)$ is an open interval containing x, it follows from (*ii*) that A is a *nbd* of x.

Example 1. *Give an example of each of the following :*

(*i*) *a set which is a nbd of each of its points.*

(*ii*) *a set which is not a nbd of any of its points.*

(*iii*) *a set which is a nbd of each of its points with the exception of one point.*

(*iv*) *a set which is a nbd of each of its points with the exception of two points.*

(*v*) *a set which is a nbd of all its points except n points* ($n \geq 1$). *Can this set be an interval ?*

Sol. (*i*) The open interval (2, 3).

(*ii*) The finite set {1, 2, 3}.

(*iii*) [2, 3) is a *nbd* of each of its points except 2.

(*iv*) [2, 3] is a *nbd* of each of its points except 2 and 3.

(*v*) The set $(0, 1) \cup \{1, 2, 3, ..., n\}$ is a *nbd* of all its points except n points 1, 2, 3, ..., n.

If $n = 1$, the set can be an interval. See part (*iii*).

If $n = 2$, the set can be an interval. See part (*iv*).

If $n > 2$, the set cannot be an interval.

Example 2. *Show that the intersection of the family of all neighbourhoods of an arbitrary point* $x \in R$ *is* $\{x\}$.

Sol. Let S denote the intersection of the family of all neighbourhoods N(x) of $x \in$ R, then $S \subset N(x)$.

If y be a point different from x and $\varepsilon = |x - y|$, then $(x - \varepsilon, x + \varepsilon)$ is a *nbd* of x that does not contain y.

$$y \notin (x-\varepsilon, x+\varepsilon) \Rightarrow y \notin N(x) \Rightarrow y \notin S.$$

Thus S does not contain any point different from x.

Hence $S = \{x\}$.

3.7. OPEN SET

Let A be a subset of R, then A is said to be open if it is a *nbd* of each of its points.

Equivalently, A is open if for each $x \in A$, $\exists\, \varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subset A$.

Remark. A is not open $\Rightarrow$ $\exists$ at least one point of A of which A is not a *nbd*, *i.e.*, $\exists$ some $x \in A$ such that for each $\varepsilon > 0$, however small, $(x - \varepsilon, x + \varepsilon) \not\subset A$.

For example. (*i*) **Every open interval (a, b) is an open set** because it is a *nbd* of each of its points.

(*ii*) The set R is open.

(*iii*) A non-empty finite set cannot be open.

(*iv*) None of the intervals $[a, b]$, $[a, b)$, $(a, b]$ is an open set. $[a, b]$ or $[a, b)$ is not a *nbd* of a. $[a, b]$ or $(a, b]$ is not a *nbd* of b.

(*v*) The empty set ϕ is open $\because$ there is no point in ϕ of which it is not a *nbd*.

(*vi*) The sets N, Z, Q are not open sets.

3.8. THEOREMS ON OPEN SETS

Theorem I. *The union of two open sets is an open set.*

Proof. Let A and B be two open sets. Let x be *any* element of $S = A \cup B$.

$$x \in S \Rightarrow x \in A \cup B \Rightarrow x \in A \text{ or } x \in B$$

$\Rightarrow$ A is a *nbd* of x or B is a *nbd* of x. ($\because$ Both A and B are open sets)

But $A \subset A \cup B$, also $B \subset A \cup B$

$\therefore$ $A \cup B = S$ is a *nbd* of x.

[$\because$ any super set of the *nbd* of a point is also a *nbd* of that point.]

Since x is any point of S, it follows that S is a *nbd* of each of its points.

Hence S is an open set.

Theorem II. *The union of an arbitrary family of open sets is an open set.*

Proof. Let $\{A_\lambda\}_{\lambda \in \Lambda}$ be an arbitrary family of open sets.

Let x be any element of $S = \bigcup_{\lambda \in \Lambda} A_\lambda$.

$$x \in S \Rightarrow x \in \bigcup_{\lambda \in \Lambda} A_\lambda \Rightarrow x \in A_\lambda \text{ for at least one } \lambda \in \Lambda$$

$\Rightarrow$ at least one A_λ is a *nbd* of x. [$\because$ each A_λ is open]

But $A_\lambda \subset S$ for all $\lambda \in \Lambda$.

$\therefore$ S is a *nbd* of x.

$\Rightarrow$ S is a *nbd* of each of its points. ($\because$ x is arbitrary)

Hence S is an open set.

Theorem III. *The intersection of two open sets is an open set.*

Proof. Let A and B be two open sets and $S = A \cap B$.

If $S = \phi$, then S is open because ϕ is open.

If $S \neq \phi$, let x be any element of S.

$$x \in S \Rightarrow x \in A \cap B \Rightarrow x \in A \text{ and } x \in B$$

$\Rightarrow$ A and B are both *nbds* of x [$\because$ A and B are open sets]

$\Rightarrow$ $A \cap B$ is a *nbd* of x.

$\Rightarrow$ S is a *nbd* of x.

Since x is any point of S, it follows that S is *nbd* of each of its points. Hence S is an open set.

Theorem IV. *The intersection of a finite number of open sets is an open set.*

Proof. Let $A_1, A_2, A_3, ..., A_n$ be n open sets and $S = \bigcap_{i=1}^{n} A_i$

If $S = \phi$, then S is open because ϕ is open.

If $S \neq \phi$, let x be any element of S.

$$x \in S \Rightarrow x \in \bigcap_{i=1}^{n} A_i \Rightarrow x \in A_i \text{ for each } i = 1, 2, 3, ..., n.$$

But A_i is an open set for each $i = 1, 2, 3, ..., n$.

$\Rightarrow$ Each A_i is a *nbd* of x.

$\Rightarrow$ For each i, $\exists\ \varepsilon_i > 0$ s.t. $(x - \varepsilon_i, x + \varepsilon_i) \subset A_i$

Let $\varepsilon = \min. \{\varepsilon_1, \varepsilon_2, \ldots\ldots, \varepsilon_n\}$, then $(x - \varepsilon, x + \varepsilon) \subset (x - \varepsilon_i, x + \varepsilon_i) \subset A_i$ *for each i*

$\Rightarrow$ $(x - \varepsilon, x + \varepsilon) \subset \bigcap_{i=1}^{n} A_i$ $\Rightarrow$ $(x - \varepsilon, x + \varepsilon) \subset S$

$\Rightarrow$ S is a *nbd* of x $\Rightarrow$ S is an open set. ($\because$ x is arbitrary)

Remark. The intersection of an infinite family of open sets may or may not be an open set as is clear from the following examples.

(*i*) Let $I_n = (0, n)$, $n \in N$. Then $\{I_n\}_{n \in N}$ is an infinite family of open sets.

$\bigcap_{n \in N} I_n = (0, 1)$ which is an open set.

(*ii*) Let $I_n = \left(-\dfrac{1}{n}, \dfrac{1}{n}\right)$, $n \in N$. Then $\{I_n\}_{n \in N}$ is an infinite family of open sets.

$\bigcap_{n \in N} I_n = \{0\}$, which being a non-empty finite set is not an open set.

Example 1. *Give an example of*

(*i*) *an open set which is not an interval.*

(*ii*) *an interval which is an open set.*

(*iii*) *an interval which is not an open set.*

(*iv*) *a set which is neither an interval nor an open set.*

Sol. (*i*) (1, 2) and (3, 4) are open sets.

$\Rightarrow$ $(1, 2) \cup (3, 4)$ is an open set but it is not an interval.

(*ii*) The open interval (1, 2) is an open set. (In fact, every open interval is an open set).

(*iii*) The closed interval [1, 2] is not an open set because it is not a *nbd* of 1 and 2 which belong to it.

(*iv*) The set N of all natural numbers is neither an interval nor an open set.

Example 2. *Is the following statement true or false ? Justify your answer. The set of rational numbers is open.*

Sol. The set Q of rational numbers is not open and the given statement is false.

Justification. Let x be any element of Q.

Q will be open if $\exists\ \varepsilon > 0$ s.t. $(x - \varepsilon, x + \varepsilon) \subset Q$.

But $(x - \varepsilon, x + \varepsilon)$ contains infinitely many (irrational) numbers which do not belong to Q.

$\therefore$ Q is not a *nbd* of x *i.e.*, Q is not a *nbd* of any of its points. Hence Q is not open.

Example 3. (*i*) *Is a super set of an open set open ?*

(*ii*) *Is a subset of an open set open ?*

Sol. (*i*) No. A super set of an open set need not be open.

For example, $[1, 2] \supset (1, 2)$.

(1, 2) is an open set whereas [1, 2] is not an open set.

(*ii*) No. A subset of an open set need not be open.

For example, $[1, 2] \subset$ ~~(0, 3)~~

(0, 3) is an open set whereas [1, 2] is not an open set.

Example 4. *Answer in Yes or No, justifying your answer in each case either by a proof or by counter example.*

(*i*) *Can a finite set be open ?* (*ii*) *Can a non-empty finite set be open ?*

(*iii*) *Is every infinite set open ?*

(*iv*) *Is the union of an arbitrary collection of open sets open ?*

(*v*) *Is the intersection of an arbitrary collection of open sets open ?*

Sol. (*i*) Yes. ϕ is a finite set and ϕ is open.

(*ii*) No. A non-empty finite set cannot be a *nbd* of any of its points and hence, is not open. ($\because$ a *nbd* is always an infinite set)

(*iii*) No. The set Q of rational numbers is an infinite set but Q is not open. (See Ex. 2).

(*iv*) Yes. See Theorem II.

(*v*) Not necessary. See Remark after Theorem IV.

Example 5. *Which of the following sets are open ? Justify your answer.*

(*i*) *The set Q of all rational numbers.* (*ii*) *The set* $\{x : 0 < x < 1\}$.

(*iii*) *The closed interval [0, 1].* (*iv*) *The set of all non-zero real numbers.*

Sol. (*i*) It is not an open set. (See Ex. 2)

(*ii*) The set $\{x : 0 < x < 1\} = (0, 1)$. Since every open interval is an open set, so is (0, 1).

(*iii*) The closed interval [0, 1] in not an open set because it is not a *nbd* of its end points 0 and 1.

(*iv*) The set of all non-zero real numbers *i.e.*, the set $(-\infty, 0) \cup (0, \infty)$ is an open set because every open ray is an open set.

Example 6. *Prove or disapprove the following statements :*

(*i*) *Every finite set is open.* (*ii*) *Every infinite set is open.*

(*iii*) *The intersection of two open sets is open.*

Sol. (*i*) Let A be a finite set.

If $A = \phi$, then A is open because ϕ is open.

If $A \neq \phi$, let $x \in A$. Then A cannot be a *nbd* of x because every *nbd* of x must be an infinite set.

$\Rightarrow$ A is not an open set.

(*ii*) The set Q of rational numbers is an infinite set but it is not open. (See Ex. 2)

(*iii*) See Theorem III.

Example 7. *Define an open set and prove that the union of an arbitrary family of open sets is open. Give an example to show that the intersection of an arbitrary family of open sets may not be open.*

Sol. (*i*) Def. See 3.7.

(*ii*) See Theorem II.

(*iii*) See Remark after Theorem IV.

Example 8. *Prove or disapprove the following statements :*

(*i*) $\{x : 0 \le x < 1\}$ *is open.* (*ii*) $\{x : 0 < x \le 1\}$ *is open.*

(*iii*) $(0, 1) \cup (1, 2)$ *is open.*

Sol. (*i*) $\{x : 0 \le x < 1\} = [0, 1)$ is not an open set because it is not a *nbd* of 0.

For any $\varepsilon > 0$, however small, $(-\varepsilon, \varepsilon) \not\subset [0, 1)$.

(*ii*) $\{x : 0 < x \le 1\} = (0, 1]$ is not an open set because it is not a *nbd* of 1.

For any $\varepsilon > 0$, however, small $(1 - \varepsilon, 1 + \varepsilon) \not\subset (0, 1]$.

(*iii*) $(0, 1)$ and $(1, 2)$ are both open sets, being open intervals.

Also, the union of two open sets is an open set.

$\therefore$ $(0, 1) \cup (1, 2)$ is open.

Example 9. *Prove that R – N and R – Z are open sets.*

Sol. (*i*) Let $x \in R - N = N^c$, then $x \notin N$, *i.e.*, x is not a natural number.

If n is the natural number nearest to x, then

$\exists\, \varepsilon = \dfrac{|x-n|}{2} > 0$ s.t. $(x - \varepsilon, x + \varepsilon)$ does not contain any natural number,

i.e., $(x - \varepsilon, x + \varepsilon) \cap N = \phi$

$\therefore$ $(x - \varepsilon, x + \varepsilon) \subset N^c$

$\Rightarrow$ N^c is a *nbd* of x. $\Rightarrow$ N^c is open ($\because$ x is arbitrary)

Hence R – N is an open set.

(*ii*) Let $x \in R - Z = Z^c$, then $x \notin Z$, *i.e.*, x is not an integer.

If n is the integer nearest to x, then $\exists\, \varepsilon = \dfrac{|x-n|}{2} > 0$ s.t. $(x - \varepsilon, x + \varepsilon)$ does not contain any point of Z

i.e., $(x - \varepsilon, x + \varepsilon) \cap Z = \phi$ $\therefore$ $(x - \varepsilon, x + \varepsilon) \subset Z$

$\Rightarrow$ Z^c is a *nbd* of x

$\Rightarrow$ Z^c is open ($\because$ x is arbitrary)

Hence R – Z is an open set.

Example 10. *Prove that every non-empty open set is a union of open intervals.*

Sol. Let S be a non-empty open set and x_λ a point of S. The set S can be thought of as the union of singletons like $\{x_\lambda\}$.

i.e., $S = \bigcup_{\lambda \in \Lambda} \{x_\lambda\}$, where Λ is the indexing set.

Since S is open, it is a *nbd* of each of its points.

$\Rightarrow$ For every x_λ, $\exists$ an open interval I_{x_λ} such that $x_\lambda \in I_{x_\lambda} \subset S$

Since each $I_{x_\lambda} \subset S$

$\therefore$ $\bigcup_{\lambda \in \Lambda} I_{x_\lambda} \subset S$ ...(1)

Also $x_\lambda \in I_{x_\lambda} \Rightarrow \{x_\lambda\} \subset I_{x_\lambda}$

$\therefore$ $S = \bigcup_{\lambda \in \Lambda} \{x_\lambda\} \subset \bigcup_{\lambda \in \Lambda} I_{x_\lambda}$ ...(2)

From (1) and (2), we have $S = \bigcup_{\lambda \in \Lambda} I_{x_\lambda}$

Example 11. *Prove that the set $\{x : 1 < x < 5, x \ne 3, 4\}$ is open.*

Sol. $A = \{x : 1 < x < 5, x \ne 3, 4\} = (1, 3) \cup (3, 4) \cup (4, 5)$ $(1, 3), (3, 4), (4, 5)$ being open intervals are open sets and the union of any number of open sets in an open set.

$\therefore$ A is an open set.

3.9. INTERIOR POINT OF A SET

A real number x is called an interior point of a set A if A is a *nbd* of x *i.e.*, if $\exists\, \varepsilon > 0$ s.t.

$$(x - \varepsilon, x + \varepsilon) \subset A.$$

For example. (*i*) Every point of the open interval (a, b) is its interior point.

(*ii*) a and b are not interior points of the closed interval $[a, b]$ since $[a, b]$ is not a *nbd* of a or b.

3.10. INTERIOR OF A SET

The set of all interior points of a set A is called the interior of A and is denoted by $A°$.

For example. (*i*) If $A = (a, b)$, then $A° = A$ because every point of A is an interior point of A.

(*ii*) If $A = [a, b]$, then $A° = (a, b)$ because every point of A is an interior point of A except the end points a and b.

(*iii*) $R° = R$ because R is a *nbd* of each of its points and therefore, every point of R is an interior point of R.

(*iv*) If A is a non-empty finite set, then $A° = \phi$.

(*v*) $N° = \phi$, $Z° = \phi$, $Q° = \phi$ because N, Z, Q are not *nbds* of any points and therefore, no point is an interior point of N, Z or Q.

3.11. THEOREMS OF INTERIOR ON A SET

Theorem I. *For any set A, A° is open.*

Proof.

Case 1. If $A° = \phi$, then $A°$ is open because ϕ is open.

Case 2. If $A° \ne \phi$, let x be any element of $A°$.

$x \in A° \Rightarrow$ A is a *nbd* of x.

$\Rightarrow$ $\exists\, \varepsilon > 0$ s.t. $I = (x - \varepsilon, x + \varepsilon) \subset A$ ...(*i*)

Let y be any element of I. Since I is an open interval, I is a *nbd* of y.

Also $I \subset A \Rightarrow A$ is a *nbd* of y.

Since y is any point of I $\therefore$ A is a *nbd* of every point of I.

$\Rightarrow$ Every point of I is an interior point of A

$\Rightarrow$ $I \subset A°$ *i.e.*, $x \in I \subset A°$

$\Rightarrow$ $A°$ is a *nbd* of x.

Since x is any point of $A°$, it follows that $A°$ is a *nbd* of each of its point. Hence $A°$ is an open set.

Theorem II. *A° is the largest open set contained in A.*

Proof. Let B be any open set contained in A *i.e.*, let $B \subset A$ where B is open.

If $B = \phi$, then $B \subset A°$ ($\because$ $\phi \subset A°$)

If $B \ne \phi$, let x be any element of B. Since B is an open set, B is a *nbd* of x.

Also $B \subset A$ $\therefore$ A is a *nbd* of x.

$\Rightarrow$ $x \in A°$

Since $x \in B \Rightarrow x \in A°$ $\therefore$ $B \subset A°$

$\Rightarrow$ Every open set contained in A is a subset of $A°$.

Also $A°$ is open.

Hence $A°$ is the largest open set contained in A.

Theorem III. *If A is any subset of R, then* $A^\circ \subset A$

Proof. Let x be any element of A°

$$x \in A^\circ \Rightarrow A \text{ is a } nbd \text{ of } x \Rightarrow x \in A$$

Since $\quad x \in A^\circ \Rightarrow x \in A \quad \therefore \quad A^\circ \subset A.$

Theorem IV. *If* $A \subset B$, *then* $A^\circ \subset B^\circ$.

Proof. Let x be any element of A°.

$$x \in A^\circ \Rightarrow A \text{ is a } nbd \text{ of } x \Rightarrow B \text{ is a } nbd \text{ of } x \qquad (\because B \supset A)$$

$\Rightarrow \quad x \in B^\circ$

Since $\quad x \in A^\circ \Rightarrow x \in B^\circ \quad \therefore \quad A^\circ \subset B^\circ.$

Theorem V. *A is open iff* $A = A^\circ$, *where A is any subset of R.*

Proof. Let A be an open set.

We know that for every subset A of R, $A^\circ \subset A$ ...(1)

Also A° is the largest open set contained in A. (Theorem II)

Since A is an open set contained in A

$\therefore \quad A \subset A^\circ$ ...(2)

From (1) and (2), $\quad A = A^\circ$

$\therefore$ A is open $\Rightarrow A = A^\circ$

Let $A^\circ = A$

Since A° is open (Theorem I)

$\therefore$ A is open.

$\therefore \quad A = A^\circ \Rightarrow$ A is open

Hence A is open iff $A = A^\circ$.

Theorem VI. *For any set A,* $(A^\circ)^\circ = A^\circ$.

Proof. We know that

A is open $\Rightarrow A^\circ = A$ (Theorem V)

Also, for any set A, A° is open

$\therefore \quad (A^\circ)^\circ = A^\circ.$

Theorem VII. A° *is equal to the union of all open subsets of A.*

Proof. Let $F = \{S : S \text{ is an open set and } S \subset A\}$

Let $B = \bigcup_{S \in F} S$. We have to prove that $A^\circ = B$

Since A° is an open set (Th. I) and $A^\circ \subset A$ (Th. III)

$\therefore$ By def. of F, $\quad A^\circ \in F.$

But $\quad B = \bigcup_{S \in F} S$

$\therefore \quad A^\circ \subset B.$ ...(1)

Since B is the union of open subsets of A, therefore, B is an open subset of A.

($\because$ the union of an arbitrary family of open sets is open)

Also A° is the largest open set contained in A

$\therefore \quad B \subset A^\circ$ ...(2)

From (1) and (2), $\quad A^\circ = B.$

Example 1. *Find the interiors of the following sets :*

(*i*) $\{1, 2, 3, 4, 5\}$ (*ii*) $[0, 1]$ (*iii*) $[0, 1] \cup [3, 5]$

(*iv*) $\left\{\frac{1}{n} : n \in N\right\}$ (*v*) Z (*vi*) $R - Z$

(*vii*) Q (*viii*) $R - Q$ (*ix*) R.

Sol. (*i*) Let $A = \{1, 2, 3, 4, 5\}$, then A is a non-empty finite set.

$\Rightarrow$ A is not a *nbd* of any point $\Rightarrow A^\circ = \phi$.

(*ii*) The closed interval [0, 1] is a *nbd* of each point of the open interval (0, 1)

$\therefore \quad [0, 1]^\circ = (0, 1)$

Note. $[0, 1]^\circ = [0, 1)^\circ = (0, 1]^\circ = (0, 1)^\circ = (0, 1).$

(*iii*) Let $A = [0, 1] \cup [3, 5]$, then A is a *nbd* of each point of $(0, 1) \cup (3, 5)$.

$\therefore \quad A^\circ = (0, 1) \cup (3, 5)$

(*iv*) Let $\quad A = \left\{\frac{1}{n} : n \in N\right\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$

If $p \in A$, then there exists no $\varepsilon > 0$ such that $(p - \varepsilon, p + \varepsilon) \subset A$

$\Rightarrow$ p is not an interior point of A

Thus A has no interior point

$\therefore \quad A^\circ = \phi$

(*v*) **Z** is not a *nbd* of any of its points

$\therefore \quad Z^\circ = \phi$

(*vi*) The set R—Z is open

$\therefore \quad (R—Z)^\circ = R—Z \quad$ *i.e.,* $\quad (Z^c)^\circ = Z^c$

(*vii*) Q is not a *nbd* of any of its points

$\therefore \quad Q^\circ = \phi$

(*viii*) $R - Q$, the set of all irrational numbers, is not a *nbd* of any of its points.

If $p \in R - Q$, then there exists no $\varepsilon > 0$ such that $(p - \varepsilon, p + \varepsilon) \subset R - Q$

since $(p - \varepsilon, p + \varepsilon)$ contains infinitely many rational numbers

$\Rightarrow$ p is not an interior point of $R - Q$

Thus $R - Q$ has no interior point

$\therefore \quad (R - Q)^\circ = \phi$

(*ix*) The set R is open $\therefore$ $R^\circ = R$.

Example 2. *Write down the interior of each of the following :*

(*i*) $\{x : 0 \le x \le 1\}$ (*ii*) $\{x : 0 < x < 1\}$ (*iii*) $\{2, 3\}$.

Sol. (*i*) Let $\quad A = \{x : 0 \le x \le 1\} = [0, 1]$

then $\quad A^\circ = (0, 1) = \{x : 0 < x < 1\}$

(*ii*) Let $\quad A = \{x : 0 < x < 1\} = (0, 1)$

then $\quad A^\circ = A = \{x : 0 < x < 1\}$

(*iii*) Let $A = \{2, 3\}$. Since A is a non-empty finite set, $A^\circ = \phi$.

Example 3. *Let A and B be two subsets of R. Prove that*

(*i*) $A^\circ \cup B^\circ \subset (A \cup B)^\circ$

Give an example to show that $A^\circ \cup B^\circ \ne (A \cup B)^\circ$

(*ii*) $(A \cap B)^\circ = A^\circ \cap B^\circ$.

Sol. (*i*) We know that $A \subset B \Rightarrow A^\circ \subset B^\circ$

Since $A \subset A \cup B$ and $B \subset A \cup B$

$\therefore$ $A^\circ \subset (A \cup B)^\circ$ and $B^\circ \subset (A \cup B)^\circ$

$\Rightarrow$ $A^\circ \cup B^\circ \subset (A \cup B)^\circ$

The inclusion cannot be replaced by equality.

For example, let $A = [0, 1], B = [1, 2]$

Then $A^\circ = (0, 1), B^\circ = (1, 2)$

$$A^\circ \cup B^\circ = (0, 1) \cup (1, 2) = (0, 2) - \{1\}$$

Also $A \cup B = [0, 2]$ so that $(A \cup B)^\circ = (0, 2)$

Clearly, $A^\circ \cup B^\circ \neq (A \cup B)^\circ$, rather $A^\circ \cup B^\circ$ is a proper subset of $(A \cup B)^\circ$.

(*ii*) We know that $A \subset B \Rightarrow A^\circ \subset B^\circ$

Since $A \cap B \subset A$ and $A \cap B \subset B$

$\therefore$ $(A \cap B)^\circ \subset A^\circ$ and $(A \cap B)^\circ \subset B^\circ$

$\Rightarrow$ $(A \cap B)^\circ \subset A^\circ \cap B^\circ$

Now, let x be any element of $A^\circ \cap B^\circ$.

Then, $x \in A^\circ \cap B^\circ \Rightarrow x \in A^\circ$ and $x \in B^\circ$

$\Rightarrow$ A is a *nbd* of x and B is a *nbd* of x

$\Rightarrow$ $A \cap B$ is a *nbd* of $x \Rightarrow x \in (A \cap B)^\circ$

Since $x \in A^\circ \cap B^\circ \Rightarrow x \in (A \cap B)^\circ$

$\therefore$ $A^\circ \cap B^\circ \subset (A \cap B)^\circ$ ...(2)

From (1) and (2), we have $(A \cap B)^\circ = A^\circ \cap B^\circ$.

Example 4. *Show by means of an example that two distinct sets A and B may have the same interior.*

Sol. Let $A = [1, 2)$ and $B = (1, 2]$ then $A^\circ = (1, 2)$ and $B^\circ = (1, 2)$

Clearly $A \neq B$, but $A^\circ = B^\circ$.

Example 5. *If A and B be non-empty subsets of R such that A is a proper subset of B, is it possible that* $A^\circ = B^\circ$? *Justify your answer.*

Sol. Yes.

Let $A = (1, 2)$ and $B = [1, 2]$

then A is a proper subset of B.

Also $A^\circ = (1, 2), B^\circ = (1, 2)$

so that $A^\circ = B^\circ$.

3.12. CLOSED SET

Let A be a subset of R, then A is said to be a closed set if its complement $A^c = R - A$ is an open set.

i.e., a set is closed if its complement is open.

For example. (*i*) $R^c = R - R = \phi$ which is open

$\Rightarrow$ R is closed.

(*ii*) $\phi^c = R - \phi = R$ which is open $\Rightarrow$ ϕ is closed.

☞ **R and ϕ are the only two sets which are both open and closed.**

(*iii*) $[a, b]^c = (-\infty, a) \cup (b, \infty)$ being the union of two open sets is itself open $\Rightarrow$ every closed interval $[a, b]$ is a closed set.

(*iv*) $(a, b)^c = (-\infty, a] \cup [b, \infty)$ is not an open set because it is not a *nbd* of a or $b \Rightarrow (a, b)$ is not a closed set.

(*v*) The sets $[a, b)$ and $(a, b]$ are neither open nor closed.

(*vi*) **The set Z of all integers is a closed set.**

Z will be a closed set if Z^c is an open set.

Let $x \in Z^c$ then $x \notin Z$ *i.e.*, x is not an integer.

If n is the integer nearest to x, then $\exists\, \varepsilon = \dfrac{|x - n|}{2} > 0$ s.t. $(x - \varepsilon, x + \varepsilon)$ does not contain any integer *i.e.*, $(x - \varepsilon, x + \varepsilon)$ does not contain any point of Z.

$\therefore$ $(x - \varepsilon, x + \varepsilon) \subset Z^c \Rightarrow Z^c$ is a *nbd* of x.

But x is any point of Z^c $\therefore$ Z^c is a *nbd* of each of its points.

$\Rightarrow$ Z^c is open $\Rightarrow$ Z is closed.

Similarly, the set N of all natural numbers (or positive integers) is a closed set.

3.13. THEOREMS ON CLOSED SETS

Theorem I. *The union of two closed sets is a closed set.*

Proof. Let A and B be two closed sets.

$\Rightarrow$ A^c and B^c are open sets.

$\Rightarrow$ $A^c \cap B^c$ is an open set.

$\Rightarrow$ $(A \cup B)^c$ is an open set [$\because$ $A^c \cap B^c = (A \cup B)^c$ De Morgan's Law]

$\Rightarrow$ $A \cup B$ is a closed set.

Theorem II. *The union of a finite number of closed sets is a closed set.*

Proof. Let $A_1, A_2, \ldots\ldots, A_n$ be n closed sets and $S = \bigcup\limits_{i=1}^{n} A_i$

$\Rightarrow$ $A_1^c, A_2^c, \ldots\ldots, A_n^c$ are n open sets

$\Rightarrow$ $\bigcap\limits_{i=1}^{n} A_i^c$ is an open set.

[$\because$ the intersection of a finite collection of open sets is an open set]

$\Rightarrow$ $(\bigcup\limits_{i=1}^{n} A_i)^c$ is an open set [$\because$ $\bigcap\limits_{i=1}^{n} A_i^c = (\bigcup\limits_{i=1}^{n} A_i)^c$ De Morgan's Law]

$\Rightarrow$ $\bigcup\limits_{i=1}^{n} A_i$ is a closed set.

Note **Theorem III.** *The union of an infinite family of closed sets need not be a closed set.*

Proof. Let $A_n = \left[\dfrac{1}{n}, 1\right]$ $\forall\, n \in N$, then $\{A_n\}_{n \in N}$ is an infinite family of closed sets.

$$A_1 = \{1\}, A_2 = [\tfrac{1}{2}, 1], A_3 = [\tfrac{1}{3}, 1], \ldots\ldots$$

$$\bigcup_{n=1}^{\infty} A_n = \{1\} \cup [\tfrac{1}{2}, 1] \cup [\tfrac{1}{3}, 1] \cup \ldots\ldots = (0, 1] \text{ which is not a closed set.}$$

Theorem IV. *The intersection of two closed sets is a closed set.*

Proof. Let A and B be two closed sets

$\Rightarrow$ A^c and B^c are two open sets. $\Rightarrow$ $A^c \cup B^c$ is an open set.

$\Rightarrow$ $(A \cap B)^c$ is an open set [$\because$ $A^c \cup B^c = (A \cap B)^c$ De Morgan's Law]

$\Rightarrow$ $A \cap B$ is a closed set.

Theorem V. *The intersection of an arbitrary family of closed sets is a closed set.*

Proof. Let $\{A_\lambda\}_{\lambda \in \Lambda}$ be an arbitrary family of closed sets.

A_λ is closed (given) $\Rightarrow$ A_λ^c is open

Since the union of an arbitrary family of open sets is an open set, $\bigcup_{\lambda \in \Lambda} A_\lambda^c$ is open

$\Rightarrow$ $(\bigcap_{\lambda \in \Lambda} A_\lambda)^c$ is open [$\because$ $\bigcup_{\lambda \in \Lambda} A_\lambda^c = (\bigcap_{\lambda \in \Lambda} A_\lambda)^c$ De Morgan's Law]

$\Rightarrow$ $\bigcap_{\lambda \in \Lambda} A_\lambda$ is a closed set.

Example 1. *Give an example to show that*

(*i*) *a subset of a closed set need not be closed*

(*ii*) *a set containing a closed set need not be closed.*

Sol. (*i*) $(2, 3) \subset [2, 3]$

Closed interval [2, 3] is a closed set whereas the open interval (2, 3) is not a closed set.

(*ii*) $(2, 5) \supset [3, 4]$

[3, 4] being a closed interval is a closed set whereas (2, 5) being an open interval is an open set.

Example 2. *Give an example of :*

(*i*) *an interval which is a closed set.* (*ii*) *an interval which is not a closed set.*

(*iii*) *a closed set other than an interval.*

(*iv*) *a set which is open as well as closed.*

(*v*) *a set which is neither open nor closed.*

(*vi*) *a set which is neither an interval nor a closed set.*

Sol. (*i*) [2, 3] (*ii*) (2, 3) (*iii*) Any finite set {2, 3}

(*iv*) R (*v*) [2, 3) (*vi*) $(2, 3) \cup (4, 5)$.

Example 3. *Show that every finite set is a closed set.*

Sol. Let A be a finite set.

If $A = \phi$, then A is closed because ϕ is closed.

If $A \neq \phi$, let $A = \{a_1, a_2, ..., a_n\}$

Consider the singleton $\{a\}$.

$\{a\}^c = (-\infty, a) \cup (a, \infty)$ being the union of two open sets is an open set $\Rightarrow$ $\{a\}$ is a closed set *i.e.*, every singleton is a closed set

Since the union of a finite number of closed sets is a closed set

$\therefore$ $\{a_1\} \cup \{a_2\} \cup ...... \cup \{a_n\}$ is a closed set.

$\Rightarrow$ $\{a_1, a_2, ..., a_n\} = A$ is a closed set.

Example 4. *The set Q of all rationals is not a closed set.*

Sol. Let x be any element of $Q^c = R - Q$, the set of irrational numbers.

For every $\varepsilon > 0$, $(x - \varepsilon, x + \varepsilon)$ contains infinitely many rational numbers.

$\therefore$ $\exists$ no open interval I s.t. $x \in I \subset Q^c$

$\Rightarrow$ Q^c is not a *nbd* of x $\Rightarrow$ Q^c is not an open set

$\Rightarrow$ Q is not a closed set.

Example 5. *Prove or disapprove the following statements :*

(*i*) *Every finite set is closed.* (*ii*) *Every infinite set is closed.*

(*iii*) *The union of an arbitrary family of closed sets is closed.*

Sol. (*i*) See Example 3.

(*ii*) Every infinite set need not be closed. For example (1, 2) is an infinite set but is not closed. Q, the set of all rationals is an infinite set which is not closed. (See Ex. 4)

(*iii*) See Theorem III.

Example 6. *Let A be a closed set and B be an open set. Show that*

(*i*) *B – A is an open set*

(*ii*) *A – B is a closed set.*

Sol. A is closed $\Rightarrow$ A^c is open ; B is open $\Rightarrow$ B^c is closed.

(*i*) $B - A = B \cap A^c$.

Since B and A^c are open, $B \cap A^c$ is open.

$\therefore$ B – A is open.

(*ii*) $A - B = A \cap B^c$

Since A and B^c are closed, $A \cap B^c$ is closed

$\therefore$ A – B is closed.

Example 7. *Which of the following sets are closed, open, neither closed nor open ?*

(*i*) $\{x : 0 \le x \le 1\}$ (*ii*) $[0, 1] \cup [2, 3]$

(*iii*) $\{x : 1 < x < 8\}$ (*iv*) $\{x : 4 \le x < 8\}$.

Sol. (*i*) $\{x : 0 \le x \le 1\} = [0, 1]$ being a closed interval is a closed set.

(*ii*) [0, 1] and [2, 3] being closed intervals are closed sets and so is their union $[0, 1] \cup [2, 3]$.

(*iii*) $\{x : 1 < x < 8\} = (1, 8)$ being an open interval is an open set.

(*iv*) $\{x : 4 \le x < 8\} = [4, 8)$ is neither closed nor open.

3.14. LIMIT POINT OF A SET

Definition 1. A point $p \in R$ is said to be a limit point of a subset S of R if every *nbd* of p has a point of S other than p.

In symbols, a point $p \in R$ is said to be a limit point of a subset S of R if for each *nbd* N or p,

$$(N \cap S) - \{p\} \neq \phi.$$

Definition 2. A point $p \in R$ is said to be a limit point of a subset S of R if every *nbd* of p contains infinitely many points of S.

Remark. *Definitions 1 and 2 are equivalent.*

Definition 1 $\Rightarrow$ **Definition 2**

Let p be a limit point of S according to definition 1, then every *nbd* N of p has a point of S other than p, *i.e.*, $N \cap S - \{p\} \neq \phi$

Suppose N contains only finitely many points of S. If

$$N \cap S - \{p\} = \{p_1, p_2, \ldots\ldots, p_n\}$$

and
$$\varepsilon = \min.\{|p - p_1|, |p - p_2|, \ldots\ldots, |p - p_n|\} > 0$$

then $(p - \varepsilon, p + \varepsilon)$ is a *nbd* of p which contains no point of S other than p, *i.e.*,

$$(p - \varepsilon, p + \varepsilon) \cap S - \{p\} = \phi$$

⇒ p is not a limit point of S according to definition 1.

⇒ we have arrived at a contradiction.

⇒ our supposition that N has only finitely many points of S is wrong.

⇒ N contains infinitely many points of S.

⇒ p is a limit point of S according to definition 2.

Definition 2 ⇒ Definition 1

Let p be a limit point of S according to definition 2, then every *nbd* N of p contains infinitely many points of S.

⇒ every *nbd* N of p contains a point of S other than p *i.e.*, $N \cap S - \{p\} \neq \phi$.

⇒ p is a limit point of S according to definition 1.

Hence, Definition 1 ⇔ Definition 2

Note 1. Limit point of a set is also called a *limiting point* or a *cluster point* or a *condensation point* or an *accumulation point* of the set.

Note 2. A finite set has no limit point.

Note 3. A limit point of S may or may not belong to S.

Note 4. In order to show that a point p is not a limit point of a set S, it is enough to find a *nbd* N of p such that

$$N \cap S = \{p\} \quad \text{or} \quad N \cap S = \phi.$$ **(Remember)**

3.15. ISOLATED POINT

A point $p \in S$ is called an isolated point of S if p is not a limit point of S.

3.16. DERIVED SET

The set of all limit points of a set S is called the derived set of S and is denoted by S′.

Thus

$$S' = \{x : x \text{ is a limit point of } S\}.$$

3.17. ADHERENT POINT

A real number p is called an adherent point of a set $S \subset R$ if every nbd of p contains a point of S.

In symbols, a point $p \in R$ is an adherent point of $S \subset R$ iff for each *nbd* N of p, $N \cap S \neq \phi$.

Note. Due to a close resemblance between the definitions of an adherent point of a set and a limit point of a set, the distinction between the two should be carefully noted.

For a point p to be a limit of a set S, every *nbd* N of p must contain a point of S *other than p*.

i.e.,
$$N \cap S - \{p\} \neq \phi$$

For a point p to be an adherent point of a set, S every *nbd* of p must contain a point of S *which can be p itself*.

i.e.,
$$N \cap S \neq \phi.$$

If $p \in S$, then p is an adherent point of S, since every *nbd* of p contains p which belongs to S.

If $p \in S'$, then p is a limit point of S and, therefore, every *nbd* of p contains a point of S other than p. Thus p is also an adherent point of S.

Clearly, a real number p is an adherent point of S iff either $p \in S$ or $p \in S'$.

Every point of S is an adherent point of S. Every limit point of S is an adherent point of S. An adherent point of S need not be a limit point of S.

3.18. CLOSURE OF A SET

The set of all adherent points of a set S is called the closure of S and is denoted by cl S or $\overline{S}$.

Thus
$$\overline{S} = S \cup S'.$$

3.19. PERFECT SET

A set S is said to be perfect if $S = S'$.

3.20. DENSE SET

Let A and B be two subsets of R such that $A \subset B$. The set A is said to be dense (or everywhere dense) in B if $B \subset \overline{A}$.

Example 1. *Find the derived set and the closure of each of the following sets :*

(i) Q *(ii) R – Q* *(iii) N*

(iv) Z *(v) φ* *(vi) R.*

Sol. (*i*) Let x be any real number. Then for each $\varepsilon > 0$, however small, $(x - \varepsilon, x + \varepsilon)$ is a *nbd* of x and it contains infinitely many rational numbers. Consequently, it contains a point of Q other than x

i.e.,
$$(x - \varepsilon, x + \varepsilon) \cap Q - \{x\} \neq \phi$$

⇒ x is a limit point of Q

⇒ every real number is a limit point of Q

⇒ $Q' = R$ and $\overline{Q} = Q \cup Q' = Q \cup R = R$

(*ii*) *Please try yourself.*

$$(R - Q)' = R \quad \text{and} \quad \overline{R - Q} = R$$

(*iii*) Let x be any real number. We know that $R = N \cup N^c$

If $x \in N$, then $(x - \frac{1}{2}, x + \frac{1}{2}) \cap N = \{x\}$

If $x \in N^c = R - N$

i.e., if x is a real number but not a natural number, let n be the natural number nearest to x. Then, for

$$\varepsilon = \frac{|x - n|}{2} > 0, \ (x - \varepsilon, x + \varepsilon) \cap N = \phi$$

Thus whatever real number x may be, there exists a *nbd* of x which either does not contain a point of N other than x itself or does not contain any point of N at all. Consequently, x is not a limit point of N.

$$\therefore \quad N' = \phi \quad \text{and} \quad \overline{N} = N \cup N' = N \cup \phi = N$$

(*iv*) *Please try yourself.*

$$Z' = \phi \quad \text{and} \quad \bar{Z} = Z$$

(*v*) Let x be any real number. Then for each $\varepsilon > 0$, however small, $(x - \varepsilon, x + \varepsilon) \cap \phi = \phi$

$\Rightarrow$ x is not a limit point of ϕ

$\Rightarrow$ No real number is a limit point of ϕ

$$\therefore \quad \phi' = \phi \quad \text{and} \quad \bar{\phi} = \phi \cup \phi' = \phi \cup \phi = \phi$$

Note. Since $\phi = \phi'$, the set ϕ is a perfect set.

(*vi*) Let x be any real number. Then for each $\varepsilon > 0$, however small, $(x - \varepsilon, x + \varepsilon)$ is a *nbd* of x and it contains infinitely many real numbers.

Thus $(x - \varepsilon, x + \varepsilon) \cap R - \{x\} \neq \phi$

$\Rightarrow$ x is a limit point of R

$\Rightarrow$ every real number is a limit point of R.

$$\therefore \quad R' = R \quad \text{and} \quad \bar{R} = R \cup R' = R \cup R = R.$$

Note. Since $R = R'$, the set R is a perfect set.

Example 2. *Find the derived set and the closure of each of the following sets :*

(*i*) (a, b) (*ii*) $(a, b]$

(*iii*) $[a, b)$ (*iv*) $[a, b]$ *where* $a, b \in R$.

Sol. (*i*) Let x be any real number.

If $x < a$, than for $0 < \varepsilon < a - x$, $(x - \varepsilon, x + \varepsilon) \cap (a, b) = \phi$

$\Rightarrow$ *Any real number* $< a$ *is not a limit point of* (a, b)

If $x > b$, then for $0 < \varepsilon < x - b$, $(x - \varepsilon, x + \varepsilon) \cap (a, b) = \phi$

$\Rightarrow$ *Any real number* $> b$ *is not a limit point of* (a, b)

If $x \in [a, b]$, then for any $\varepsilon > 0$, $(x - \varepsilon, x + \varepsilon)$ is a *nbd* of x and $(x - \varepsilon, x + \varepsilon) \cap (a, b) = (c, d)$ where

$$c = \max. \{x - \varepsilon, a\} \text{ and } d = \min \{x + \varepsilon, b\}$$

The interval (c, d) contains infinitely many points of (a, b) and consequently, x is a limit point of (a, b).

$\Rightarrow$ *every point of* $[a, b]$ *is a limit point of* (a, b)

$$\therefore \quad (a, b)' = [a, b] \quad \text{and} \quad \overline{(a, b)} = (a, b) \cup (a, b)' = (a, b) \cup [a, b] = [a, b].$$

Note. We can also proceed like this.

If $x \in [a, b]$ then $x = a$ or $x = b$ or $x \in (a, b)$

If $x = a$, then for every $\varepsilon > 0$, $(x - \varepsilon, x + \varepsilon) = (a - \varepsilon, a + \varepsilon)$ contains infinitely many points of (a, b) to the right of a.

If $x = b$, then for every $\varepsilon > 0$, $(x - \varepsilon, x + \varepsilon) = (b - \varepsilon, b + \varepsilon)$ contains infinitely many points of (a, b) to the left of b.

If $x \in (a, b)$, then for every $\varepsilon > 0$, $(x - \varepsilon, x + \varepsilon)$ contains infinitely many points of (a, b).

Thus, if $x \in [a, b]$, then for every $\varepsilon > 0$, $(x - \varepsilon, x + \varepsilon)$ is a *nbd* of x containing infinitely many points of (a, b).

Consequently, x is a limit point of (a, b).

(*ii*) *Please try yourself.*

$$(a, b]' = \overline{(a, b]} = [a, b].$$

(*iii*) *Please try yourself.*

$$[a, b)' = \overline{[a, b)} = [a, b].$$

(*iv*) *Please try yourself.*

$$[a, b]' = \overline{[a, b]} = [a, b].$$

Example 3. *If* x *is a limit point of* A *and* $A \subset B$, *then* x *is also a limit point of* B.

Sol. x is a limit point of A.

$\Rightarrow$ every *nbd* of x contains *a point of* A other than x

$\Rightarrow$ every *nbd* of x contains *a point of* B other than x $(\because A \subset B)$

$\Rightarrow$ x is a limit point of B.

Example 4. *If* p *is a limit point of a set* $E \subset R$, *then every neighbourhood of* p *contains infinitely many points of* E.

Sol. Suppose N is a *nbd* of p and N contains only finitely many points of E. If

$$N \cap E - \{p\} = \{p_1, p_2, \ldots\ldots, p_n\}$$

and $$\varepsilon = \min. \{\,|p - p_1|\,, |p - p_2|\,, \ldots\ldots, |p - p_n|\,\} > 0$$

then $(p - \varepsilon, p + \varepsilon)$ is a *nbd* of p which contains no point of E other than p *i.e.*, $(p - \varepsilon, p + \varepsilon) \cap E - \{p\} = \phi$

$\Rightarrow$ p is not a limit point of E which is a contradiction

$\Rightarrow$ N contains infinitely many points of E.

Example 5. *Prove that a real number* x *is a limit point of a set* S *iff for each* $n \in N$, *the open interval* $\left(x - \frac{1}{n}, x + \frac{1}{n}\right)$ *contains a point of* S *other than* x, *i.e.*, $\left(x - \frac{1}{n}, x + \frac{1}{n}\right) \cap S - \{x\} \neq \phi$.

Sol. Let x be a limit point of S, then for every $\varepsilon > 0$, $(x - \varepsilon, x + \varepsilon)$ contains a point of S other than x

i.e., $$(x - \varepsilon, x + \varepsilon) \cap S - \{x\} \neq \phi$$

Taking $\varepsilon = \frac{1}{n} > 0$, $n \in N$, we have $\left(x - \frac{1}{n}, x + \frac{1}{n}\right) \cap S - \{x\} \neq \phi$ for each $n \in N$.

Conversely, for each $n \in N$, let the open interval $\left(x - \frac{1}{n}, x + \frac{1}{n}\right)$ contain a point of S other than x.

For every $\varepsilon > 0$, we can choose a natural number n such that

$$n > \frac{1}{\varepsilon} \quad i.e., \quad \frac{1}{n} < \varepsilon \quad \text{and} \quad -\frac{1}{n} > -\varepsilon$$

$$\Rightarrow \quad x + \frac{1}{n} < x + \varepsilon \quad \text{and} \quad x - \frac{1}{n} > x - \varepsilon$$

$$\Rightarrow \quad \left(x - \frac{1}{n}, x + \frac{1}{n}\right) \subset (x - \varepsilon, x + \varepsilon)$$

Since $\left(x - \frac{1}{n}, x + \frac{1}{n}\right)$ contains a point of S other than x, so does $(x - \varepsilon, x + \varepsilon)$.

Now, for every $\varepsilon > 0$, $(x - \varepsilon, x + \varepsilon) \cap S - \{x\} \neq \phi$

$\therefore$ x is a limit point of S.

Example 6. *Prove that 0 is the only limit point of the set*

$$S = \left\{\frac{1}{n} : n \in N\right\}.$$

Sol. For each $\varepsilon > 0$, $(-\varepsilon, \varepsilon)$ is a *nbd* of 0.

By Archimedean property of reals, for each $\varepsilon > 0$, $\exists\, n \in \mathrm{N}$ such that $n > \frac{1}{\varepsilon}$

$$\Rightarrow \quad \frac{1}{n} < \varepsilon \;\Rightarrow\; -\varepsilon < 0 < \frac{1}{n} < \varepsilon \;\Rightarrow\; \frac{1}{n} \in (-\varepsilon, \varepsilon)$$

Thus every *nbd* of 0 contains a point of S, namely $\frac{1}{n}$.

$\Rightarrow$ 0 is a limit point of S.

Uniqueness. $\mathrm{S} = \left\{\frac{1}{n} : n \in \mathrm{N}\right\} \subset (0, 1]$.

We shall show that no real number other than 0 is a limit point of S. *Let x be any non-zero real number.*

Following cases arise :

Case (*i*) If $x < 0$, then $(-\infty, 0)$ is a *nbd* of x which contains no point of S,

i.e., $(-\infty, 0) \cap \mathrm{S} = \phi$.

$\therefore$ x is not a limit point of S.

Case (*ii*) If $x > 1$, then $(1, \infty)$ is a *nbd* of x which does not contain any point of S,

i.e., $(1, \infty) \cap \mathrm{S} = \phi$.

$\therefore$ x is not a limit point of S.

Case (*iii*) If $x = 1$, then $(\frac{1}{2}, \infty)$ is a *nbd* of x which contains no point of S other than x,

i.e., $(\frac{1}{2}, \infty) \cap \mathrm{S} - \{1\} = \phi$.

$\therefore$ x is not a limit point of S.

Case (*iv*) If $0 < x < 1$, then $\frac{1}{x} > 0$

$\therefore$ $\exists$ a unique natural number n such that

$$n \le \frac{1}{x} < n+1 \;\Rightarrow\; \frac{1}{n} \ge x > \frac{1}{n+1} \;\Rightarrow\; \frac{1}{n+1} < x \le \frac{1}{n} < \frac{1}{n-1}$$

$\Rightarrow$ The *nbd* $\left(\frac{1}{n+1}, \frac{1}{n-1}\right)$ of x contains only one point $\frac{1}{n}$ of S *i.e.,* a finite number of points of S.

$\therefore$ x is not a limit point of S.

Hence 0 is the only limit point of S.

Note. If $\mathrm{S} = \left\{\frac{1}{n} : n \in \mathrm{N}\right\}$, then $\mathrm{S}' = \{0\}$.

Example 7. *Find S′ where* $S = \left\{\frac{1}{n} : n \in Z, n \ne 0\right\}$.

Sol. *Please try yourself.* **[Ans.** $\mathrm{S}' = \{0\}$**]**

[Hint. $\mathrm{S} \subset [-1, 1]$. For uniqueness, consider.

Case (*i*) $x < -1$, *nbd* $(-\infty, -1)$

Case (*ii*) $x > 1$, *nbd* $(1, \infty)$

Case (*iii*) $x = 1$, *nbd* $(\frac{1}{2}, \infty)$

Case (*iv*) $x = -1$, *nbd* $(-\infty, -\frac{1}{2})$

Case (*v*) $0 < x < 1$, *nbd* $\left(\frac{1}{n+1}, \frac{1}{n-1}\right)$

Case (*vi*) $-1 < x < 0$ so that $0 < -x < 1$ and $-\frac{1}{x} > 0$. $\exists$ unique $n \in \mathrm{N}$ s.t.

$$n \le -\frac{1}{x} < n+1 \quad \Rightarrow \quad -\frac{1}{n} \le x < -\frac{1}{n+1}$$

$$\Rightarrow \quad -\frac{1}{n-1} < -\frac{1}{n} \le x < -\frac{1}{n+1}$$

$\Rightarrow$ The *nbd* $\left(-\frac{1}{n-1}, -\frac{1}{n+1}\right)$ of x contains only one point $-\frac{1}{n}$ of S.**]**

Example 8. *Prove that a finite set has no limit point.*

Sol. Let $\mathrm{S} = \{p_1, p_2, \ldots\ldots, p_n\}$ be a finite subset of R. Let p be any real number.

If we choose $\varepsilon = \min. \{ |p - p_1|, |p - p_2|, \ldots\ldots, |p - p_n| \}$, then $(p - \varepsilon, p + \varepsilon)$ is a *nbd* of p which contains no element of S, *i.e.,* $(p - \varepsilon, p + \varepsilon) \cap \mathrm{S} = \phi$

$\therefore$ p is not a limit point of S.

Since p is arbitrary, S has no limit point.

Example 9. *Find the derived set of each of the following :*

(*i*) $(1, \infty)$ (*ii*) $(-\infty, -1)$ (*iii*) $\left\{\frac{1+(-1)^n}{n} : n \in N\right\}$

(*iv*) $\{r\sqrt{2} : r \in Q\}$ (*v*) $\left\{a + \frac{1}{n} : a \in R, n \in N\right\}$.

Sol. (*i*) Let x be any real number.

If $x < 1$, then for $0 < \varepsilon < 1 - x$, $(x - \varepsilon, x + \varepsilon) \cap (1, \infty) = \phi$.

$\Rightarrow$ Any real number < 1 is not a limit point of $(1, \infty)$.

If $x \in [1, \infty)$, then for every $\varepsilon > 0$, $(x - \varepsilon, x + \varepsilon)$ contains infinitely many points of $(1, \infty)$ to the right of 1.

$\Rightarrow$ Every element of $[1, \infty)$ is a limit point of $(1, \infty)$.

$\therefore$ $(1, \infty)' = [1, \infty)$.

(*ii*) *Please try yourself.* **[Ans.** $(-\infty, -1]$**]**

(*iii*) Let $\mathrm{S} = \left\{\frac{1+(-1)^n}{n} : n \in \mathrm{N}\right\}$

when n is odd, $\frac{1+(-1)^n}{n} = \frac{1-1}{n} = 0$

when n is even, $\frac{1+(-1)^n}{n} = \frac{1+1}{n} = \frac{2}{n}$

$$\therefore \qquad S = \{0\} \cup \left\{\frac{1}{n} : n \in N\right\} \subset [0, 1].$$

Now proceeding as in Example 6, we have $S' = \{0\}$.

(*iv*) Let $S = \{r\sqrt{2} : r \in Q\}$.

Let x be any real number, then for each $\varepsilon > 0$, $(x - \varepsilon, x + \varepsilon)$ is a *nbd* of x.

Now $\qquad x - \varepsilon < x + \varepsilon \Rightarrow \dfrac{x-\varepsilon}{\sqrt{2}} < \dfrac{x+\varepsilon}{\sqrt{2}}$

Since between any two distinct real numbers, there lie infinitely many rational numbers, therefore, $\exists$ infinitely many rational numbers r such that

$$\frac{x-\varepsilon}{\sqrt{2}} < r < \frac{x+\varepsilon}{\sqrt{2}} \Rightarrow x - \varepsilon < r\sqrt{2} < x + \varepsilon$$

$\Rightarrow$ $(x - \varepsilon, x + \varepsilon) \cap S$ contains infinitely many elements of S.

$\Rightarrow$ x is a limit point of S.

Since x is arbitrary $S' = R$.

(*v*) Let $\qquad S = \left\{a + \dfrac{1}{n} : a \in R, n \in N\right\} = \left\{a + 1, a + \dfrac{1}{2}, a + \dfrac{1}{3}, \ldots\ldots\right\}$

Let x be any real number.

Following cases arise :

Case (*i*) If $x < a$, then $(-\infty, a)$ is a *nbd* of x such that $\quad (-\infty, a) \cap S = \phi \quad \therefore \quad x \notin S'$

Case (*ii*) If $x > a + 1$, then $(a + 1, \infty)$ is a *nbd* of x such that $(a + 1, \infty) \cap S = \phi \quad \therefore \quad x \notin S'$

Case (*iii*) If $x = a + 1$, then $\left(a + \dfrac{1}{2}, \infty\right)$ is a *nbd* of x which contains only one element of S, namely

$$x = a + 1. \quad \therefore \quad x \notin S'.$$

Case (*iv*) If $a < x < a + 1$ then $0 < x - a < 1 \Rightarrow \dfrac{1}{x-a} > 0$

$\therefore$ $\exists$ unique $m \in N$ each that

$$m < \frac{1}{x-a} < m + 1 \Rightarrow \frac{1}{m} > x - a > \frac{1}{m+1}$$

$$\Rightarrow \qquad a + \frac{1}{m+1} < x < a + \frac{1}{m}$$

$\Rightarrow$ $\left(a + \dfrac{1}{m+1}, a + \dfrac{1}{m}\right)$ is a *nbd* of x and $\left(a + \dfrac{1}{m+1}, a + \dfrac{1}{m}\right) \cap S = \phi \quad \therefore \quad x \notin S'$.

Case (*v*) If $x = a$, then for every $\varepsilon > 0$, $\exists\, m \in N$ such that $m\varepsilon > 1$ *i.e.*, $\varepsilon > \dfrac{1}{m}$

$$\therefore \qquad a + \varepsilon > a + \frac{1}{m} \Rightarrow a - \varepsilon < a < a + \frac{1}{m} < a + \varepsilon$$

$\Rightarrow$ every *nbd* $(a - \varepsilon, a + \varepsilon)$ of a contains a point $a + \dfrac{1}{m}$ of S which is different from a.

$\Rightarrow \qquad a \in S'$

Hence a is the only limit point of S.

Note. $S = \left\{a + \dfrac{1}{n} : a \in R, n \in N\right\} \Rightarrow S' = \{a\}$.

Example 10. *Prove that the set of limit points of the set*

$$A = \left\{\frac{1}{p} + \frac{1}{q} : p, q \in N\right\} \textit{ is } B = \{0\} \cup \left\{\frac{1}{q} : q \in N\right\}.$$

Sol. Let x be any real number. Following cases arise :

Case (*i*) If $x < 0$, then $(x, 0)$ is a *nbd* of x and does not contain any positive number. As such, it does not contain any point of A.

i.e., $\qquad (x, 0) \cap A = \phi \qquad \therefore \qquad x \notin A'$.

Case (*ii*) If $x = 0$, then for every $\varepsilon > 0$, $\exists\, p \in N$ such that

$$\frac{\varepsilon}{2} > \frac{1}{q} \;\; \forall\, q \geq p \Rightarrow \varepsilon > \frac{1}{p} + \frac{1}{q} > 0 \;\; \forall\, q \geq p$$

$$\Rightarrow \qquad -\varepsilon < 0 < \frac{1}{p} + \frac{1}{q} < \varepsilon \;\; \forall\, q \geq p$$

$\Rightarrow$ every *nbd* $(-\varepsilon, \varepsilon)$ of 0 contains infinitely many points $\dfrac{1}{p} + \dfrac{1}{q}$ $(\forall\, q \geq p)$ of A.

$\Rightarrow \qquad 0 \in A'$.

Case (*iii*) If $x = \dfrac{1}{q}$, $q \in N$, then for every $\varepsilon > 0$, $\exists\, m \in N$ such that

$$\varepsilon > \frac{1}{p} \;\forall\, p \geq m \Rightarrow -\varepsilon < -\frac{1}{p} \;\forall\, p \geq m$$

$$\Rightarrow \qquad \frac{1}{q} - \varepsilon < \frac{1}{q} - \frac{1}{p} < \frac{1}{q} < \frac{1}{p} + \frac{1}{q} < \frac{1}{q} + \varepsilon$$

$\Rightarrow$ every *nbd* $\left(\dfrac{1}{q} - \varepsilon, \dfrac{1}{q} + \varepsilon\right)$ of $\dfrac{1}{q}$ contains infinitely many points $\dfrac{1}{p} + \dfrac{1}{q}$ $(\forall\, p \geq m)$ of A

$$\Rightarrow \qquad \frac{1}{q} \in A' \;\forall\, q \in N.$$

Case (*iv*) If $0 < x < 2$ and $x \neq \dfrac{1}{q}$, $q \in N$, then $\dfrac{1}{x} > 0$.

$\Rightarrow$ $\exists$ unique $n \in N$ such that

$$n + 1 > \frac{1}{x} > n \Rightarrow \frac{1}{n+1} < x < \frac{1}{n}$$

$$\Rightarrow \qquad x - \frac{1}{n+1} > 0 \Rightarrow \left(x - \frac{1}{n+1}\right)^{-1} > 0$$

$\Rightarrow$ $\exists\, m \in N$ such that $m + 1 > \left(x - \dfrac{1}{n+1}\right)^{-1} \geq m$

$$\Rightarrow \quad \frac{1}{m+1} < x - \frac{1}{n+1} \le \frac{1}{m} \quad \Rightarrow \quad \frac{1}{m+1} + \frac{1}{n+1} < x \le \frac{1}{m} + \frac{1}{n+1}$$

$\Rightarrow \left(\frac{1}{m+1} + \frac{1}{n+1}, 2\right)$ is a *nbd* of x and contains only finitely many points $\left(\frac{1}{p} + \frac{1}{q}\right)$ for $p = 1, 2, \ldots, m$; $q = 1, 2, \ldots, n$ except $p = m, q = n$ simultaneously) of A.

$\Rightarrow \qquad x \notin A'$.

Case (*v*) If $x \ge 2$, then $\left(x - \frac{1}{4}, x + \frac{1}{4}\right)$ is a *nbd* of x and contains at the most $2 \in A$.

$\Rightarrow \qquad x \notin A'$.

Hence $\qquad B = A' = \{0\} \cup \left\{\frac{1}{q} : q \in N\right\}$.

Example 11. *Find out the interior and closure of the following sets in R. Also find out which of the following sets are closed, open or dense in R. Justify your answer in each case.*

(*i*) *A* : ~~*a finite set*~~ (*ii*) *Q : the set of rationals*

(*iii*) *R – Q : the set of irrationals* (*iv*) *Z : the set of integers*

(*v*) *N : the set of natural numbers.*

Sol. (*i*) $A^\circ = \phi$

$\bar{A} = A \cup A' = A \cup \phi = A$

A is a closed set.

$\bar{A} \ne R \Rightarrow$ A is not dense in R.

(*ii*) $Q^\circ = \phi$

$\bar{Q} = Q \cup Q' = Q \cup R = R$

Q is neither open nor closed.

$\bar{Q} = R \Rightarrow$ Q is dense in R.

(*iii*) $(R - Q)^\circ = \phi$

$\overline{R - Q} = (R - Q) \cup (R - Q)' = (R - Q) \cup R = R$

R – Q is neither open nor closed.

$\overline{R - Q} = R \Rightarrow$ R – Q is dense in R.

(*iv*) $Z^\circ = \phi$

$\bar{Z} = Z \cup Z' = Z \cup \phi = Z$

Z is a closed set.

$\bar{Z} \ne R \Rightarrow$ Z is not dense in R.

(*v*) $N^\circ = \phi$

$\bar{N} = N \cup N' = N \cup \phi = N$

N is a closed set.

$\bar{N} \ne R \Rightarrow$ N is not dense in R.

Example 12. *Give one example of each of the following :*

(*i*) *an infinite set having no limit point.*

(*ii*) *an infinite set having one limit point.*

(*iii*) *a set having two limit points.*

(*iv*) *a set having an infinite number of limit points.*

(*v*) *a set every point of which is a limit point.*

(*vi*) *a set with only $\sqrt{2}$ as a limit point.*

(*vii*) *a set with only 0 as a limit point.*

Sol. (*i*) The set N of all natural numbers is an infinite set having no limit point.

(*ii*) The set $\left\{\frac{1}{n} : n \in N\right\}$ is an infinite set having only one limit point, namely 0.

(*iii*) The set $\left\{\frac{1}{n} : n \in N\right\} \cup \left\{1 - \frac{1}{n} : n \in N\right\}$ has two limit points, namely 0 and 1.

(*iv*) The sets Q, R (1, 2) have an infinite number of limits points.

(*v*) Every point of the closed interval [1, 2] is a limit point.

(*vi*) The set $\left\{\sqrt{2} + \frac{1}{n} : n \in N\right\}$ has only $\sqrt{2}$ as a limit point.

(*vii*) The set $\left\{\frac{1}{n} : n \in N\right\}$ has only 0 as a limit point.

Example 13. *Give an example of*

(*i*) *a set whose only limit point is not an element of the set.*

(*ii*) *a set having + 1 and – 1 as the only limit points.*

Sol. (*i*) The only limit point of the set $S = \left\{\frac{1}{n} : n \in N\right\}$ is 0 and $0 \notin S$.

(*ii*) The set $\left\{1 + \frac{1}{n} : n \in N\right\} \cup \left\{-1 + \frac{1}{n} : n \in N\right\}$ has only two limit points, + 1 and – 1.

Example 14. *Give an example of*

(*i*) *an unbounded set having limit points.*

(*ii*) *a bounded set having limit points.*

(*iii*) *an unbounded set having no limit point.*

(*iv*) *a bounded set having no limit point.*

(*v*) *an infinite set having a finite number of limit points.*

Sol. (*i*) The set Q is unbounded and $Q' = R$

(*ii*) The set [1, 2] is bounded and $[1, 2]' = [1, 2]$

(*iii*) The set Z is unbounded and $Z' = \phi$

(*iv*) Every finite set S is bounded and $S' = \phi$

(*v*) The set $S = \left\{\frac{1}{n} : n \in N\right\}$ is an infinite set having only one (*i.e.*, finitely many) limit point 0.

Example 15. *Give an example to show that for two sets A and B, the derived sets A′ and B′ can be equal without the sets being identical.*

Sol. Let $A = [1, 2]$ and $B = (1, 2)$

Then $A \neq B$ $[\because\ 1 \in A \text{ but } 1 \notin B]$

$A' = [1, 2],\ B' = [1, 2]$

$\therefore$ $A' = B'$ but $A \neq B$.

Example 16. (*i*) *Show that Q is dense in R.*

(*ii*) *Show that $Q^c = R - Q$ is dense in R.*

Sol.

[**Note.** Let A and B be two subsets of R such that $A \subset B$. Then A is said to be dense (or everywhere dense) in B if $B \subset \bar{A}$.

A set S is said to be dense in R if $R \subset \bar{S}$

But $\bar{S} \subset R$ $[\because\ R$ is the universal set$]$

$\therefore$ S is dense in R if $\bar{S} = R$.]

(*i*) Let x be any real number. Then for each $\varepsilon > 0$, however small, $(x - \varepsilon, x + \varepsilon)$ is a *nbd* of x and it contains infinitely many rational numbers. Consequently, x is a limit point of Q.

Since x is arbitrary, $Q' = R$

$$\therefore \quad \bar{Q} = Q \cup Q' = Q \cup R = R$$

$\Rightarrow$ Q is dense in R.

(*ii*) *Please try yourself.*

3.21. THEOREMS ON LIMIT POINTS

Theorem I. (*Bolzano-Weierstrass Theorem*)

Every infinite and bounded subset of R has a limit point.

Or

If S is an infinite bounded subset of R, then $S' \neq \phi$.

Proof. Let S be an infinite bounded subset of R.

(*i*) S is bounded $\Rightarrow$ $\exists$ real numbers k and K such that $k \le s \le K\ \forall\ s \in S$

(*ii*) Let a set T be defined as follows

$$T = \{t : t > \text{finitely many elements of } S\}.$$

(*iii*) **To prove that $T \neq \phi$.**

$k \le s\ \forall\ s \in S \Rightarrow k$ is greater than no element of S $\Rightarrow k \in T \Rightarrow T \neq \phi$

(*iv*) **To prove that T is bounded above.**

For any $\varepsilon > 0$, $K + \varepsilon > K \ge s \quad \forall\ s \in S \quad \Rightarrow K + \varepsilon \notin T, K \notin T$

$\Rightarrow \quad \forall\ t \in T, t < K \quad \Rightarrow$ T is bounded above.

$\because$ T is a non-empty bounded above subset of R

$\therefore$ T has the *l.u.b.* say u.

(*v*) **To prove that u is a limit point of S.**

Let $(u - \varepsilon, u + \varepsilon)$ be *any nbd* of u.

u is *l.u.b.* of T $\Rightarrow$ $\exists$ some $t \in T$ s.t. $t > u - \varepsilon$, $\varepsilon > 0$.

Now $t \in T \Rightarrow t >$ finitely many elements of S

$\Rightarrow u - \varepsilon >$ finitely many elements of S

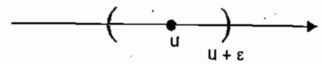
$\Rightarrow$ finitely many elements of S lie to the left of $u - \varepsilon$

$\Rightarrow$ **infinitely many elements of S lie to the right of $u - \varepsilon$** ...(1)

Also $u = l.u.b.$ of T $\Rightarrow u + \varepsilon \notin T$

$\Rightarrow u + \varepsilon >$ infinitely many elements of S

$\Rightarrow$ **infinitely many elements of S lie to the left of $u + \varepsilon$** ...(2)



Combining (1) and (2), $(u - \varepsilon, u + \varepsilon)$ has infinitely many elements of S. But $(u - \varepsilon, u + \varepsilon)$ is *any nbd* of u.

$\therefore$ Every *nbd* of u has infinitely many elements of S.

Hence u is a limit point of S.

Remark. We know that a finite set has no limit point. Also, the set Z is unbounded and has no limit point. Therefore, the conditions 'infinite and bounded' in the statement of the theorem are necessary.

Theorem II. *If S is an infinite bounded above subset of R and $u = l.u.b.$ S such that $u \notin S$, then $u \in S'$, i.e., u is a limit point of S.*

Proof. For each $\varepsilon > 0$, $(u - \varepsilon, u + \varepsilon)$ is a *nbd* u.

Since u = l.b.u. S, $u - \varepsilon$ is not an upper bound of S.

$\therefore$ $\exists$ some $x \in S$ such that $x > u - \varepsilon$.

Also $x < u < u + \varepsilon$ $[\because\ x \in S$ and $u \notin S, x \neq u]$

$\therefore$ $u - \varepsilon < x < u + \varepsilon$

$\Rightarrow$ $(u - \varepsilon, u + \varepsilon) \cap S - \{u\}$ contains at least one point x of S.

$\Rightarrow$ u is a limit point of S.

Theorem III. *If S is an infinite bounded below subset of R and $l = g.l.b.$ S such that $l \notin S$, then $l \in S'$, i.e., l is a limit point of S.*

Proof. For each $\varepsilon > 0$, $(l - \varepsilon, l + \varepsilon)$ is a *nbd* of l,

Since, $l = g.l.b.$ S, $l + \varepsilon$ is not a lower bound of S.

$\therefore$ $\exists$ some $x \in S$ such that $x < l + \varepsilon$.

Also $x > l > l - \varepsilon$ $[\because\ x \in S$ and $l \notin S, x \neq l]$

$\therefore$ $l - \varepsilon < x < l + \varepsilon$.

$\Rightarrow$ $(l - \varepsilon, l + \varepsilon) \cap S - \{l\}$ contains at least one point x of S.

$\Rightarrow$ l is a limit point of S.

Theorem IV. *The derived set of an infinite bounded subset of R is bounded.*

Proof. Let S be an infinite bounded subset of R, then $\exists$ real numbers h, k such that $S \subset [h, k]$.

Since S is infinite and bounded, $S' \neq \phi$. (Bolzano-Weierstrass Theorem)

We shall show that no element of S′ (*i.e.*, no limit point of S) is less than h or greater than k.

If $x < h$, then for $\varepsilon = h - x > 0$, $(x - \varepsilon, x + \varepsilon)$ is a *nbd* of x containing no element of $[h, k]$ and hence, containing no element of S.

$\therefore$ $x \notin S'$

If $x > k$, then for $\varepsilon = x - k > 0$, $(x-\varepsilon, x+\varepsilon)$ is a *nbd* of x containing no element of $[h, k]$ and hence, containing no element of S.

$\therefore \quad x \notin S'$

Thus $\quad x \notin [h, k] \Rightarrow x \notin S'$

$\Rightarrow$ All the limit points of S lie in $[h, k]$

$\Rightarrow \quad S' \subset [h, k]$

$\Rightarrow$ S′ is bounded.

Note. Bounds of S are also the bounds of S′.

Remark. An unbounded set may not have a bounded derived set. For example, the set Q is unbounded and $Q' = R$ is also unbounded.

Theorem V. *If S is an infinite bounded set and $l = g.l.b.\ S$, $u = l.u.b.\ S$, then $S' \subset [l, u]$.*

Proof. See Theorem IV. Here $S \subset [l, u]$.

Theorem VI. *Every infinite bounded set has the greatest and the smallest limit points, i.e., the derived set of an infinite bounded set attains its bounds.*

Proof. Let S be an infinite bounded set, then $\exists$ real numbers h, k such that $S \subset [h, k]$.

$S \subset [h, k] \Rightarrow S' \subset [h, k]$

Since S is infinite and bounded, $S' \neq \phi$. (Bolzano-Weierstrass Theorem)

Now S′ is non-empty and bounded.

$\therefore$ By completness axiom, S′ has infimum as well as supermum.

Let $\quad \inf. S' = l$ and $\sup S' = u$.

We shall show that l and u are limit points of S, *i.e.*, $l, u \in S'$.

$l = \inf. S' \quad \Rightarrow$ for any $\varepsilon > 0$, $\exists$ some $x \in S'$ such that

$l \le x < l + \varepsilon \quad \Rightarrow \quad l - \varepsilon < l \le x < l + \varepsilon$

$\Rightarrow \quad x \in (l-\varepsilon, l+\varepsilon) \quad \Rightarrow (l-\varepsilon, l+\varepsilon)$ is a *nbd* of $x \in S'$.

$\Rightarrow$ $(l-\varepsilon, l+\varepsilon)$ is a *nbd* of a limit point of S.

$\Rightarrow$ $(l-\varepsilon, l+\varepsilon)$ contains infinitely many elements of S.

$\Rightarrow$ every *nbd* of l contains infinitely many elements of S.

$\Rightarrow$ l is a limit point of S.

$\Rightarrow$ $l \in S'$. Similarly $u \in S'$.

Note. The smallest and the greatest members l and u of the derived set S′ of an infinite and bounded set S always exist. They are usually denoted by $\underline{\lim}$ S and $\overline{\lim}$ S respectively and are called the inferior (or lower) limit of S and the superior (or upper) limit of S.

Theorem VII. *If $A, B \subset R$, then*

(i) $A \subset B \Rightarrow A' \subset B'$ *(ii)* $(A \cup B)' = A' \cup B'$

(iii) $(A \cap B)' \subset A' \cap B'$ *(iv)* $A'' \subset A'$.

Proof. *(i)* If $A' = \phi$ then $A' \subset B'$, since empty set is a subset of every set.

If $A' \neq \phi$, let $x \in A'$ and N be any *nbd* of x.

$\Rightarrow$ N contains infinitely many points of A.

$\Rightarrow$ N contains infinitely many points of B. $(\because A \subset B)$

$\Rightarrow$ x is a limit point of B. *i.e.*, $x \in B'$

Since $\quad x \in A' \Rightarrow x \in B' \quad \therefore \quad A' \subset B'$

(ii) Since $\quad A \subset A \cup B$ and $B \subset A \cup B$

$\therefore \quad A' \subset (A \cup B)'$ and $B' \subset (A \cup B)'$ [By Theorem VII *(i)*]

Consequently $\quad A' \cup B' \subset (A \cup B)'$ ...*(i)*

Now we proceed to show that $(A \cup B)' \subset A' \cup B'$.

If $(A \cup B)' = \phi$ then evidently $(A \cup B)' \subset A' \cup B'$.

If $(A \cup B)' \neq \phi$, let $x \in (A \cup B)'$.

$\Rightarrow$ x is a limit point of $A \cup B$.

$\Rightarrow$ every *nbd* of x contains infinitely many points of $A \cup B$.

$\Rightarrow$ every *nbd* of x contains infinitely many points of A or of B.

$\Rightarrow$ x is a limit point of A or a limit point of B.

$\Rightarrow \quad x \in A'$ or $x \in B' \Rightarrow x \in A' \cup B'$

Since $\quad x \in (A \cup B)' \Rightarrow x \in A' \cup B'$

$\therefore \quad (A \cup B)' \subset A' \cup B'$ ...(2)

From (1) and (2), it follows that $(A \cup B)' = A' \cup B'$

Thus the derived set of the union of sets is ~~equal to the union~~ of their derived sets.

Remark. To show that $(A \cup B)' \subset A' \cup B'$, we can also proceed as follows.

If possible, suppose $(A \cup B)' \not\subset A' \cup B'$, then $\exists$ some $x \in (A \cup B)'$ such that $x \notin A' \cup B'$.

$\Rightarrow$ $x \notin A'$ and $x \notin B'$

$\Rightarrow$ x is not a limit point of A and x is not a limit point of B.

$\Rightarrow$ $\exists$ *nbds* N_1, N_2 of x such that N_1 contains no point of A other than x and N_2 contains no point of B other than x *i.e.*, $N_1 \cap A - \{x\} = \phi$ and $N_2 \cap B - \{x\} = \phi$

Since $\quad N_1 \cap N_2 \subset N_1$ and $N_1 \cap N_2 \subset N_2$

$\therefore \quad (N_1 \cap N_2) \cap A - \{x\} = \phi$ and $(N_1 \cap N_2) \cap B - \{x\} = \phi$

$\Rightarrow$ $\exists$ a *nbd* $N_1 \cap N_2$ of x which contains no point other than x of A or B and thus of $A \cup B$

$\Rightarrow$ x is not a limit point of $(A \cup B)$

$\Rightarrow$ $x \notin (A \cup B)'$ which leads to a contradiction. [$\because$ we started with $x \in (A \cup B)'$]

$\Rightarrow$ Our assertion that $(A \cup B)' \not\subset A' \cup B'$ is wrong

$\Rightarrow \quad (A \cup B)' \subset A' \cup B'$.

(iii) $\quad A \cap B \subset A \Rightarrow (A \cap B)' \subset A'$

$A \cap B \subset B \Rightarrow (A \cap B)' \subset B'$

$\therefore \quad (A \cap B)' \subset A' \cap B'$

Note. *$(A \cap B)'$ and $A' \cap B'$ may not be equal.*

For example, let $A = (1, 2)$, $B = (2, 3)$, then $A \cap B = \phi$ $\therefore$ $(A \cap B)' = \phi$

Also $\quad A' = [1, 2]$, $B' = [2, 3]$

$\therefore \quad A' \cap B' = \{2\}$ so that $(A \cap B)' \neq A' \cap B'$.

Remark. A′ is the set of all limit points of A.

A″ is the set of all limit points of A′.

For example, $Q' = R \Rightarrow Q'' = R' = R$

(iv) If $A'' = \phi$, then $A'' \subset A'$ since $\phi \subset A'$.

If $A'' \neq \phi$, let $x \in A''$. Then x is a limit point of A′.

$\Rightarrow$ For any $\varepsilon > 0$, $(x-\varepsilon, x+\varepsilon) \cap A' - \{x\} \neq \phi$

Let $y \in (x-\varepsilon, x+\varepsilon) \cap A' - \{x\}$, then $y \in (x-\varepsilon, x+\varepsilon) \cap A'$ and $y \notin \{x\}$

$\Rightarrow \quad y \in (x-\varepsilon, x+\varepsilon), y \in A'$ and $y \neq x$.

Now $(x-\varepsilon, x+\varepsilon)$ being an open interval is a *nbd* of each of its points.

$\Rightarrow$ $(x-\varepsilon, x+\varepsilon)$ is a *nbd* of y

Also $y \in A'$, so that every *nbd* of y contains infinitely many points of A

$\Rightarrow$ $(x-\varepsilon, x+\varepsilon)$ contains infinitely many points of A

$\Rightarrow$ $(x-\varepsilon, x+\varepsilon) \cap A$ is an infinite set

$\Rightarrow$ $(x-\varepsilon, x+\varepsilon) \cap A - \{x\} \neq \phi$

$\Rightarrow$ x is a limit point of A $\Rightarrow$ $x \in A'$

Since $\quad x \in A'' \Rightarrow x \in A' \quad \therefore \quad A'' \subset A'$.

Theorem VIII. *A is bounded* $\Rightarrow$ $\bar{A}$ *is bounded.*

Proof. A is bounded, therefore, $\exists$ real numbers h, k such that $A \subset [h, k]$

Also $\quad A \subset [h, k] \Rightarrow A' \subset [h, k]$ [Theorem IV]

$\therefore \quad A \cup A' \subset [h, k]$

$\Rightarrow \quad \bar{A} \subset [h, k] \quad \therefore \quad \bar{A}$ is bounded.

Theorem IX. *For any set A, A′ is a closed set.*

Proof. To prove that A′ is a closed set, we shall prove that $(A')^c$ is an open set.

Let x be any element of $(A')^c$ $\quad x \in (A')^c \Rightarrow x \notin A'$

$\Rightarrow$ x is not a limit point of A

$\Rightarrow$ $\exists$ a *nbd* $I = (x-\varepsilon, x+\varepsilon)$ of x such that $I \cap A - \{x\} = \phi$

Let $y \in I$, then I being an open interval is an open set

$\Rightarrow$ I is a *nbd* of y. Also $I \cap A - \{x\} = \phi$

$\Rightarrow$ y is not a limit point of A $\Rightarrow y \notin A' \Rightarrow y \in (A')^c$

Since $\quad y \in I \Rightarrow y \in (A')^c$

$\therefore \quad I = (x-\varepsilon, x+\varepsilon) \subset (A')^c$

$\Rightarrow$ $(A')^c$ is a *nbd* of x.

Since x is any element of $(A')^c$, it follows that $(A')^c$ is a *nbd* of each of its points.

$\Rightarrow$ $(A')^c$ is an open set $\Rightarrow$ A′ is a closed set.

Theorem X. *For any set A,* $\bar{A}$ *is closed.*

Proof. To prove that $\bar{A}$ is closed, we shall show that $(\bar{A})^c$ is open.

Let x be any element of $(\bar{A})^c$.

$$x \in (\bar{A})^c \Rightarrow x \notin \bar{A} \Rightarrow x \notin A \cup A' \Rightarrow x \notin A \text{ and } x \notin A'$$

$\Rightarrow \quad \exists$ a *nbd* $I = (x-\varepsilon, x+\varepsilon)$ of x such that $I \cap A = \phi$

Let $y \in I$, then I being an open interval is an open set.

$\Rightarrow$ I is a *nbd* of y. Also $I \cap A = \phi$

$\Rightarrow$ y is not a limit point of A

$\Rightarrow$ $y \notin A'$. Also $y \notin A$.

$y \notin A$ and $y \notin A' \Rightarrow y \notin A \cup A' \Rightarrow y \notin \bar{A} \Rightarrow y \in (\bar{A})^c$

Since $\quad y \in I \Rightarrow y \in (\bar{A})^c$

$\therefore \quad I = (x-\varepsilon, x+\varepsilon) \subset (\bar{A})^c$

$\Rightarrow$ $(\bar{A})^c$ is a *nbd* of x

$\Rightarrow$ $(\bar{A})^c$ is an open set $\quad$ ($\because$ x is arbitrary)

$\Rightarrow$ $\bar{A}$ is a closed set.

Theorem XI. $A \subset B \Rightarrow \bar{A} \subset \bar{B}$.

Proof. $\quad A \subset B \Rightarrow A' \subset B'$ [Theorem VII (*i*)]

Let x be any element of $\bar{A}$. Then

$$x \in \bar{A} \Rightarrow x \in A \cup A' \Rightarrow x \in A \text{ or } x \in A'$$

$\Rightarrow \quad x \in B$ or $x \in B'$ $\quad$ [$\because$ $A \subset B, A' \subset B'$]

$\Rightarrow \quad x \in B \cup B' \Rightarrow x \in \bar{B}$

Hence $\quad \bar{A} \subset \bar{B}$.

Theorem XII. *A is a closed set iff* $A = \bar{A}$.

i.e., A is a closed set iff A contains all its limit points.

Proof. If $A = \bar{A}$, then A is closed because $\bar{A}$ is closed. [Theorem X]

Conversely, let A be a closed set. We have to prove that

$$A = \bar{A}.$$

$$\bar{A} = A \cup A' \supset A \Rightarrow A \subset \bar{A} \qquad ...(1)$$

[To prove that $\bar{A} \subset A$ *i.e.*, $A \cup A' \subset A$, we shall prove that $A' \subset A$, since $A \subset A$ always]

If $A' = \phi$ then $A' \subset A$

If $A' \neq \phi$, let $x \in A'$

Suppose $x \notin A$, then $x \in A^c$

$\because$ A is a closed set.

$\therefore$ A^c is an open set.

$\therefore$ A^c is a *nbd* of x.

Also $x \in A'$ $\Rightarrow$ x is a limit point of A

$\Rightarrow$ every *nbd* of x contains infinitely many points of A

$\Rightarrow$ A^c contains infinitely many points of A

$\Rightarrow$ $A^c \cap A \neq \phi$

which of course is wrong. Thus, our supposition that $x \notin A$ does not hold.

$\therefore \quad x \in A$

Since $\quad x \in A' \Rightarrow x \in A \quad \therefore \quad A' \subset A$

$\Rightarrow$ Also $\quad A \subset A \quad \therefore \quad A \cup A' \subset A$

$\Rightarrow \quad \bar{A} \subset A \qquad ...(2)$

Combining (1) and (2), $A = \bar{A}$.

Theorem XIII. *For any set A,* $\bar{A}$ *is the smallest closed set containing A.*

Proof. We know that for any set A, $\bar{A}$ is closed and $\bar{A} \supset A$.

Let B be **any other** closed set containing A.

$$B \supset A \Rightarrow A \subset B \Rightarrow \bar{A} \subset \bar{B}$$

$\Rightarrow \qquad \bar{A} \subset B \qquad [\because$ B is closed $\therefore B = \bar{B}]$

Hence $\bar{A}$ is the smallest closed set containing A.

Remark. In view of Theorem XIII, the second part of Theorem XII can be proved as follows.

$\bar{A}$ is the smallest closed set containing A.
Also A is a closed set and $A \supset A$

$\therefore \qquad \bar{A} \subset A$

Theorem XIV. *A non-empty bounded closed set contains its supremum as well as infimum.*

Proof. Let A be a non-empty bounded closed set.

If A is finite, then sup A = its greatest member and inf A = its smallest member, both of which belong to A.

If A is infinite, let sup A = u

If $u \notin A$, then $\qquad u \in A'$

But $\qquad A' \subset A \qquad [\because$ A is closed$]$

$\therefore\ u \in A$, which is a contradiction.

Hence $u \in A$. Similarly inf A = $l \in A$.

Theorem XV. *For any set A,* $\overline{(\bar{A})} = \bar{A}$

Proof. We know that for any set A, $\bar{A}$ is a closed set.

$\therefore \qquad (\bar{A})' \subset \bar{A} \qquad$ [a closed set contains all its limit points]

Now $\qquad \overline{(\bar{A})} = \bar{A} \cup (\bar{A})' \qquad [\because \bar{B} = B \cup B']$

$\qquad = \bar{A} \qquad [\because B \subset A \Rightarrow A \cup B = A]$

Hence $\qquad \overline{(\bar{A})} = \bar{A}$.

Theorem XVI. *If A, B are subsets of R, then*

(*i*) $\overline{A \cup B} = \bar{A} \cup \bar{B}$ $\qquad$ (*ii*) $\overline{A \cap B} \subset \bar{A} \cap \bar{B}$

Proof. (*i*)

$$\begin{aligned}
\overline{A \cup B} &= (A \cup B) \cup (A \cup B)' && [\because \bar{S} = S \cup S'] \\
&= (A \cup B) \cup (A' \cup B') && [\because (A \cup B)' = A' \cup B'] \\
&= A \cup (B \cup A') \cup B' && \text{[union of sets is associative]} \\
&= A \cup (A' \cup B) \cup B' && \text{[union of sets is commutative]} \\
&= (A \cup A') \cup (B \cup B') && \text{[union of sets is associative]} \\
&= \bar{A} \cup \bar{B} && [\because S \cup S' = \bar{S}]
\end{aligned}$$

(*ii*) $\qquad A \cap B \subset A \Rightarrow \overline{A \cap B} \subset \bar{A}$

$\qquad A \cap B \subset B \Rightarrow \overline{A \cap B} \subset \bar{B}$

$\therefore \qquad \overline{A \cap B} \subset \bar{A} \cap \bar{B}$

The inclusion cannot be replaced by equality. For example,

if $\quad A = (0, 1)$ and $B = (1, 2)$, then $A \cap B = \phi \ \therefore\ \overline{A \cap B} = \phi$

Also $\bar{A} = [0, 1]$ and $\bar{B} = [1, 2]$, so that $\bar{A} \cap \bar{B} = \{1\}$

Thus $\quad \overline{A \cap B} \neq \bar{A} \cap \bar{B}$.

Theorem XVII. *If $A \subset R$, then $\bar{A}$ is the intersection of all closed sets containing A.*

Proof. Let $\{F_\lambda\}_{\lambda \in \Lambda}$ be the family of closed sets containing A and $S = \bigcap_{\lambda \in \Lambda} F_\lambda$.

Since $\bar{A}$ is a closed set containing A. $\therefore \bar{A} \in \{F_\lambda\}_{\lambda \in \Lambda}$.

Also S is the intersection of the family.

$\Rightarrow$ S is a subset of every member of the family

$\Rightarrow \qquad S \subset \bar{A} \qquad$ ...(1)

Now, $\forall \lambda \in \Lambda$, F_λ is a closed set and since the intersection of every family of closed sets is closed, therefore S is a closed set.

Also $\qquad \forall \lambda \in \Lambda, F_\lambda \supset A \Rightarrow S \supset A$

Thus S is a closed set containing A.

But $\bar{A}$ is the smallest closed set containing A

$\therefore \qquad \bar{A} \subset S \qquad$ ...(2)

From (1) and (2), we have $\bar{A} = S$.

Theorem XVIII. *The g.l.b. and the l.u.b. of a set A are also the g.l.b. and the l.u.b. of $\bar{A}$ and are contained in $\bar{A}$ according as A is bounded below or above.*

Proof. Let A be a bounded below subset of R. Then A has the g.l.b. l (say).

If $l \in A$, then $\qquad l \in \bar{A} = A \cup A'$

If $l \notin A$, then $l \in A'$ so that $\quad l \in \bar{A} = A \cup A'$

Thus $l \in \bar{A}$ in both cases.

Now A is bounded below by l

$\Rightarrow$ A′ is bounded below by l. $\qquad (\because$ bounds of A are bounds of A′)

$\Rightarrow \bar{A} = A \cup A'$ is bounded below by l

$\Rightarrow \bar{A}$ has g.l.b. l_1 (say).

$\Rightarrow \qquad l_1 \in \overline{(\bar{A})} = \bar{A}$

[since in the first part, we have proved that $\Rightarrow l = \text{g.l.b. } A \Rightarrow l \in \bar{A}$]

Now $\qquad A \subset \bar{A} \Rightarrow \text{g.l.b. } \bar{A} \leq \text{g.l.b. } A \Rightarrow l_1 \leq l \qquad [\because \forall x \in A, x \geq l]$

If possible, let $l_1 < l$, then $l_1 \notin A$

Take $\varepsilon = \dfrac{l - l_1}{2} > 0$, then $(l_1 - \varepsilon, l_1 + \varepsilon)$ is a *nbd* of l_1 and contains no point of A,

i.e., $\qquad (l_1 - \varepsilon, l_1 + \varepsilon) \cap A = \phi$

$\therefore \quad l_1 \notin A'$. Also $l_1 \notin A$

$\Rightarrow \quad l_1 \notin \bar{A}$ which is a contradiction. $\qquad [\because l_1 \in \bar{A}]$

$\Rightarrow \quad l_1 \nless l \ \therefore\ l_1 = l$

Hence $\quad \text{g.l.b. } A = \text{g.l.b. } \bar{A} \in \bar{A}$

When A is bounded above, one can similarly show that $\text{l.u.b. } A = \text{l.u.b. } \bar{A} \in \bar{A}$.

Theorem XIX. *A set is closed iff its complement is open.*

Proof. Let A be a closed set. We shall show that A^c is open.

Let x be any element of A^c.

$x \in A^c \Rightarrow x \notin A$

$\Rightarrow x \notin A'$ [$\because$ A is closed $\therefore$ $A' \subset A$]

$\Rightarrow x$ is not a limit point of A

$\Rightarrow \exists$ a *nbd* N of x such that $N \cap A = \phi$

$\Rightarrow$ N contains elements of A^c only

$\Rightarrow N \subset A^c$

$\Rightarrow A^c$ is a *nbd* of x

$\Rightarrow A^c$ is a *nbd* of each of its points ($\because$ x is arbitrary)

$\Rightarrow A^c$ is open.

Conversely, let A^c be an open set. We shall show that A is closed (*i.e.*, A contains all its limit points, *i.e.*, $A' \subset A$).

Suppose $x \in A'$. If possible, let $x \in A^c$.

$\because$ A^c is open $\therefore$ A^c is a *nbd* of x

$\Rightarrow \exists$ a *nbd* N such that $x \subset N \subset A^c$

$\Rightarrow N \cap A = \phi$

$\Rightarrow x \notin A'$ which is a contradiction.

$\therefore$ $x \in A' \Rightarrow x \notin A^c$

$\Rightarrow$ $x \in A' \Rightarrow x \in A$

$\Rightarrow$ $A' \subset A$

$\Rightarrow$ A contains all its limit points

$\Rightarrow$ A is closed.

3.22. COMPACT SET

A non-empty subset of R is said to be compact if it is closed and bounded.

For example. (*i*) Every finite set being closed and bounded is a compact set.

(*ii*) The closed interval $[a, b]$, where $a, b \in R$, being closed and bounded is a compact set. No other type of interval is compact.

(*iii*) Consider the set $A = (-1, 1) \cup \left\{-1-\frac{1}{n} : n \in N\right\} \cup \left\{1+\frac{1}{n} : n \in N\right\}$

Since $A \subset [-2, 2]$, A is bounded.

The only limit points of A are -1 and 1, both of which belong to A. Since A contains all its limit points, A is closed.

A is bounded and closed $\Rightarrow$ A is compact.

Example 1. *Prove that the union of a finite family of compact sets is compact.*

Sol. Let $\{A_i\}_{1 \le i \le n}$ be a finite family of compact sets. Then each A_i is closed and bounded, $1 \le i \le n$.

Let $S = \bigcup_{i=1}^{n} A_i$.

Since the union of a finite family of closed sets is a closed set,

$\therefore$ *S is closed.*

Also $A_i \subset [a_i, b_i]$, $1 \le i \le n$.

If $a = \min. \{a_1, a_2, ..., a_n\}$ and $b = \max. \{b_1, b_2, ..., b_n\}$

then $S \subset [a, b]$ $\therefore$ *S is bounded.*

Now S is closed and bounded $\Rightarrow$ S is compact.

Example 2. *Prove that the intersection of an arbitrary family of compact sets, containing at least one point in common, is compact.*

Sol. Let $\{A_\lambda\}_{\lambda \in \Lambda}$ be an arbitrary family of compact sets. Then each A_λ is closed and bounded for $\lambda \in \Lambda$.

Let $S = \bigcap_{\lambda \in \Lambda} A_\lambda$.

Since the intersection of an arbitrary family of closed sets is a closed set. $\therefore$ S is closed.

Also $S \subset A_\lambda$ $\forall$ $\lambda \in \Lambda$ and each A_λ is bounded

$\therefore$ S is bounded.

Now S is closed and bounded $\Rightarrow$ S is compact.

Example 3. *Given $S = \{1, 1.1, 0.9, 1.01, 0.99, 1.001, 0.999, ...\}$*

(*i*) *Is the set S bounded ?*

(*ii*) *Does the set S have l.u.b. and g.l.b. ? If so, determine them.*

(*iii*) *Does the set S attain its bounds ?*

(*iv*) *Find the interior of S.*

(*v*) *Does the set S have any limit point ? If so, determine them.*

(*vi*) *Is S closed ?*

(*vii*) *Is S a compact set ?*

Sol. (*i*) $S = \{1\} \cup \left\{1 \pm \frac{1}{10^n} : n \in N\right\} \subset [0.9, 1.1] \Rightarrow$ S is bounded.

(*ii*) S is a non-empty bounded subset of R

$\therefore$ S has the l.u.b. and the g.l.b.

1.1 is an upper bound of S and $1.1 \in S$ $\Rightarrow$ l.u.b. S = 1.1

0.9 is a lower bound of S and $0.9 \in S$ $\Rightarrow$ g.l.b. S = 0.9

(*iii*) Yes. l.u.b. S = $1.1 \in S$ and g.l.b. S = $0.9 \in S$.

(*iv*) Let $x \in S$. For any $\varepsilon > 0$, $(x-\varepsilon, x+\varepsilon)$ is a *nbd* of x. Since $(x-\varepsilon, x+\varepsilon)$ contains infinitely many points which are not in S.

$\therefore$ S is not a *nbd* of x.

$\Rightarrow$ S is not a *nbd* of any of its points. $\Rightarrow$ $S° = \phi$

(*v*) Yes. The set has one limit point, namely 1.

(*vi*) Since the only limit point $1 \in S$, $\therefore$ S is closed.

(*vii*) S is closed and bounded $\Rightarrow$ S is a compact set.

Example 4. *Which of the following sets are compact :*

(*i*) $[0, 1]$ (*ii*) $[-1, 1] \cup [2, 3]$ (*iii*) N

(*iv*) Z (*v*) Q (*vi*) $(0, 5]$

(*vii*) $[2, \infty)$.

Sol. (*i*) The set [0, 1] is closed and bounded. $\Rightarrow$ [0, 1] is a compact set

(*ii*) The set $[-1, 1] \cup [2, 3]$ being the union of two closed and bounded sets is closed and bounded.

$\Rightarrow$ $[-1, 1] \cup [2, 3]$ is a compact set.

(*iii*) N is closed but not bounded. $\Rightarrow$ N is not a compact set.

(*iv*) Z is closed but not bounded. $\Rightarrow$ Z is not a compact set.

(*v*) Q is neither closed nor bounded. $\Rightarrow$ Q is not a compact set.

(*vi*) The set (0, 5] is bounded but not closed $\Rightarrow$ (0, 5] is not a compact set.

(*vii*) The set $[2, \infty)$ is neither closed nor bounded $\Rightarrow$ $[2, \infty)$ is not a compact set.

Note. A set S is compact if it is both bounded *and* closed.

A set S is not compact if it is *either* not bounded *or* not closed.

Example 5. *Give an example of a closed non-compact subset of R.*

Sol. N (or Z) is a closed subset of R. Since N (or Z) is not bounded, N (or Z) is not compact.

Example 6. *Show by means of an example that a bounded subset of R may fail to be compact.*

Sol. (0, 1] is a bounded subset of R. Since it is not closed, it is not compact.

Example 7. *Show that the set $\{1^2, 2^2, 3^2, ..., (123)^2\}$ is compact.*

Sol. Let $A = \{1^2, 2^2, 3^2, ..., (123)^2\}$

Since A is finite subset of R, A is closed and bounded.

$\Rightarrow$ A is compact.

3.23. THEOREM

A set is compact if and only if every infinite subset of the set has a limit point in the set.

Proof. Let A be a compact set, then A is closed and bounded.

Let B be an infinite subset of A.

We have to prove that B has a limit point in A

Now $B \subset A$ and A is bounded $\Rightarrow$ B is bounded.

B is infinite and bounded $\Rightarrow$ $B' \neq \phi$ (Bolzano Weierstrass Theorem)

Let p be a limit point of B.

Also $B \subset A$ $\Rightarrow$ $B' \subset A'$

$\therefore$ $p \in B'$ $\Rightarrow$ $p \in A'$

Since A is a closed set, $A' \subset A$ $\therefore$ $p \in A$

Hence $p \in B'$ $\Rightarrow$ $p \in A$

Conversely. Let A be a set such that every infinite subset of A has a limit point in A. We shall show that A is closed and bounded.

If A is not closed, then A does not contain all its limit points. Therefore, $\exists$ a real number x_0 which is a limit point of A but not a point of A.

Now, for every $\cancel{n \in N}$, $\exists$ points $x_n \in A$ such that

$$|x_n - x_0| < \frac{1}{n} \quad i.e., \quad x_0 - \frac{1}{n} < x_n < x_0 + \frac{1}{n}$$

i.e., $$x_n \in \left(x_0 - \frac{1}{n}, x_0 + \frac{1}{n}\right)$$

Since $x_0 \notin A$ and $x_n \in A$, $x_n \neq x_0$

Let $B = \{x_n : n \in N\}$, then $B \subset A$ and B is infinite.

For each $n \in N$, $\left(x_0 - \frac{1}{n}, x_0 + \frac{1}{n}\right)$ has a point x_n of A and hence of B other than x_0.

$\Rightarrow$ x_0 is a limit point of B.

We shall show that x_0 is the only limit point of B.

If possible, let $p \neq x_0$ be also a limit point of B.

Suppose $p > x_0$. Let $\varepsilon = p - x_0 > 0$

Consider the *nbd* $\left(p - \frac{\varepsilon}{3}, p + \frac{\varepsilon}{3}\right)$ of p.

$\exists\, m \in N$ such that $\frac{1}{n} < \frac{\varepsilon}{3}\ \forall\, n \geq m$

$\therefore$ $$-\frac{1}{n} > -\frac{\varepsilon}{3}$$

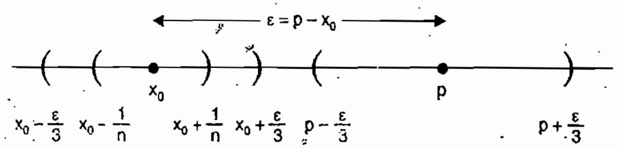
$$x_0 + \frac{1}{n} < x_0 + \frac{\varepsilon}{3}$$

$$x_0 - \frac{1}{n} < x_0 - \frac{\varepsilon}{3}$$

Thus $\left(x_0 - \frac{1}{n}, x_0 + \frac{1}{n}\right) \subset \left(x_0 - \frac{\varepsilon}{3}, x_0 + \frac{\varepsilon}{3}\right)\ \forall\, n \geq m$

$\Rightarrow$ $$x_n \in \left(x_0 - \frac{\varepsilon}{3}, x_0 + \frac{\varepsilon}{3}\right)\ \forall\, n \geq m$$

$\Rightarrow$ Infinitely many points of B lie in $\left(x_0 - \frac{\varepsilon}{3}, x_0 + \frac{\varepsilon}{3}\right)$



Also $$x_0 + \frac{\varepsilon}{3} = p - \varepsilon + \frac{\varepsilon}{3} = p - \frac{2}{3}\varepsilon < p - \frac{\varepsilon}{3} < p + \frac{\varepsilon}{3}$$

$\Rightarrow$ $$\left(x_0 - \frac{\varepsilon}{3}, x_0 + \frac{\varepsilon}{3}\right) \cap \left(p - \frac{\varepsilon}{3}, p + \frac{\varepsilon}{3}\right) = \phi$$

$\therefore$ $$x_n \in \left(p - \frac{\varepsilon}{3}, p + \frac{\varepsilon}{3}\right)$$

for at the most a finite number of values of n, namely 1, 2, 3,......, $m - 1$.

$\Rightarrow$ a *nbd* of p contains at the most a finite number of points of B.

$\Rightarrow$ p is not a limit point of B.

Thus x_0 is the only limit point of an infinite subset B of A. By the given hypothesis, $x_0 \in A$ which contradicts the assumption that $x_0 \notin A$.

$\Rightarrow$ Our supposition is wrong.

$\Rightarrow$ A contains all its limit points.

$\Rightarrow$ A is closed.

Now, suppose A is not a bounded set. Let A be not bounded above.

Find $x_1 \in A$ such that $x_1 > 1$. Then find $x_2 \in A$ such that $x_2 > \max.\{2, x_1\}$. Continuing like this, we obtain an increasing sequence $<x_n>$ of distinct elements of A such that $x_n > n$ for each $n \in N$.

The set $B = \{x_n : n \in N\}$ is an infinite subset of A and has no limit point.

[Because if x is a limit point of B, then infinitely many elements of B lie in $(x - 1, x + 1)$. But if m is an integer $> x + 1$, then for all $n \geq m$, $x_n \geq x_m > m > x + 1$. That is at most $x_1, x_2, \ldots\ldots, x_{m-1}$ can belong to $(x - 1, x + 1)$, which is a contradiction].

This contradicts the given hypothesis that every infinite subset of A has a limit point in A. Therefore, A is bounded above. Similarly we can show that A is bounded below.

Thus A is a bounded set.

Hence A is closed and bounded *i.e.*, A is a compact set.

3.24. OPEN COVER

Let S be a non-empty subset of R. A family $\{A_\lambda\}_{\lambda \in \Lambda}$ of subsets of R is said to be a *cover* of S if

$$\bigcup_{\lambda \in \Lambda} A_\lambda \supset S$$

Thus $\qquad x \in S \Rightarrow x \in \bigcup_{\lambda \in \Lambda} A_\lambda$.

$\Rightarrow$ $x \in A_\lambda$ for at least one $\lambda \in \Lambda$ *i.e.*, every point of S is covered by at least one member of the family. Hence the word *cover* has literal meaning.

If Λ is a finite set, the cover $\{A_\lambda\}_{\lambda \in \Lambda}$ is said to be a *finite cover*. If Λ is an infinite set, the cover $\{A_\lambda\}_{\lambda \in \Lambda}$ is said to be an *infinite cover*.

If each member of a cover $\{A_\lambda\}_{\lambda \in \Lambda}$ is an open set, then the cover is called an *open cover*.

Thus, a family $\{A_\lambda\}_{\lambda \in \Lambda}$ of open sets is said to be an **open cover** of a set S if $S \subset \bigcup_{\lambda \in \Lambda} A_\lambda$, *i.e.*, each element of S belongs to some A_λ, $\lambda \in \Lambda$

Let $\{A_\lambda\}_{\lambda \in \Lambda}$ be an open cover of a set S. If there exists a subset $\Lambda' \subset \Lambda$ such that the sub-family $\{A_\lambda\}_{\lambda \in \Lambda'}$ also covers S, then the sub-family $\{A_\lambda\}_{\lambda \in \Lambda'}$ is called a *sub-cover* of the open cover $\{A_\lambda\}_{\lambda \in \Lambda}$.

Example. Let $A_n = (-n, n)$ where $n \in N$. Every member of the family $\{A'_n\}_{n \in N}$ is an open interval and, therefore, an open set. The family $\{A_n\}_{n \in N}$ is an open cover of R. Also, the cover is infinite.

Let $A_n' = (-2n, 2n)$ where $n \in N$. Every member of the family $\{A'_n\}_{n \in N}$ is an open interval and, therefore, an open set. The family $\{A'_n\}_{n \in N}$ is an open cover of R. This cover is also infinite.

Since $\{A_n\}_{n \in N}$ and $\{A'_n\}_{n \in N}$ are both open covers of R and $\{A'_n\}_{n \in N}$ is a sub-family of $\{A_n\}_{n \in N}$, therefore, $\{A'_n\}_{n \in N}$ is a sub-cover of $\{A_n\}_{n \in N}$.

There is no finite sub-family of $\{A_n\}_{n \in N}$ that covers R.

3.25. HEINE-BOREL PROPERTY

A subset A of R is said to have the Heine-Borel property if every open cover of A has a finite sub-cover.

3.26. THEOREM

If a set A satisfies the Heine-Borel property, then any closed subset of A also satisfies the Heine-Borel property.

Proof. Let A satisfy the Heine-Borel property.

Also let B be a closed subset of A.

We shall prove that B also satisfies the Heine-Borel property.

Suppose $\{B_\lambda\}_{\lambda \in \Lambda}$ is an open cover of B.

$\therefore \qquad B \subset \bigcup_{\lambda \in \Lambda} B_\lambda$

$\Rightarrow \qquad B^c \cup B \subset B^c (\bigcup_{\lambda \in \Lambda} B_\lambda)$ where B^c is open, since B is closed

$\Rightarrow \qquad R \subset B^c \cup (\bigcup_{\lambda \in \Lambda} B_\lambda) \qquad [\because B^c \cup B = R)$

$\Rightarrow \qquad A \subset B^c \cup (\bigcup_{\lambda \in \Lambda} B_\lambda) \qquad [\because A \subset R]$

$\Rightarrow$ The family F consisting of B^c and $\{B_\lambda\}_{\lambda \in \Lambda}$ is an open cover of A.

But A has the Heine-Borel property.

$\therefore$ F has a finite open sub-cover F_1 consisting of B^c and $B_{\lambda_1}, B_{\lambda_2}, \ldots\ldots, B_{\lambda_n}$.

$\Rightarrow \qquad A \subset B^c \cup B_{\lambda_1} \cup B_{\lambda_2} \cup \ldots\ldots \cup B_{\lambda_n}$

$\Rightarrow \qquad B \subset B^c \cup B_{\lambda_1} \cup B_{\lambda_2} \cup \ldots\ldots \cup B_{\lambda_n} \qquad [\because B \subset A]$

$\Rightarrow \qquad B \subset B_{\lambda_1} \cup B_{\lambda_2} \cup \ldots\ldots \subset B_{\lambda_n} \qquad [\because B \cap B^c = \phi$, *i.e.*, no element of B is present in B^c]

$\Rightarrow \qquad \{B_{\lambda_1}, B_{\lambda_2}, \ldots\ldots B_{\lambda_n}\}$ is a cover of B.

Thus, the open cover $\{B_\lambda\}_{\lambda \in \Lambda}$ of B has a finite sub-cover $\{B_{\lambda_1}, B_{\lambda_2}, \ldots\ldots B_{\lambda_n}\}$

Hence B also satisfies the Heine-Borel property.

3.27. HEINE-BOREL THEOREM

Statement. *A set A is compact if and only if A has the Heine-Borel property.*

Proof. Let A be a compact set, then A is bounded and closed.

Let $\quad a = \text{g.l.b. } A \quad$ and $\quad b = \text{l.u.b. } A$, then $A \subset [a, b]$

If $a = b$, then $A = \{a\}$ and every open cover of A contains an open *nbd* of a. This *nbd* is then the sub-cover. Thus A has the Heine-Borel property.

Now, let $a \neq b$ and $[a, b] = I$

We shall prove that I satisfies the Heine-Borel property.

Suppose I does not have the Heine-Borel property. Then, there exists a family F of open sets which covers I, but no finite sub-family of which covers I.

Divide I into two equal closed intervals I′ and I″, where $I' = \left[a, \frac{a+b}{2}\right]$ and $I'' = \left[\frac{a+b}{2}, b\right]$

Then at least one of these I′ and I″ cannot be covered by finitely many members of F.

Let I_1 be that one of I′ and I″ which is not covered by finitely many members of F.

Length of $\quad I_1 = l(I_1) = \frac{1}{2}(b-a)$.

Again, divide I_1 into two equal closed intervals I_1' and I_1''. Then at least one of these I_1' and I_1'' cannot be covered by finitely many members of F. Let I_2 be that one of I_1' and I_1'' which is not covered by finitely many members of F.

Length of $\quad I_2 = l(I_2) = \frac{1}{2^2}(b-a)$

Continuing like this, we get a sequence $\{I_n\}$ of closed intervals such that

(*i*) no I_n can be covered by finitely many members of F

(*ii*) $I = I_0 \supset I_1 \supset I_2 \supset \ldots\ldots$ *i.e.*, $I_{n+1} \subset I_n \ \forall\, n$

(*iii*) length of $I_n = l(I_n) = \frac{1}{2^2}(b-a) \to 0$ as $n \to \infty$

$\therefore$ By Nested Interval Property of sequences, $\bigcup_{n=1}^{\infty} I_n$ is a singleton. Let $\bigcup_{n=1}^{\infty} I_n = \{x\}$, then $x \in I_0 = I$ also.

Since the family F is an open cover of I, $\exists$ an open set $B \in F$ such that $x \in B$

$\Rightarrow$ B is a *nbd* of x

$\Rightarrow$ $\exists\, \varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subset B$

Now, $\quad l(I_n) \to 0$ as $n \to \infty$

$\therefore$ $\exists$ a natural number m such that $l(I_m) < \varepsilon$ and $I_m \subset (x-\varepsilon, x+\varepsilon) \subset B$

Thus I_m is covered by a single member B of F, which is a contradiction (since no I_n can be covered by finitely many members of F).

$\Rightarrow$ Our supposition is wrong.

$\Rightarrow$ I has the Heine-Borel property.

Since A is a closed subset of I, therefore, A also has the Heine-Borel property.

Conversely. *Let A have the Heine-Borel property.* We shall show that A is compact *i.e.*, A is bounded and closed *A is bounded.*

We know that $\{I_n\}_{n \in N}$ where $I_n = (-n, n)$ of open intervals covers R and, therefore, covers A which is a subset of R.

But A has the Heine-Borel property *i.e.*, every open cover of A has a finite sub-cover.

$\therefore$ There exist finitely many natural numbers $n_1, n_2, \ldots\ldots, n_k$ such that the finite family $\{I_{n_1}, I_{n_2}, \ldots\ldots, I_{n_k}\}$ covers A.

If $\quad M = \max. \{n_1, n_2, \ldots\ldots, n_k\}$, then

$A \subset (-M, M)$. Hence A is bounded.

A is closed.

Suppose A is not closed. Then there exists an infinite subset B of A which has no limit point in A.

Let $x \in A$ and $x \notin B$ *i.e.*, $x \in A - B$. Since x is not a limit point of B (because no limit point of B is in A), there exists an open interval G_x around x which does not contain any point of B.

Let $y \in B$, then y is not a limit point of B (because, if it were a limit point of B, it would have been in A). There exists an open interval H_y around y containing only one point, namely y, of B. Clearly H_y are infinite in number as B is infinite.

Since the points belonging to A are either in A − B or in B

[$\because$ $A = (A - B) \cup B$], *the family of open intervals G_x and H_y forms an open cover of A. This family has no finite sub-cover,* because, if we omit any H_y, the corresponding point y is left uncovered.

This is a contradiction, because A has the Heine-Borel property, therefore, every open cover of A must have a finite sub-cover.

$\therefore$ Our supposition is wrong. Hence A is closed.

4
Sequences

4.1. SEQUENCE

A sequence is a function whose domain is the set N of all natural numbers whereas the range may be any set S. In other words, a sequence in a set S is a rule which assigns to each natural number a unique element of S.

4.2. REAL SEQUENCE

A real sequence is a function whose domain is the set N of all natural numbers and range a subset of the set R of real numbers.

Symbolically $f : N \to R$ (or $x : N \to R$ or $a : N \to R$) is a real sequence.

Note 1. In this chapter, we shall study only real sequences. Therefore by a 'sequence' we shall mean a 'real sequence'.

Note 2. If $x : N \to R$ be a sequence, the image of $n \in N$ instead of denoting it by $x(n)$, we shall generally denote it by x_n. Thus x_1, x_2, x_3 etc. are the real numbers associated to 1, 2, 3 etc. by this mapping. Also, the sequence $x : N \to R$ is denoted by $\{x_n\}$ or $< x_n >$.

$x_1, x_2, \ldots$ are called the first, second... terms of the sequence. The mth and nth terms x_m and x_n for $m \neq n$ are treated as distinct even if $x_m = x_n$ *i.e.*, the terms occurring at different positions are treated as distinct terms even if they have the same value.

4.3. RANGE OF A SEQUENCE

The set of all **distinct** terms of a sequence is called its range.

Note. In a sequence $\{x_n\}$, since $n \in N$ and N is an infinite set, **the number of terms of a sequence is always infinite.** The range of a sequence may be a finite set.

e.g., if $x_n = (-1)^n$, then $\{x_n\} = \{-1, 1, -1, 1, \ldots\}$

The range of sequence $\{x_n\} = \{-1, 1\}$ which is a finite set.

4.4. CONSTANT SEQUENCE

A sequence $\{x_n\}$ defined by $x_n = c \in R \ \forall \ n \in N$ is called a constant sequence.

Thus $\{x_n\} = \{c, c, c, \ldots\}$ is a constant sequence with range $= \{c\}$, a singleton.

4.5. BOUNDED AND UNBOUNDED SEQUENCE

Bounded above sequence. A sequence $\{a_n\}$ is said to be bounded above if $\exists$ a real number K such that

$$a_n \leq K \ \forall \ n \in N.$$

i.e., if the range of the sequence is bounded above.

Bounded below sequence. A sequence $\{a_n\}$ is said to be bounded below if $\exists$ a real number k such that

$$a_n \geq k \quad \forall \ n \in N$$

i.e., if the range of the sequence is bounded below.

Bounded Sequence. A sequence is said to be bounded if it is bounded above as well as below.

Thus a sequence $\{a_n\}$ is bounded if $\exists$ two real numbers k and K, ($k \leq K$), such that

$$k \leq a_n \leq K \quad \forall \ n \in N$$

i.e., if the range of the sequence is bounded.

A sequence is said to be *unbounded* if it is not bounded.

Unbounded above sequence. A sequence $\{a_n\}$ is said to be unbounded above if it is not bounded above *i.e.*, if for every real number K, $\exists \ m \in N$ such that $a_m > K$.

Unbounded below sequence. A sequence $\{a_n\}$ is said to be unbounded below if it is not bounded below *i.e.* if for every real number k, $\exists \ m \in N$ such that $a_m < k$.

Example (*i*) The sequence $\{a_n\}$ defined by $a_n = \frac{1}{n}$ is bounded, since $0 < a_n \leq 1$.

(*ii*) The sequence $\{a_n\}$ defined by $a_n = n$ is bounded below, because $a_n \geq 1 \ \forall \ n \in N$. It is not bounded above, because there exists no real number K such that

$$a_n \leq K \quad \forall \ n \in N.$$

(*iii*) The sequence $< (-1)^n >$ is bounded, since $-1 \leq a_n \leq 1 \quad \forall \ n \in N$.

(*iv*) The sequence $< -n >$ is bounded above, because $a_n \leq -1 \quad \forall \ n \in N$. It is not bounded below, because there exists no real number k such that $a_n \geq k \quad \forall \ n \in N$.

(*v*) Every constant sequence is bounded.

(*vi*) The sequence $\{a_n\}$ defined by $a_n = (-1)^n \cdot n$ is neither bounded above nor bounded below.

4.6. THEOREM

A sequence $\{a_n\}$ is bounded iff $\exists$ a positive real number M such that $| a_n | \leq M \ \forall \ n \in N$.

Proof. Necessary part

Let $\{a_n\}$ be bounded. Then $\exists$ two real numbers h and k such that

$$h \leq a_n \leq k \quad \forall \ n \in N \qquad \ldots(1)$$

Let M = Max. $\{ | h |, | k | \}$, then $| h | \leq M$ and $| k | \leq M$

$$\Rightarrow \quad -M \leq h \leq M \quad \text{and} \quad -M \leq k \leq M \qquad \ldots(2)$$

From (1) and (2), we have

$$-M \leq h \leq a_n \leq k \leq M \quad \forall \ n \in N$$

$$\Rightarrow \quad -M \leq a_n \leq M \quad \forall \ n \in N$$

$$\Rightarrow \quad | a_n | \leq M \quad \forall \ n \in N$$

Sufficient part

Let M be a positive real number such that

$$| a_n | \leq M \quad \forall \ n \in N$$

Then $\quad -M \leq a_n \leq M \quad \forall \ n \in N$

$\Rightarrow \quad \{a_n\}$ is bounded.

Note. *The above theorem is used as a definition of a bounded sequence and should be committed to memory.*

4.7. LEAST UPPER BOUND AND GREATEST LOWER BOUND OF A SEQUENCE

(a) **Least upper bound of a sequence.** If a sequence $\{a_n\}$ is bounded above, then $\exists$ a real number K_1 such that $a_n \le K_1 \quad \forall\, n \in N$

K_1 is called an upper bound of the sequence.

If $K_1 < K_2$, then $\quad a_n < K_2, \quad \forall\, n \in N.$

$\Rightarrow$ K_2 is also an upper bound of the sequence.

$\Rightarrow$ Any number $> K_1$ is also an upper bound of the sequence.

Therefore, if a sequence is bounded above, it has infinitely many upper bounds.

Of all the upper bounds of the sequence, if K is the least, then K is called the least upper bound (l.u.b.) of the sequence.

It has the following properties :

1. It is an upper bound of the sequence. $\Rightarrow$ $a_n \le K \quad \forall\, n \in N$
2. Given $\varepsilon > 0$, $\quad K - \varepsilon < K$.

Since K is the **least upper bound**, $K - \varepsilon$ is not even an upper bound.

$\Rightarrow$ $\exists$ at least one +ve integer m such that $a_m \nleq K - \varepsilon \Rightarrow a_m > K - \varepsilon$

(b) **Greatest lower bound of a sequence.** If a sequence $\{a_n\}$ is bounded below, then $\exists$ a real number k_1 such that $k_1 \le a_n \quad \forall\, n \in N$

k_1 is called a lower bound of the sequence.

If $k_2 < k_1$, then $\quad k_2 < a_n, \quad \forall\, n \in N.$

$\Rightarrow$ k_2 is also a lower bound of the sequence.

$\Rightarrow$ any number $< k_1$ is also a lower bound of the sequence

$\therefore$ If a sequence is bounded below, it has infinitely many lower bounds.

Of all the lower bounds of the sequence, if k is the greatest, then k is called the greatest lower bound (g.l.b.) of the sequence.

It has the following properties :

1. It is a lower bound of the sequence, $\Rightarrow$ $k \le a_n \quad \forall\, n \in N$
2. Given $\varepsilon > 0$, $\quad k + \varepsilon > k$

Since k is the greatest lower bound, $k + \varepsilon$ is not even a lower bound.

$\Rightarrow$ $\exists$ at least one +ve integer m such that $k + \varepsilon \nleq a_m \Rightarrow k + \varepsilon > a_m$ or $a_m < k + \varepsilon$.

4.8. LIMIT OF A SEQUENCE

Let $\{a_n\}$ be a sequence and $l \in R$. The real number l is said to be the limit of the sequence $\{a_n\}$ if to each $\varepsilon > 0$, $\exists\, m \in N$ (m depending on ε) such that $|a_n - l| < \varepsilon \quad \forall\, n \ge m.$

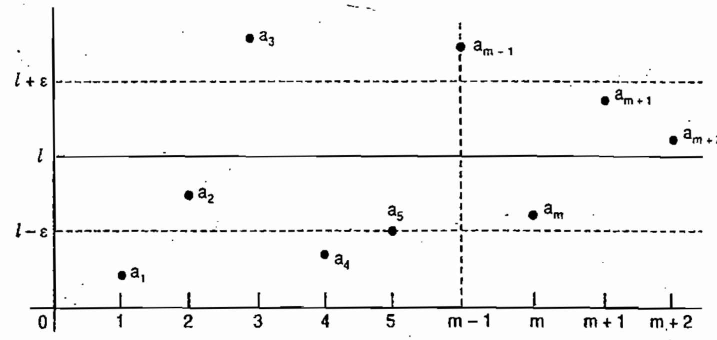
If l is the limit of $\{a_n\}$, then we write $a_n \to l$ as $n \to \infty$ or $\lim\limits_{n \to \infty} a_n = l$.

Note. $\quad |a_n - l| < \varepsilon \quad \forall\, n \ge m$

$\Rightarrow \quad l - \varepsilon < a_n < l + \varepsilon \quad \forall\, n \ge m$

$\Rightarrow \quad a_n \in (l - \varepsilon, l + \varepsilon) \quad \forall\, n \ge m$

The definition of the limit of a sequence asserts that given any $\varepsilon < 0$, however small, all the terms of the sequence, except the first $m - 1$ terms, lie in the interval $(l - \varepsilon, l + \varepsilon)$. The first $m - 1$ terms of the sequence may be scattered anywhere. The number $m - 1$ of the terms left out of the interval $(l - \varepsilon, l + \varepsilon)$ depends upon the size of ε. The smaller the size of ε, the larger will be the number of terms left out of the interval $(l - \varepsilon, l + \varepsilon)$.



4.9. CONVERGENT SEQUENCE

If $\lim\limits_{n \to \infty} a_n = l$, then we say that the sequence $\{a_n\}$ converges to l.

Equivalently, a sequence $\{a_n\}$ is said to converge to (or tend to) a real number l if given $\varepsilon > 0$, however small, $\exists$ a positive integer m (depending on ε) such that $|a_n - l| < \varepsilon\ \forall\, n \ge m$.

The real number l is called the limit of the sequence $\{a_n\}$.

4.10. THEOREM

Every convergent sequence has a unique limit.

Or

A sequence cannot converge to more than one limit.

Proof. If possible, let a sequence $\{a_n\}$ converge to two distinct real numbers l and l'.

Let $\varepsilon = \frac{1}{2}\,|l - l'|$. Since $l \ne l'$, $|l - l'| > 0$ so that $\varepsilon > 0$.

Now the sequence $\{a_n\}$ converges to l

$\Rightarrow$ Given $\varepsilon > 0$, $\exists$ a positive integer m_1 such that $\quad |a_n - l| < \dfrac{\varepsilon}{2} \quad \forall\, n \ge m_1$

Also the sequence $\{a_n\}$ converges to l'

$\Rightarrow$ Given $\varepsilon > 0$, $\exists$ a positive integer m_2 such that $\quad |a_n - l'| < \dfrac{\varepsilon}{2} \quad \forall\, n \ge m_2$

Let $\quad m = \max.\ \{m_1, m_2\}$

Then $\quad |a_n - l| < \dfrac{\varepsilon}{2}$

and $\quad |a_n - l'| < \dfrac{\varepsilon}{2} \quad \forall\, n \ge m \quad$ ...(1)

Now $\quad |l - l'| = |(l - a_n) + (a_n - l')| \le |l - a_n| + |a_n - l'|$

$\quad = |a_n - l| + |a_n - l'| \quad$ [$\because\ |-x| = |x|$]

$\quad < \dfrac{\varepsilon}{2} + \dfrac{\varepsilon}{2} = \varepsilon \quad \forall\, n \ge m \quad$ [Using (1)]

$\therefore \quad |l - l'| < \varepsilon \quad \forall\, n \ge m$

which contradicts the assumption that $\varepsilon = \frac{1}{2}\,|l - l'|$

$\Rightarrow$ Our supposition is wrong. Hence $l = l'$.

4.11. DIVERGENT SEQUENCE

(*i*) A sequence $\{a_n\}$ is said to diverge to $+\infty$ if given any positive real number K, however larger there exists a positive integer m (depending on K) such that $a_n > K \ \forall\ n \geq m$

and we write $\lim\limits_{n\to\infty} a_n = \infty$ or $a_n \to \infty$ as $n \to \infty$.

(*ii*) A sequence $\{a_n\}$ is said to diverge to $-\infty$ if given any positive real number K, however large, there exists a positive integer m (depending on K) such that $a_n < -K \ \forall\ n \geq m$

and we write $\lim\limits_{n\to\infty} a_n = -\infty$ or $a_n \to -\infty$ as $n \to \infty$.

Equivalently, a sequence $\{a_n\}$ is said to diverge to $-\infty$ if given any negative real number k, however small, there exists a positive integer m (depending on k) such that

$$a_n < k \quad \forall\ n \geq m.$$

(*iii*) A sequence $\{a_n\}$ is said to be a divergent sequence if it diverges to $+\infty$ or $-\infty$.

i.e., if $a_n \to \infty$ or $a_n \to -\infty$.

Examples. (*i*) The sequences $\{n\}$ and $\{n^2\}$ diverge to ∞.

(*ii*) The sequences $\{-n\}$ and $\{-n^2\}$ diverge to $-\infty$.

4.12. OSCILLATORY SEQUENCES

If a sequence $\{a_n\}$ neither converges to a finite number nor diverges to $+\infty$ or $-\infty$, it is called an oscillatory sequence. Oscillatory sequences are of two types :

(*i*) A bounded sequence which does not converge is said to **oscillate finitely.**

For example, consider the sequence $\{(-1)^n\}$.

Here $a_n = (-1)^n$.

It is a bounded sequence.

$$\underset{n\to\infty}{\text{Lt}}\ a_{2n} = \underset{n\to\infty}{\text{Lt}}\ (-1)^{2n} = 1$$

$$\underset{n\to\infty}{\text{Lt}}\ a_{2n+1} = \underset{n\to\infty}{\text{Lt}}\ (-1)^{2n+1} = -1.$$

Thus $\underset{n\to\infty}{\text{Lt}}\ a_n$ does not exist $\Rightarrow$ the sequence does not converge.

Hence this sequence oscillates finitely.

(*ii*) An unbounded sequence which does not diverge is said to **oscillate infinitely.**

For example, consider the sequence $\{(-1)^n n\}$.

Here $a_n = (-1)^n n$.

It is an unbounded sequence.

$$\underset{n\to\infty}{\text{Lt}}\ a_{2n} = \underset{n\to\infty}{\text{Lt}}\ (-1)^{2n}.\,2n = \underset{n\to\infty}{\text{Lt}}\ 2n = +\infty$$

$$\underset{n\to\infty}{\text{Lt}}\ a_{2n+1} = \underset{n\to\infty}{\text{Lt}}\ (-1)^{2n+1}(2n+1) = \underset{n\to\infty}{\text{Lt}}\ -(2n+1) = -\infty.$$

Thus the sequence does not diverge.

Hence this sequence oscillates **infinitely.**

Note. When we say $\underset{n\to\infty}{\text{Lt}}\ a_n = l$, it means $\underset{n\to\infty}{\text{Lt}}\ a_{2n} = \underset{n\to\infty}{\text{Lt}}\ a_{2n+1} = l$

Similarly $\underset{n\to\infty}{\text{Lt}}\ a_n = +\infty$ means $\underset{n\to\infty}{\text{Lt}}\ a_{2n} = \underset{n\to\infty}{\text{Lt}}\ a_{2n+1} = -\infty.$

4.13. NULL SEQUENCE

A sequence $\{a_n\}$ is said to be a null sequence if it converges to zero *i.e.*, if $\lim\limits_{n\to\infty} a_n = 0$.

For example, the sequences $\left\{\frac{1}{n}\right\}, \left\{\frac{1}{n^2}\right\}, \left\{\frac{1}{2^n}\right\}$ and $\left\{\frac{(-)^{n-1}}{n}\right\}$ are null sequences.

4.14. MONOTONIC SEQUENCES

(*i*) A sequence $\{a_n\}$ is said to be monotonically increasing

if $\qquad a_{n+1} \geq a_n \quad \forall\ n \in N.$

i.e., if $\qquad a_1 \leq a_2 \leq a_3 \leq \ldots\ldots \leq a_n \leq a_{n+1} \leq \ldots\ldots$

(*ii*) A sequence $\{a_n\}$ is said to be **monotonically decreasing**

if $\qquad a_{n+1} \leq a_n \ \forall\ n \in N.$

i.e., if $\qquad a_1 \geq a_2 \geq a_3 \geq \ldots\ldots \geq a_n \geq a_{n+1} \geq \ldots\ldots$

(*iii*) A sequence $\{a_n\}$ is said to be **monotonic** if it is either monotonically increasing or monotonically decreasing.

(*iv*) A sequence $\{a_n\}$ is said to be **strictly monotonically increasing if**

$$a_{n+1} > a_n \quad \forall\ n \in N.$$

(*v*) A sequence $\{a_n\}$ is said to be **strictly monotonically decreasing** if

$$a_{n+1} < a_n \quad \forall\ n \in N.$$

(*vi*) A sequence $\{a_n\}$ is said to be **strictly monotonic** if it is either strictly monotonically increasing or strictly monotonically decreasing.

4.15. THEOREM

Every convergent sequence is bounded.

Proof. Let $\{a_n\}$ be a convergent sequence, converging to l.

For $\varepsilon = 1$, there exists a positive integer m such that

$$|a_n - l| < 1 \qquad \forall\ n \geq m$$

$$\Rightarrow \qquad l - 1 < a_n < l + 1 \ \forall\ n \geq m$$

Let $k = \min.\ \{a_1, a_2, \ldots\ldots, a_{m-1}, l - 1\}$ and $K = \max.\ \{a_1, a_2, \ldots\ldots, a_{m-1}, l + 1\}$.

Then $\qquad k \leq a_n \leq K \qquad \forall\ n \in N.$

$\Rightarrow$ The sequence $\{a_n\}$ is a bounded sequence.

Note 1. *The converse of the above theorem is not true i.e.*, a bounded sequence is not necessarily convergent. For example, consider the sequence $\{a_n\}$ defined by $a_n = (-1)^n$. Clearly this sequence is bounded. -1 is its g.l.b. and 1 is its l.u.b. but it is not convergent.

Note 2. From Theorem 4.15 and Note 1, we conclude that "*boundedness is a necessary condition for the convergence of a sequence*". If a sequence is not bounded, it cannot be convergent.

For example, the sequence $\{n^2\}$ is not convergent because it is unbounded.

4.16. THEOREM

If the sequence $\{a_n\}$ converges to l, then the sequence $\{\,|a_n|\,\}$ converges to $|\,l\,|$.

Proof. The sequence $\{a_n\}$ converges to l

$\Rightarrow$ given $\varepsilon > 0$, $\exists$ a positive integer m such that $|a_n - l| < \varepsilon\ \forall\ n \geq m$

Since $\qquad \big||a_n| - |l|\big| \leq |a_n - l|$

$\therefore \quad || a_n - l || < \varepsilon \ \forall \ n \geq m$

$\Rightarrow \quad \lim_{n \to \infty} | a_n | = | l |$

Hence the sequence $\{ | a_n | \}$ converges to $\{ | l | \}$.

Note. *The converse of the above theorem is not true.*

For example, if $a_n = (-1)^n$, then $| a_n | = 1$ is a constant sequence.

Obviously, the sequence $\{ | a_n | \}$ converges to 1 but the sequence $\{a_n\}$ is not convergent.

4.17. THEOREM

(i) *If sequence $\{a_n\}$ diverges to ∞, then $\{a_n\}$ is bounded below but unbounded above.*

(ii) *If a sequence $\{a_n\}$ diverges to $-\infty$, then $\{a_n\}$ is bounded above but unbounded below.*

Proof. (i) The sequence $\{a_n\}$ diverges to ∞

$\Rightarrow$ For any positive real number K, however large, there exists a positive integer m such that

$$a_n > K \ \forall \ n \geq m$$

$\Rightarrow$ Out of the infinitely many terms of the sequence, only finitely many terms are $\leq K$

$\Rightarrow$ The sequence $\{a_n\}$ is not bounded above

Now, taking K = 1, we have $\quad a_n > 1 \quad \forall \ n \geq m$

Let $\quad k = \min. \{a_1, a_2, ..., a_{m-1}, 1\}$, then $\quad a_n \geq k \quad \forall \ n \in N$

$\Rightarrow$ The sequence $\{a_n\}$ is bounded below.

(ii) The sequence $\{a_n\}$ diverges to $-\infty$

$\Rightarrow$ For any negative real number k, however small, there exists a positive integer m such that

$$a_n < k \quad \forall \ n \geq m$$

$\Rightarrow$ Out of the infinitely many terms of the sequence, only finitely many terms are $\geq k$.

$\Rightarrow$ The sequence $\{a_n\}$ is not bounded below.

Now, taking $k = -1$, we have $\quad a_n < -1 \quad \forall \ n \geq m$

Let $\quad K = \max. \{a_1, a_2, ..., a_{m-1}, -1\}$, then $\quad a_n \leq K \quad \forall \ n \in N$

$\Rightarrow$ The sequence (a_n) is bounded above.

Note. The converses of these theorems are not true.

(i) Consider the sequence $\{a_n\}$ where $a_n = \begin{cases} n & \text{if } n \text{ is even} \\ \dfrac{1}{n} & \text{if } n \text{ is odd} \end{cases}$

The sequence is bounded below by zero and unbounded above but does not diverge to ∞.

(ii) Consider the sequence $\{a_n\}$ where $a_n = \begin{cases} -n & \text{if } n \text{ is even} \\ -\dfrac{1}{n} & \text{if } n \text{ is odd} \end{cases}$

The sequence is bounded above by zero and unbounded below but does not diverge to $-\infty$.

4.18. CONVERGENCE OF MONOTONIC SEQUENCES

Theorem I. (i) *Every monotonically increasing sequence which is bounded above converge to its least upper bound.*

(ii) *Every monotonically decreasing sequence which is bounded below converges to its greatest lower bound.*

Proof. (i) Let $\{a_n\}$ be a monotonically increasing sequence which is bounded above. Let u be the l.u.b. of the sequence $\{a_n\}$.

We shall show that $\{a_n\}$ converges to u.

Let $\varepsilon > 0$ be given.

Since $u - \varepsilon < u$, therefore, $u - \varepsilon$ is not an upper bound of $\{a_n\}$

$\Rightarrow \ \exists$ a positive integer m such that $a_m > u - \varepsilon$

Since $\{a_n\}$ is monotonically increasing

$\therefore \quad a_n \geq a_m > u - \varepsilon \quad \forall \ n \geq m$

$\Rightarrow \quad a_n > u - \varepsilon \quad \forall \ n \geq m \quad$ ...(1)

Also, u is the l.u.b. of $\{a_n\}$

$\Rightarrow \quad a_n \leq u + \varepsilon \quad \forall \ n \in N$

$\Rightarrow \quad a_n < u + \varepsilon \quad \forall \ n \in N \quad$ ...(2)

From (1) and (2), $\quad u - \varepsilon < a_n < u + \varepsilon \quad \forall \ n \geq m$

$\Rightarrow \quad | a_n - u | < \varepsilon \quad \forall \ n \geq m$

$\Rightarrow \quad \lim_{n \to \infty} a_n = u$

$\Rightarrow \ \{a_n\}$ converges to u.

(ii) Let $\{a_n\}$ be a monotonically decreasing sequence which is bounded below. Let l be the g.l.b. of sequence $\{a_n\}$

We shall show that $\{a_n\}$ converges to l.

Let $\varepsilon > 0$ be given.

Since $l + \varepsilon > l$, therefore, $l + \varepsilon$ is not a lower bound of $\{a_n\}$

$\Rightarrow \ \exists$ a positive integer m such that $a_m < l + \varepsilon$

Since $\{a_n\}$ is monotonically decreasing

$\therefore \quad a_n \leq a_m < l + \varepsilon \quad \forall \ n \geq m$

$\Rightarrow \quad a_n < l + \varepsilon \quad \forall \ n \geq m \quad$ ...(1)

Also, l is the g.l.b. of $\{a_n\}$

$\Rightarrow \quad a_n \geq l > l - \varepsilon \quad \forall \ n \in N$

$\Rightarrow \quad a_n > l - \varepsilon \quad \forall \ n \in N \quad$ ...(2)

From (1) and (2), $\quad l - \varepsilon < a_n < l + \varepsilon \quad \forall \ n \geq m$

$\Rightarrow \quad | a_n - l | < \varepsilon \quad \forall \ n \geq m$

$\Rightarrow \quad \lim_{n \to \infty} a_n = l$

$\Rightarrow \ \{a_n\}$ converges to l.

Theorem II. *The necessary and sufficient condition for the convergence of a monotonic sequence is that it is bounded.*

Proof. The condition is necessary (*i.e.*, a convergent sequence is bounded).

Reproduce Theorem 4.15.

The condition is sufficient (*i.e.*, a monotonic sequence is bounded

$\Rightarrow$ the sequence is convergent).

Let $\{a_n\}$ be a monotonic bounded sequence.

Then it is either monotonically increasing or monotonically decreasing. Also it is bounded above *as well as* below.

(*i*) Suppose $\{a_n\}$ is a bounded monotonically increasing sequence. Then $\{a_n\}$ is bounded above.

Reproduce Theorem I (*i*).

(*ii*) Suppose $\{a_n\}$ is a bounded monotonically decreasing sequence. Then $\{a_n\}$ is bounded below.

Reproduce Theorem I (*ii*).

Exercise. Prove that a bounded increasing sequence $\{a_n\}$ of reals is convergent.

4.19. THEOREM

(*i*) *Every monotonically increasing sequence which is not bounded above diverges to* ∞.

(*ii*) *Every monotonically decreasing sequence which is not bounded below diverges to* $-\infty$.

Proof. (*i*) Let $\{a_n\}$ be a monotonically increasing sequence which is not bounded above. Then given any $K > 0$, however large, $\exists$ a positive integer m such that $a_m > K$.

Since $\{a_n\}$ is monotonically increasing

$\therefore \quad a_n \ge a_m > K \quad \forall\, n \ge m$

$\Rightarrow \quad a_n > K \quad \forall\, n \ge m$

$\Rightarrow \quad \lim\limits_{n\to\infty} a_n = \infty$

$\Rightarrow \quad \{a_n\}$ diverges to ∞.

(*ii*) Let $\{a_n\}$ be a monotonically decreasing sequence which is not bounded below. Then given any $k > 0$, however large, $\exists$ a positive integer m such that $a_m < -k$

Since $\{a_n\}$ is monotonically decreasing

$\therefore \quad a_n \le a_m < -k \quad \forall\, n \ge m$

$\Rightarrow \quad a_n < -k \quad \forall\, n \ge m$

$\Rightarrow \quad \lim\limits_{n\to\infty} a_n = -\infty$

$\Rightarrow \quad \{a_n\}$ diverges to $-\infty$.

4.20. THEOREM

Every monotonic sequence either converges or diverges

OR A monotonic sequence is never oscillatory.

Proof. Let $\{a_n\}$ be a monotonic sequence, then $\{a_n\}$ is either monotonically increasing or monotonically decreasing.

Case I. Let $\{a_n\}$ be a monotonically increasing sequence.

If $\{a_n\}$ is bounded above, then by Theorem I (*i*), it converges to its l.u.b.

If $\{a_n\}$ is not bounded above, then by **Theorem 4.19, it diverges to** $+\infty$.

Case II. Let $\{a_n\}$ be a monotonically decreasing sequence.

If $\{a_n\}$ **is bounded below, then by Theorem I** (*ii*), **it converges to its g.l.b.**

If $\{a_n\}$ **is not bounded below, then by Theorem 4.19, it diverges to** $-\infty$.

4.21. NESTED INTERVAL PROPERTY (*or Cantor Intersection Theorem*)

Statement. *Let* $\{I_n = [a_n, b_n]\}$ *be a sequence of closed intervals such that*

(*i*) $I_{n+1} \subset I_n \ \forall\, n \in N$

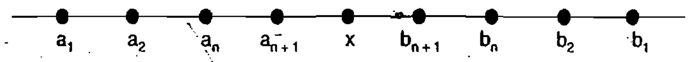
(*ii*) *Length of the nth interval* $= l(I_n) = b_n - a_n \to 0$ *as* $n \to \infty$

then there exists a unique number x such that $x \in I_n \ \forall\, n$

i.e., $\bigcap\limits_{n=1}^{\infty} I_n$ *is a singleton.*

[**Explanation.** The condition (*i*) of the statement implies that each interval contains the succeeding interval *i.e.,* the sequence is monotonically decreasing.

$$I_1 \supset I_2 \supset I_3 \ldots\ldots \supset I_n \supset I_{n+1} \supset \ldots\ldots$$



The condition (*ii*) implies that their length is shrinking to zero. Then the theorem asserts that their exists a point which is common to all the intervals and that such a point is unique.]

Proof. Since $I_{n+1} \subset I_n$ *i.e.,* $[a_{n+1}, b_{n+1}] \subset [a_n, b_n] \quad \forall\, n \in N$

$\therefore \quad a_n \le a_{n+1} \le b_{n+1} \le b_n \quad \forall\, n \in N$

$\Rightarrow \quad \{a_n\}$ is monotonically increasing and $\{b_n\}$ is monotonically decreasing.

Also $\quad I_n \subset I_1 \quad \forall\, n \in N$

$\Rightarrow \quad a_1 \le a_n \le b_n \le b_1 \quad \forall\, n \in N$

$\Rightarrow \quad a_n \le b_1$ and $b_n \ge a_1 \quad \forall\, n \in N$

$\Rightarrow$ The sequence $\{a_n\}$ is bounded above by b_1 and the sequence $\{b_n\}$ is bounded below by a_1.

Thus $\{a_n\}$ is monotonically increasing and bounded above while $\{b_n\}$ is monotonically decreasing and bounded below.

By monotone convergence theorem, the sequence $\{a_n\}$ converges to its l.u.b. a (say) and the sequence $\{b_n\}$ converges to its g.l.b. b (say).

Since $\quad b_n = (b_n - a_n) + a_n \ \forall\, n$

$\therefore \quad \lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} (b_n - a_n) + \lim\limits_{n\to\infty} a_n$

$\Rightarrow \quad b = 0 + a \qquad [\because \lim\limits_{n\to\infty} (b_n - a_n) = 0]$

$\Rightarrow \quad a = b = x$ (say)

Now x is the l.u.b. of the sequence $\{a_n\}$

$\therefore \quad a_n \le x \quad \forall\, n$

Also x is the g.l.b. of the sequence $\{b_n\}$

$\therefore \quad b_n \ge x \quad \forall\, n$

Combining $\quad a_n \le x \le b_n \quad \forall\, n$

$\Rightarrow \quad x \in [a_n, b_n] \quad \forall\, n$

$\Rightarrow \quad x \in I_n \quad \forall\, n$

$\Rightarrow \quad x \in \bigcap\limits_{n=1}^{\infty} I_n$

$\Rightarrow \quad \exists$ a number common to all the intervals.

If possible, suppose x and y are two distinct numbers common to all the intervals.

Then $\quad x \in [a_n, b_n] \quad \forall\, n$ and $y \in [a_n, b_n] \quad \forall\, n$

If $x < y$, then $\quad a_n \le x < y \le b_n \quad \forall\, n$

$\Rightarrow \quad b_n - a_n \ge y - x$

Let $\varepsilon = y - x > 0$, then $b_n - a_n \ge \varepsilon \quad \forall\, n$ ...(1)

Also $\quad \lim\limits_{n \to \infty} (b_n - a_n) = 0$

$\Rightarrow$ for $\varepsilon = y - x$, $\exists$ a positive integer m such that $|(b_n - a_n) - 0| < \varepsilon \quad \forall\, n \ge m$

$\Rightarrow$ $b_n - a_n < \varepsilon \quad \forall\, n \ge m$ which contradicts (1).

Hence x is the only number common to all I_n, *i.e.*, $\bigcap\limits_{n=1}^{\infty} I_n = \{x\}$.

Note. The word 'closed' in the statement of the above theorem cannot be dropped *i.e.*, the intersection of a decreasing sequence of open intervals may be empty.

For example, let $I_n = \left(0, \frac{1}{n}\right)$, $n \in N$, then $\bigcap\limits_{n=1}^{\infty} I_n = \phi$.

For, if $x \le 0$, then $x \notin I_n$ for any n and if $x > 0$, then by Archimedean property of reals, $\exists$ a positive integer m such that $m > \frac{1}{x}$.

Now $\quad m > \frac{1}{x} \Rightarrow \frac{1}{m} < x \Rightarrow x \notin I_m \quad \therefore \quad x \notin \bigcap\limits_{n=1}^{\infty} I_n$.

4.22. THEOREM

A sequence $\{a_n\}$ is convergent if given $\varepsilon > 0$, there exists a positive integer m such that

$$|a_n - a_m| < \varepsilon \text{ whenever } n \ge m.$$

Proof. Let the sequence $\{a_n\}$ converge to l. The given $\varepsilon > 0$, there must exist a positive integer m such that $|a_n - l| < \frac{\varepsilon}{2}$ whenever $n \ge m$

In particular, $|a_m - l| < \frac{\varepsilon}{2}$

$$\therefore \quad |a_n - a_m| = |(a_n - l) - (a_m - l)| \le |a_n - l| + |a_m - l|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \text{ whenever } n \ge m.$$

Note. The converse of above theorem is also true which we shall prove later on as *Cauchy convergence criterion.*

ILLUSTRATIVE EXAMPLES

Example 1. *By definition show that*

(*i*) *The sequence* $\left\{\frac{1}{n}\right\}$ *converges to 0* (*ii*) *The sequence* $\left\{\frac{1}{n^2}\right\}$ *converges to 0*

(*iii*) *The sequence* $\left\{\frac{1}{3^n}\right\}$ *converges to 0*

(*iv*) *The sequence* $<\sqrt{n+1} - \sqrt{n}>$ *is a null sequence.*

Sol. (*i*) Let $a_n = \frac{1}{n}$. Let $\varepsilon > 0$ be given.

Now $\quad |a_n - 0| = \left|\frac{1}{n} - 0\right| = \left|\frac{1}{n}\right| = \frac{1}{n} < \varepsilon$ if $n > \frac{1}{\varepsilon}$.

If m is a positive integer $> \frac{1}{\varepsilon}$, then

$$|a_n - 0| < \varepsilon \quad \forall\, n \ge m \quad \Rightarrow \quad \lim_{n \to \infty} a_n = 0$$

$\Rightarrow$ The sequence $\{a_n\}$ *i.e.*, $\left\{\frac{1}{n}\right\}$ converges to 0.

(*ii*) Let $a_n = \frac{1}{n^2}$. Let $\varepsilon > 0$ be given.

Now $\quad |a_n - 0| = \left|\frac{1}{n^2} - 0\right| = \left|\frac{1}{n^2}\right| = \frac{1}{n^2} < \varepsilon$ if $n^2 > \frac{1}{\varepsilon}$ *i.e.*, if $n > \frac{1}{\sqrt{\varepsilon}}$.

If m is a positive integer $> \frac{1}{\sqrt{\varepsilon}}$, then $|a_n - 0| < \varepsilon \ \forall\, n \ge m$

$\Rightarrow \quad \lim\limits_{n \to \infty} a_n = 0$

$\Rightarrow$ The sequence $\{a_n\}$ *i.e.*, $\left\{\frac{1}{n^2}\right\}$ converges to 0.

(*iii*) Let $a_n = \frac{1}{3^n}$. Let $\varepsilon > 0$ be given.

Now $\quad |a_n - 0| = \left|\frac{1}{3^n} - 0\right| = \left|\frac{1}{3^n}\right| = \frac{1}{3^n} < \varepsilon$ if $3^n > \frac{1}{\varepsilon}$.

i.e., if $\quad n \log 3 > \log\left(\frac{1}{\varepsilon}\right)$ [$\because$ the logarithmic function is an increasing function]

i.e., if $\quad n > \dfrac{\log\left(\frac{1}{\varepsilon}\right)}{\log 3}$ [$\because$ $\log 3 > 0$]

If m is a positive integer $> \dfrac{\log\left(\frac{1}{\varepsilon}\right)}{\log 3}$, then $|a_n - 0| < \varepsilon \quad \forall\, n \ge m$

$\Rightarrow \quad \lim\limits_{n \to \infty} a_n = 0$

$\Rightarrow$ The sequence $\{a_n\}$ *i.e.*, $\left\{\frac{1}{3^n}\right\}$ converges to 0.

(*iv*) Let $\quad a_n = \sqrt{n+1} - \sqrt{n}$

$$= \frac{(\sqrt{n-1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} = \frac{(n+1) - n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}}$$

Let $\varepsilon > 0$ be given.

Now $\quad |a_n - 0| = \left|\frac{1}{\sqrt{n+1} + \sqrt{n}}\right| = \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{\sqrt{n} + \sqrt{n}} = \frac{1}{2\sqrt{n}} < \frac{1}{\sqrt{n}} < \varepsilon$

if $$\sqrt{n} > \frac{1}{\varepsilon} \quad i.e., \quad \text{if} \quad n > \frac{1}{\varepsilon^2}$$

If m is a positive integer $> \frac{1}{\varepsilon^2}$, then

$$| a_n - 0 | < \varepsilon \quad \forall\, n \ge m \quad \Rightarrow \quad \lim_{n\to\infty} a_n = 0$$

$\Rightarrow$ The sequence $< a_n >$ is a null sequence.

Example 2. *By definition show that*

(i) $\lim\limits_{n\to\infty} \frac{2n+1}{n+3} = 2$ (ii) $\lim\limits_{n\to\infty} \frac{n+1}{n} = 1$

(iii) $\lim\limits_{n\to\infty} \frac{n^2+1}{2n^2+3} = \frac{1}{2}$ (iv) $\lim\limits_{n\to\infty} \frac{n^2+3n+5}{2n^2+5n+7} = \frac{1}{2}$

(v) $\lim\limits_{n\to\infty} \frac{3+2\sqrt{n}}{\sqrt{n}} = 2.$

Sol. (i) Let $\varepsilon > 0$ be given. $\left|\frac{2n+1}{n+3} - 2\right| = \left|\frac{-5}{n+3}\right| = \frac{5}{n+3} < \frac{5}{n} < \varepsilon$ if $\frac{n}{5} > \frac{1}{\varepsilon}$ i.e., if $n > \frac{5}{\varepsilon}$.

If m is a positive integer $> \frac{5}{\varepsilon}$, then

$$\left|\frac{2n+1}{n+3} - 2\right| < \varepsilon \;\forall\, n \ge m \quad \Rightarrow \quad \lim_{n\to\infty} \frac{2n+1}{n+3} = 2.$$

(ii) *Please try yourself.*

(iii) Let $\varepsilon > 0$ be given. $\left|\frac{n^2+1}{2n^2+3} - \frac{1}{2}\right| = \left|\frac{-1}{2(2n^2+3)}\right| = \frac{1}{2(2n^2+3)}$

$$< \frac{1}{2n^2+3} < \frac{1}{2n^2} < \frac{1}{n^2} < \varepsilon \quad \text{if} \quad n^2 > \frac{1}{\varepsilon} \quad i.e., \quad \text{if} \quad n > \frac{1}{\sqrt{\varepsilon}}.$$

If m is a positive integer $> \frac{1}{\sqrt{\varepsilon}}$, then

$$\left|\frac{n^2+1}{2n^2+3} - \frac{1}{2}\right| < \varepsilon \;\forall\, n \ge m \quad \Rightarrow \quad \lim_{n\to\infty} \frac{n^2+1}{2n^2+3} = \frac{1}{2}.$$

(iv) Let $\varepsilon > 0$ be given.

$$\left|\frac{n^2+3n+5}{2n^2+5n+7} - \frac{1}{2}\right| = \left|\frac{n+3}{2(2n^2+5n+7)}\right| = \frac{n+3}{2(2n^2+5n+7)}$$

$$\le \frac{n+3}{2(2n^2+5n+7)} \;(\forall\, n \ge 1) = \frac{2n}{2n^2+5n+7}$$

$$< \frac{2n}{2n^2} = \frac{1}{n} < \varepsilon \text{ if } n > \frac{1}{\varepsilon}$$

If m is a positive integer $> \frac{1}{\varepsilon}$, then

$$\left|\frac{n^2+3n+5}{2n^2+5n+7} - \frac{1}{2}\right| < \varepsilon \;\forall\, n \ge m \quad \Rightarrow \quad \lim_{n\to\infty} \frac{n^2+3n+5}{2n^2+5n+7} = \frac{1}{2}.$$

(v) *Please try yourself.*

Example 3. *By definition show that the sequence* $\{a_n\}$, *given by*

(i) $a_n = \frac{3n-1}{4n+5}$ *converges to* $\frac{3}{4}$ (ii) $a_n = \frac{2n-7}{3n+2}$ *converges to* $\frac{2}{3}$

(iii) $a_n = \frac{3n+4}{2n+1}$ *converges to* $\frac{3}{2}$.

Sol. (i) Let $\varepsilon > 0$ be given

$$\left|a_n - \frac{3}{4}\right| = \left|\frac{3n-1}{4n+5} - \frac{3}{4}\right| = \left|\frac{-19}{4(4n+5)}\right| = \frac{19}{4(4n+5)} < \frac{19}{4n+5}$$

$$< \frac{19}{4n} < \frac{19}{n} < \varepsilon \text{ if } \frac{n}{19} > \frac{1}{\varepsilon} \text{ i.e., if } n > \frac{19}{\varepsilon}$$

If m is a positive integer $> \frac{19}{\varepsilon}$, then

$$| a_n - \tfrac{3}{4} | < \varepsilon \;\forall\, n \ge m \quad \Rightarrow \quad \lim_{n\to\infty} a_n = \tfrac{3}{4}$$

$\Rightarrow$ The sequence $\{a_n\}$ converges to $\frac{3}{4}$.

(ii) *Please try yourself.*

(iii) *Please try yourself.*

Example 4. *Show that the sequence* $< (-1)^n >$ *does not converge.*

Sol. Let $a_n = (-1)^n$. Let us assume that $< a_n >$ converges. Taking $\varepsilon = \frac{1}{2}$, there must exist a positive integer m such that $| a_n - a_m | < \frac{1}{2} \;\forall\, n \ge m$

But on taking $n = m + 1$, we see that

$$| a_n - a_m | = | (-1)^{m+1} - (-1)^m | = | (-1)^m (-1-1) | = | (-1)^m | \, | -2 | = 2.$$

Thus we have a contradiction. Hence the sequence $< (-1)^n >$ cannot converge.

Example 5. (i) *If* $x_n = 1 + \frac{(-1)^n}{2n}$, *find the least positive integer m such that*

$$| x_n - 1 | < \frac{1}{10^3} \quad \forall\, n > m.$$

(ii) *If* $a_n = 2 + \frac{(-1)^n}{n}$, *find the least positive integer m such that*

$$| a_n - 2 | < \frac{1}{10^4} \quad \forall\, n > m.$$

Sol. (i) $$| x_n - 1 | = \left|\frac{(-1)^n}{2n}\right| = \frac{1}{2n}$$

$$\therefore \quad | x_n - 1 | < \frac{1}{10^3} \Rightarrow \frac{1}{2n} < \frac{1}{10^3} \Rightarrow 2n > 1000 \Rightarrow n > 500$$

$\therefore$ Taking $m = 500$, we have $|x_n - 1| < \frac{1}{10^3} \quad \forall n > m$

(ii) *Please try yourself.* [**Ans.** $m = 100$]

Example 6. *By definition show that*

(i) *the sequence* $\{n\}$ *diverges to* ∞. (ii) *the sequence* $\{-n^2\}$ *diverges to* $-\infty$.

(iii) *the sequence* $\{n^2\}$ *diverges to* ∞. (iv) *the sequence* $\left\{\log \frac{1}{n}\right\}$ *diverges to* $-\infty$.

(v) *the sequence* $\{2^n\}$ *diverges to* ∞.

Sol. (i) Let K > 0 be given. Let $a_n = n$.

By Archimedian property, $\exists$ a positive integer m such that $m > K$.

For all $n \geq m$ and $m > K$, we have $n > K$

$\therefore \quad n > K \; \forall n \geq m$

$\Rightarrow \quad a_n > K \; \forall n \geq m \quad \Rightarrow \quad \lim_{n \to \infty} a_n = \infty$

$\Rightarrow$ The sequence $\{a_n\}$ *i.e.*, $\{n\}$ diverges to ∞.

(ii) Let K > 0 be given. Let $a_n = -n^2$

$a_n < -K \quad \Rightarrow \quad -n^2 < -K$

$\Rightarrow \quad n^2 > K \quad \Rightarrow \quad n > \sqrt{K}$

If m is a positive integer $> \sqrt{K}$, then

$a_n < -K \quad \forall n \geq m \quad \Rightarrow \quad \lim_{n \to \infty} a_n = -\infty$

$\Rightarrow$ The sequence $\{a_n\}$ *i.e.*, $\{-n^2\}$ diverges to $-\infty$.

(iii) *Please try yourself.*

(iv) Let K > 0 be given. Let $a_n = \log \frac{1}{n}$

$a_n < -K \quad \Rightarrow \quad \log \frac{1}{n} < -K$

$\Rightarrow \quad -\log n < -K \quad \Rightarrow \quad \log n > K \quad \Rightarrow \quad n > e^K$.

If m is a positive integer $> e^K$, then

$a_n < -K \quad \forall n \geq m \quad \Rightarrow \quad \lim_{n \to \infty} a_n = -\infty$

$\Rightarrow$ The sequence $\{a_n\}$ *i.e.*, $\left\{\log \frac{1}{n}\right\}$ diverges to $-\infty$.

(v) Let K > 0 be given. Let $a_n = 2^n$

$a_n > K \quad \Rightarrow \quad 2^n > K$

$\Rightarrow \quad n \log 2 > \log K \quad \Rightarrow \quad n > \frac{\log K}{\log 2} \quad [\because \log 2 > 0]$

If m is a positive integer $> \frac{\log K}{\log 2}$, then

$a_n > K \; \forall n \geq m \quad \Rightarrow \quad \lim_{n \to \infty} a_n = \infty$

$\Rightarrow$ The sequence $\{a_n\}$ *i.e.*, $\{2^n\}$ diverges to ∞.

Example 7. *If* $a_n = c$ *for all* $n \in N$ *is a constant sequence, then* $\lim_{n \to \infty} a_n = c$.

Sol. $a_n = c$ for all $n \quad \Rightarrow \quad a_n - c = 0$ for all n

$\therefore$ For each $\varepsilon > 0$ and for all $m \in N$, we have

$$|a_n - c| = 0 < \varepsilon \quad \forall n \geq m \quad \Rightarrow \quad \lim_{n \to \infty} a_n = c.$$

Example 8. *Discuss the boundedness of the following sequences* $< a_n >$ *where* a_n *is given by*

(i) $a_n = 3$ (ii) $a_n = (-1)^n . 5$ (iii) $a_n = \frac{2n+3}{3n+4}$

(iv) $a_n = \left(1 + \frac{1}{n}\right)^n$ (v) $a_n = n$th *prime* (vi) $a_n = \frac{1}{2^n}$

(vii) $a_n = n^3$ (viii) $a_n = -n^2$ (ix) $a_n = 1 + (-1)^n$

(x) $a_n = \frac{n}{n+1}$ (xi) $a_n = \frac{n}{n^2+1}$

(xii) $a_n = 1 + \frac{1}{3} + \frac{1}{3^2} + \ldots\ldots + \frac{1}{3^n}$ (xiii) $a_n = \frac{1}{n^2} + \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \ldots\ldots + \frac{1}{(2n)^2}$.

Sol. (i) $a_n = 3$ defines a constant sequence.

Range of $< a_n >$ is $\{3\}$, a singleton, which is bounded ($\because$ every finite set is bounded)

$\Rightarrow$ $< a_n >$ **is bounded.**

(ii) $$a_n = (-1)^n . 5 = \begin{cases} -5 \text{ if } n \text{ is odd} \\ 5 \text{ if } n \text{ is even} \end{cases}$$

$\Rightarrow$ Range of $< a_n >$ is $\{-5, 5\}$ a finite set, which is bounded. $\Rightarrow$ $< a_n >$ **is bounded.**

(iii) $$a_n = \frac{2n+3}{3n+4}$$

$$1 - a_n = 1 - \frac{2n+3}{3n+4} = \frac{n+1}{3n+4} > 0 \quad \forall n \quad \Rightarrow \quad a_n < 1 \quad \forall n$$

$\Rightarrow$ $< a_n >$ **is bounded above.**

Also $\quad a_n > 0 \quad \forall n$

$\Rightarrow$ $< a_n >$ **is bounded below. Hence** $< a_n >$ **is bounded.**

(iv) $$a_n = \left(1 + \frac{1}{n}\right)^n = 1 + {}^nC_1 . \frac{1}{n} + {}^nC_2 . \frac{1}{n^2} + {}^nC_3 . \frac{1}{n^3} + \ldots\ldots + {}^nC_n . \frac{1}{n^n}$$

(expanding by Binomial Theorem for a positive integral index)

$$= 1 + n . \frac{1}{n} + \frac{n(n-1)}{2!} . \frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!} . \frac{1}{n^3} + \ldots + \frac{n(n-1)(n-2)\ldots\ldots 1}{n!} \frac{1}{n^n}$$

$$= 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \ldots\ldots$$

$$+ \frac{1}{n!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\ldots\ldots\left(1 - \frac{n-1}{n}\right) \quad \ldots(1)$$

$$< 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots\ldots + \frac{1}{n!} \qquad \left|\ \because\ 1 - \frac{k}{n} < 1 \text{ for } k = 1, 2, \ldots\ldots, n-1\right.$$

$$= 1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \ldots\ldots + \frac{1}{1.2.3\ldots n}$$

$$< 1 + 1 + \frac{1}{2} + \frac{1}{2.2} + \ldots\ldots + \frac{1}{2.2\ldots2} = 1 + \left(1 + \frac{1}{2} + \frac{1}{2^2} + \ldots\ldots + \frac{1}{2^{n-1}}\right)$$

$$= 1 + \frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}} \qquad \left|\ \because \text{ in a G.P. } S_n = \frac{a(1-r^n)}{1-r} \text{ Here } a = 1,\ r = \frac{1}{2}\right.$$

$$= 1 + 2\left(1 - \frac{1}{2^n}\right)$$

$$= 3 - \frac{1}{2^{n-1}} < 3 \quad \forall\, n$$

$\Rightarrow$ $a_n < 3 \quad \forall\, n$

$\Rightarrow$ **$< a_n >$ is bounded above.**

Also
$$a_n = 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \ldots\ldots + \frac{1}{n!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\ldots\ldots\left(1 - \frac{n-1}{n}\right)$$

$$\geq 1 + 1 = 2 \qquad \text{[from (1)]}$$

$\Rightarrow$ $a_n \geq 2\ \forall\, n$

$\Rightarrow$ **$< a_n >$ is bounded below.** **Hence $< a_n >$ is bounded.**

(*v*) a_n = nth prime.

Since every prime number is ≥ 2, $\therefore$ $a_n \geq 2\ \ \forall\, n$

$\Rightarrow$ **$< a_n >$ is bounded below.**

Now, let K be any positive real number, however large

Then $\exists\, m \in$ N s.t. $a_n >$ K $\forall\, n \geq m$

$\Rightarrow$ **$< a_n >$ is not bounded above. Hence $< a_n >$ is not bounded.**

(*vi*) $a_n = \dfrac{1}{2^n}$

Clearly $0 < a_n \leq \frac{1}{2}\ \ \forall\, n \in$ N $\Rightarrow$ **$< a_n >$ is bounded.**

(*vii*) $a_n = n^3 \geq 1\ \ \forall\, n \in$ N $\Rightarrow$ **$< a_n >$ is bounded below.**

Now, let K be any positive real number, however large.

Then $a_n > \text{K} \Rightarrow n^3 > \text{K} \Rightarrow n > \text{K}^{1/3}$

If m is a positive integer $> \text{K}^{1/3}$, then $a_n >$ K $\forall\, n \geq m$

$\Rightarrow$ **$< a_n >$ is not bounded above. Hence $< a_n >$ is not bounded.**

(*viii*) $a_n = -n^2 < 0\ \ \forall\, n \in$ N $\Rightarrow$ **$< a_n >$ is bounded above.**

Now, let K be any positive real number, however large.

Then $a_n < -\text{K} \Rightarrow -n^2 < -\text{K} \Rightarrow n^2 > \text{K} \Rightarrow n > \sqrt{\text{K}}$

If m is a positive integer $> \sqrt{\text{K}}$, then $a_n < -$K $\forall\, n \geq m$

$\Rightarrow$ **$< a_n >$ is not bounded below. Hence $< a_n >$ is not bounded.**

(*ix*) $a_n = 1 + (-1)^n = \begin{cases} 0 \text{ if } n \text{ is odd} \\ 2 \text{ if } n \text{ is even} \end{cases}$

$\Rightarrow$ Range of $< a_n >$ is {0, 2}, a finite set, which is bounded. $\Rightarrow$ $< a_n >$ is bounded.

(*x*) $a_n = \dfrac{n}{n+1} > 0\ \ \forall\, n \in$ N $\Rightarrow$ **$< a_n >$ is bounded below.**

Now $n < n + 1 \quad \forall\, n$

$\Rightarrow$ $\dfrac{n}{n+1} < 1 \quad \forall\, n \Rightarrow a_n < 1\ \forall\, n$

$\Rightarrow$ **$< a_n >$ is bounded above. Hence $< a_n >$ is bounded.**

(*xi*) $a_n = \dfrac{n}{n^2+1} > 0 \quad \forall\, n \in$ N $\Rightarrow$ **$< a_n >$ is bounded below.**

Now $n \leq n^2 < n^2 + 1 \quad \forall\, n$

$\Rightarrow$ $\dfrac{n}{n^2+1} < 1 \quad \forall\, n \Rightarrow a_n < 1\ \ \forall\, n$

$\Rightarrow$ **$< a_n >$ is bounded above. Hence $< a_n >$ is bounded.**

(*xii*) $a_n = 1 + \dfrac{1}{3} + \dfrac{1}{3^2} + \ldots\ldots + \dfrac{1}{3^n}$

$$= \frac{1\left[1 - \left(\frac{1}{3}\right)^{n+1}\right]}{1 - \frac{1}{3}} = \frac{3}{2}\left(1 - \frac{1}{3^{n+1}}\right) = \frac{3}{2} - \frac{1}{2 \times 3^n} < \frac{3}{2} \quad \forall\, n$$

Also $a_n > 1\ \ \forall\, n$

Since $1 < a_n < \dfrac{3}{2}\ \ \forall\, n$ $\therefore$ **$< a_n >$ is bounded.**

(*xiii*) $a_n = \dfrac{1}{n^2} + \dfrac{1}{(n+1)^2} + \dfrac{1}{(n+2)^2} + \ldots\ldots + \dfrac{1}{(2n)^2}$ [*Note that a_n is the sum of $(n + 1)$ terms*]

$$< \frac{1}{n^2} + \frac{1}{n^2} + \frac{1}{n^2} + \ldots\ldots + \frac{1}{n^2} \qquad \left|\ \because\ (n+k)^2 > n^2 \text{ for } k = 1, 2, \ldots\ldots, n \quad \therefore\ \frac{1}{(n+k)^2} < \frac{1}{n^2}\right.$$

$$= \frac{n+1}{n^2} = \frac{1}{n} + \frac{1}{n^2} \leq 2 \quad \forall\, n \in \text{N}$$

Also $a_n > 0 \quad \forall\, n$ ($\because$ each term is +ve)

Since $0 < a_n \leq 2 \quad \forall\, n$ $\therefore$ **$< a_n >$ is bounded.**

Example 9. *Prove that the sequences, whose nth terms are given below, are monotonic. Find out whether they are increasing or decreasing :*

(*i*) $\dfrac{1}{2} + \dfrac{1}{2^2} + \dfrac{1}{2^3} + \ldots\ldots + \dfrac{1}{2^n}$ (*ii*) $\dfrac{1}{n} + \dfrac{1}{n+1} + \dfrac{1}{n+2} + \ldots\ldots + \dfrac{1}{2n-1}$

(*iii*) $\dfrac{3n+7}{4n+8}$ (*iv*) $\dfrac{2n+7}{3n+8}$ (*v*) $1 + \dfrac{1}{1!} + \dfrac{1}{2!} + \ldots\ldots + \dfrac{1}{(n+1)!}$

(*vi*) $\frac{1}{2n+5}$ (*vii*) $-\frac{1}{2n+1}$ (*viii*) $2n+7$

(*ix*) $\frac{1}{n}$ (*x*) $1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots+\frac{1}{n}$ (*xi*) $\left(1+\frac{1}{n}\right)^n$

(*xii*) $a_1 = 1$ *and* $a_n = \sqrt{2+a_{n-1}}$, $n \geq 2$.

Sol. (*i*) $$a_n = \frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots\ldots+\frac{1}{2^n}$$

$$a_{n+1} = \frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots\ldots+\frac{1}{2^n}+\frac{1}{2^{n+1}}$$

$\therefore$ $$a_{n+1}-a_n = \frac{1}{2^{n+1}} > 0 \quad \forall\, n \Rightarrow a_{n+1} > a_n \quad \forall\, n$$

$\Rightarrow$ $< a_n >$ is monotonically increasing and hence monotonic.

(*ii*) $$a_n = \frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\ldots\ldots+\frac{1}{2n-1}$$

$$a_{n+1} = \frac{1}{n+1}+\frac{1}{n+2}+\ldots\ldots+\frac{1}{2n-1}+\frac{1}{2n}+\frac{1}{2n+1}$$

$\therefore$ $$a_{n+1}-a_n = \frac{1}{2n}+\frac{1}{2n+1}-\frac{1}{n} = \frac{1}{2n+1}-\frac{1}{2n} = \frac{-1}{2n(2n+1)} < 0 \;\forall\, n$$

$\Rightarrow$ $a_{n+1} < a_n \quad \forall\, n$

$\Rightarrow$ $< a_n >$ is monotonically decreasing and hence monotonic.

(*iii*) $$a_n = \frac{3n+7}{4n+8} = \frac{3n+7}{4(n+2)}$$

$$a_{n+1} = \frac{3n+10}{4(n+3)}$$

$\therefore$ $$a_{n+1}-a_n = \frac{3n+10}{4(n+3)}-\frac{3n+7}{4(n+2)} = \frac{(3n+10)(n+2)-(3n+7)(n+3)}{4(n+3)(n+2)}$$

$$= \frac{-1}{4(n+3)(n+2)} < 0 \quad \forall\, n$$

$\therefore$ $a_{n+1} < a_n \quad \forall\, n$

$\Rightarrow$ $< a_n >$ is monotonically decreasing and hence monotonic.

(*iv*) *Please try yourself.* **[Ans.** Monotonically decreasing]

(*v*) $$a_n = 1+\frac{1}{1!}+\frac{1}{2!}+\ldots\ldots+\frac{1}{(n-1)!}$$

$$a_{n+1} = 1+\frac{1}{1!}+\frac{1}{2!}+\ldots\ldots+\frac{1}{(n-1)!}+\frac{1}{n!}$$

$\therefore$ $$a_{n+1}-a_n = \frac{1}{n!} > 0 \quad \forall\, n$$

$\Rightarrow$ $a_{n+1} > a_n \quad \forall\, n$

$\Rightarrow$ $< a_n >$ is monotonically increasing and hence monotonic.

(*vi*) *Please try yourself.* **[Ans.** Monotonically decreasing]
(*vii*) *Please try yourself.* **[Ans.** Monotonically increasing]
(*viii*) *Please try yourself.* **[Ans.** Monotonically increasing]
(*ix*) *Please try yourself.* **[Ans.** Monotonically decreasing]
(*x*) *Please try yourself.* **[Ans.** Monotonically increasing]

(*xi*) $$a_n = \left(1+\frac{1}{n}\right)^n = 1+n\cdot\frac{1}{n}+\frac{n(n+1)}{2!}\cdot\frac{1}{n^2}+\ldots\ldots+\frac{n(n-1)\ldots\ldots 1}{n!}\cdot\frac{1}{n^n}$$

(By Binomial Theorem for a positive integral index)

$$= 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\ldots\ldots$$
$$+\frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\ldots\left(1-\frac{n-1}{n}\right) \qquad \ldots(1)$$

$$a_{n+1} = 1+1+\frac{1}{2!}\left(1-\frac{1}{n+1}\right)+\frac{1}{3!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)$$
$$+\ldots\ldots+\frac{1}{n+1!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\ldots\ldots\left(1-\frac{n}{n+1}\right) \qquad \ldots(2)$$

Now $\frac{1}{n+1} < \frac{1}{n} \quad \forall\, n$

$\Rightarrow$ $\frac{k}{n+1} < \frac{k}{n}$ where $k \in \mathbb{N}$

$\Rightarrow$ $-\frac{k}{n+1} > -\frac{k}{n} \Rightarrow 1-\frac{k}{n+1} > 1-\frac{k}{n}$, $k = 1, 2, \ldots\ldots, n$

$\therefore$ From (1) and (2), we have $a_{n+1} > a_n \quad \forall\, n$

$\Rightarrow$ $< a_n >$ is monotonically increasing and hence monotonic.

(*xii*) $a_1 = 1$

Also $a_2 = \sqrt{2+a_1} = \sqrt{2+1} = \sqrt{3} > 1 = a_1 \Rightarrow a_2 > a_1$.

Now suppose $a_n > a_{n-1}$ for some $n \in \mathbb{N}$, $n > 1$.

Then $2+a_n > 2+a_{n-1}$

$\Rightarrow$ $\sqrt{2+a_n} > \sqrt{2+a_{n-1}} \Rightarrow a_{n+1} > a_n$.

$\therefore$ By mathematical induction, $a_{n+1} > a_n \quad \forall\, n$

$\Rightarrow$ $< a_n >$ is monotonically increasing and hence monotonic.

Example 10. *Discuss the convergence of the sequence* $< a_n >$ *where :*

(*i*) $a_n = \frac{n+1}{n}$ (*ii*) $a_n = \frac{n}{n^2+1}$ (*iii*) $a_n = 1+\frac{1}{3}+\frac{1}{3^2}+\ldots\ldots+\frac{1}{3^n}$.

Sol. (*i*) $$a_n = \frac{n+1}{n} = 1+\frac{1}{n}$$

$$a_{n+1} = 1+\frac{1}{n+1}$$

$\therefore$ $$a_{n+1}-a_n = \frac{1}{n+1}-\frac{1}{n} = \frac{-1}{n(n+1)} < 0 \quad \forall\, n.$$

$\Rightarrow$ $<a_n>$ is monotonically decreasing.

Also $\quad a_n = 1 + \frac{1}{n} > 1 \quad \forall\, n \quad \Rightarrow \quad <a_n>$ is bounded below.

Since $<a_n>$ is monotonically decreasing and bounded below, it is convergent.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1 \quad \therefore \quad <a_n> \text{ converges to } 1.$$

(ii) $$a_n = \frac{n}{n^2+1}$$

$$a_{n+1} = \frac{n+1}{(n+1)^2+1}$$

$\therefore$ $$a_{n+1} - a_n = \frac{n+1}{(n+1)^2+1} - \frac{n}{n^2+1} = \frac{(n+1)(n^2+1) - n(n^2+2n+2)}{(n^2+2n+2)(n^2+1)}$$

$$= \frac{-n^2-n+1}{(n^2+2n+2)(n^2+1)} < 0 \quad \forall\, n$$

$\Rightarrow$ $\quad a_{n+1} < a_n \ \forall\, n$

$\Rightarrow$ $<a_n>$ is monotonically decreasing.

Also $\quad a_n = \frac{n}{n^2+1} > 0 \quad \forall\, n \quad \Rightarrow \quad <a_n>$ is bounded below

Since $<a_n>$ is monotonically decreasing and bounded below, it is convergent.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n^2+1} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1+\frac{1}{n^2}} = 0$$

$\therefore$ $<a_n>$ converges to 0.

(iii) $$a_n = 1 + \frac{1}{3} + \frac{1}{3^2} + \ldots\ldots + \frac{1}{3^n}$$

$$a_{n+1} = 1 + \frac{1}{3} + \frac{1}{3^2} + \ldots\ldots + \frac{1}{3^n} + \frac{1}{3^{n+1}}$$

$\therefore$ $$a_{n+1} - a_n = \frac{1}{3^{n+1}} > 0 \quad \forall\, n$$

$\Rightarrow$ $\quad a_{n+1} > a_n \quad \forall\, n.$

$\Rightarrow$ $<a_n>$ is monotonically increasing.

Also $$a_n = 1 + \frac{1}{3} + \frac{1}{3^2} + \ldots\ldots + \frac{1}{3^n}$$

[G.P. of $(n+1)$ terms with $a = 1$, $r = \frac{1}{3}$]

$$= \frac{1\left[1-\left(\frac{1}{3}\right)^{n+1}\right]}{1-\frac{1}{3}} = \frac{3}{2}\left(1-\frac{1}{3^{n+1}}\right) = \frac{3}{2} - \frac{1}{2\times 3^n} < \frac{3}{2} \quad \forall\, n.$$

$\Rightarrow$ $<a_n>$ is bounded above.

Since $<a_n>$ is monotonically increasing and bounded above, it is convergent.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3}{2}\left(1-\frac{1}{3^{n+1}}\right) = \frac{3}{2}(1-0) = \frac{3}{2}$$

$\Rightarrow$ $<a_n>$ converges to $\frac{3}{2}$.

Example 11. *Prove that the sequence* $\left\{\frac{2n-7}{3n+2}\right\}$ *is.*

(i) monotonically increasing *(ii) bounded and*

(iii) tends to the limit $\frac{2}{3}$.

Sol. Let $\quad a_n = \frac{2n-7}{3n+2}$

(i) $$a_{n+1} = \frac{2n-5}{3n+5}$$

$\therefore$ $$a_{n+1} - a_n = \frac{2n-5}{3n+5} - \frac{2n-7}{3n+2}$$

$$= \frac{(2n-5)(3n+2)-(2n-7)(3n+5)}{(3n+5)(3n+2)} = \frac{(6n^2-11n-10)-(6n^2-11n-35)}{(3n+5)(3n+2)}$$

$$= \frac{25}{(3n+5)(3n+2)} > 0 \quad \forall\, n$$

$\Rightarrow$ $\quad a_{n+1} > a_n \quad \forall\, n$

$\Rightarrow$ $<a_n>$ is monotonically increasing.

(ii) The given sequence is $\left\{-1, -\frac{3}{8}, -\frac{1}{11}, \frac{1}{14}, \frac{3}{17}, \ldots\ldots\right\}$

Clearly $\quad a_n \geq -1 \quad \forall\, n$

Also $\quad 1 - a_n = 1 - \frac{2n-7}{3n+2} = \frac{n+9}{3n+2} > 0 \quad \forall\, n$

$\Rightarrow$ $\quad a_n < 1 \quad \forall\, n$

Thus $\quad -1 \leq a_n < 1 \quad \forall\, n$

$\Rightarrow$ The sequence $<a_n>$ is bounded.

(iii) Since $<a_n>$ is monotonically increasing and bounded above, it converges.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n-7}{3n+2} = \lim_{n\to\infty} \frac{2-\frac{7}{n}}{3+\frac{2}{n}} = \frac{2}{3}$$

$\therefore$ $<a_n>$ converges to $\frac{2}{3}$.

Example 12. *Prove that the sequence whose nth term is* $\frac{3n+4}{2n+1}$

(i) is monotonically decreasing *(ii) is bounded and*

(iii) converges to $\frac{3}{2}$.

Sol. *Please try yourself.*

Example 13. (*a*) *Prove that the sequence* $\{a_n\}$ *defined by*

$$a_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots\ldots + \frac{1}{n!}$$

is convergent and that $2 \leq \underset{n\to\infty}{Lt}\, a_n \leq 3$.

(*b*) *Prove that the sequence* $< a_n >$ *defined by the relation* $a_1 = 1$,

$$a_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots\ldots + \frac{1}{(n-1)!}, (n \geq 2)$$

converges.

Sol. (*a*)
$$a_{n+1} - a_n = \left(1 + \frac{1}{1!} + \frac{1}{2!} + \ldots\ldots + \frac{1}{(n+1)!}\right) - \left(1 + \frac{1}{1!} + \frac{1}{2!} + \ldots\ldots + \frac{1}{n!}\right)$$

$$= \frac{1}{(n+1)!} > 0 \quad \forall\, n$$

$\Rightarrow$ $a_{n+1} > a_n \quad \forall\, n$

$\therefore$ $\{a_n\}$ is a monotonically increasing sequence.

Also
$$a_n = 1 + \frac{1}{1} + \frac{1}{1.2} + \frac{1}{1.2.3} + \ldots\ldots + \frac{1}{1.2.3\ldots\ldots n}$$

$$< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots\ldots + \frac{1}{2^{n-1}} = 1 + \frac{\left(1 - \frac{1}{2^n}\right)}{1 - \frac{1}{2}} = 1 + 2 - \frac{1}{2^{n-1}}$$

$\Rightarrow$ $a_n < 3 - \frac{1}{2^{n-1}} < 3 \quad \forall\, n$

Now, $\{a_n\}$ is monotonically increasing and bounded above

$\therefore$ It converges to its l.u.b. which is ≤ 3

$\therefore$ $\underset{n\to\infty}{\text{Lt}}\, a_n \leq 3$

Since
$$a_n = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots\ldots + \frac{1}{n!}$$

$\geq 2\ \forall\, n$ $\quad\left[\text{Equality holds for } a_1 = 1 + \frac{1}{1!} = 2\right]$

$\therefore$ $\underset{n\to\infty}{\text{Lt}}\, a_n \geq 2$

Hence $2 \leq \underset{n\to\infty}{\text{Lt}}\, a_n \leq 3$.

(*b*) *Please try yourself.*

Example 14. *Show that the sequence* $\{a_n\}$ *defined by* $a_n = \left(1 + \frac{1}{n}\right)^n$ *is convergent.*

Sol. Since n is a + ve integer, we have, by Binomial Theorem

$$a_n = \left(1 + \frac{1}{n}\right)^n = 1 + n \cdot \frac{1}{n} + \frac{n(n-1)}{2!} \cdot \frac{1}{n^2} + \ldots\ldots + \frac{n(n-1)\ldots 2.1}{n!} \cdot \frac{1}{n^n}$$

$$= 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \ldots\ldots$$
$$+ \frac{1}{n!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\ldots\ldots\left(1 - \frac{n-1}{n}\right)$$

Changing n to $n + 1$

$$a_{n+1} = 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n+1}\right) + \frac{1}{3!}\left(1 - \frac{1}{n+1}\right)\left(1 - \frac{2}{n+1}\right) + \ldots\ldots$$
$$+ \frac{1}{(n+1)!}\left(1 - \frac{1}{n+1}\right)\left(1 - \frac{2}{n+1}\right)\ldots\ldots\left(1 - \frac{n}{n+1}\right)$$

Now $\quad \frac{1}{n+1} < \frac{1}{n} \Rightarrow -\frac{1}{n+1} > -\frac{1}{n} \Rightarrow 1 - \frac{1}{n+1} > 1 - \frac{1}{n}$.

Similarly $\quad 1 - \frac{k}{n+1} > 1 - \frac{k}{n}$ for $k = 2, \ldots\ldots, n - 1$

It follows that $\quad a_{n+1} \geq a_n\ \forall\, n$

$\Rightarrow$ $\{a_n\}$ is monotonically increasing.

Also $a_n = 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \ldots + \frac{1}{n!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\ldots\ldots\left(1 - \frac{n-1}{n}\right)$

$$< 1 + 1 + \frac{1}{2!} + \frac{1}{3!}\ \ldots\ldots + \frac{1}{n!} = 1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \ldots\ldots + \frac{1}{1.2.3\ldots n}$$

$$< 1 + 1 + \frac{1}{2} + \frac{1}{2.2} + \ldots\ldots + \frac{1}{2.2\ldots\ldots 2} = 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots\ldots + \frac{1}{2^{n-1}}$$

$$= 1 + \frac{1\left(1 - \frac{1}{2^n}\right)}{1 - \frac{1}{2}} = 1 + 2\left(1 - \frac{1}{2^n}\right) = 3 - \frac{1}{2^{n-1}} < 3\ \forall\, n$$

$\Rightarrow$ $\{a_n\}$ is bounded above.

Since $\{a_n\}$ is monotonically increasing and bounded above, it is covergent.

Note. Clearly $\quad a_n \geq 2 \quad \therefore \quad 2 \leq a_n < 3\ \forall\, n$

$\Rightarrow$ $\quad 2 \leq \lim\limits_{n\to\infty} a_n \leq 3 \Rightarrow 2 \leq \lim\limits_{n\to\infty}\left(1 + \frac{1}{n}\right)^n \leq 3$

This limit is denoted by e.

Example 15. (*i*) *Show that the sequence* $\{a_n\}$ *defined by* $a_1 = \sqrt{2}$ *and* $a_{n+1} = \sqrt{2a_n}$ *converges to 2.*

(*ii*) *Show that the sequence* $\{S_n\}$ *defined by* $s_{n+1} = \sqrt{3\,s_n}$, $s_1 = 1$ *converges to 3.*

(*iii*) *If* $S_{n+1} = \sqrt{7S_n}$, $S_1 = 1$, *prove that* $< S_n >$ *is covergent. What is its limit ?*

Sol. (*i*) $a_1 = \sqrt{2},\ a_2 = \sqrt{2a_1} = \sqrt{2\sqrt{2}}$

$\because$ $\sqrt{2} > 1 \ \therefore\ 2\sqrt{2} > 2 \ \Rightarrow\ \sqrt{2\sqrt{2}} > \sqrt{2} \Rightarrow a_2 > a_1.$

Suppose $a_{n+1} > a_n$, then $\sqrt{2a_{n+1}} > \sqrt{2a_n} \ \Rightarrow\ a_{n+2} > a_{n+1}.$

Thus $a_{n+1} > a_n \ \Rightarrow\ a_{n+2} > a_{n+1}.$

By mathematical induction, the sequence $\{a_n\}$ is monotonically *increasing*.

Now $a_1 = \sqrt{2} < 2$

Suppose $a_n < 2$, then $a_{n+1} = \sqrt{2a_n} < \sqrt{2.2} = 2$

Thus $a_n < 2 \ \Rightarrow\ a_{n+1} < 2.$

$\therefore$ By mathematical induction, $a_n < 2 \quad \forall\, n$

$\Rightarrow$ $\{a\}$ is *bounded above.*

Since $\{a_n\}$ is monotonically increasing and bounded above, it is convergent.

Let $\underset{n \to \infty}{\text{Lt}}\ a_n = l$ then $\lim\limits_{n \to \infty} a_{n+1} = l$

$\because$ $a_{n+1} = \sqrt{2a_n} \quad \therefore\ a^2_{n+1} = 2a_n.$

Proceeding to the limit as $n \to \infty$, $l^2 = 2l \Rightarrow l(l-2) = 0 \Rightarrow l = 0$ or 2

But $l \neq 0$ $\because$ the sequence $\{a_n\}$ is monotonically increasing.

$\therefore$ $\sqrt{2} = a_1 < a_2 < a_3 \ldots\ldots$

$\Rightarrow$ $\{a_n\}$ is bounded below by $\sqrt{2}$ $\Rightarrow$ $\underset{n \to \infty}{\text{Lt}}\ a_n \geq \sqrt{2}.$

Hence $l = 2.$

(*ii*) *Please try yourself.*

(*iii*) *Please try yourself.* **[Ans. 7]**

Example 16. *Show that the sequence $\{x_n\}$ where $x_1 = 1$ and $x_n = \sqrt{2 + x_{n-1}}$, $\forall\, n \geq 2$ is convergent and converges to 2.*

Sol. $x_1 = 1,\ x_n = \sqrt{2 + x_{n-1}}$

$\therefore$ $x_2 = \sqrt{2 + x_1} = \sqrt{2+1} = \sqrt{3} > 1 = x_1 \ \therefore\ x_2 > x_1$

Suppose $x_n > x_{n-1} \quad \Rightarrow\ 2 + x_n > 2 + x_{n-1}$

$\Rightarrow$ $\sqrt{2 + x_n} > \sqrt{2 + x_{n-1}} \quad \Rightarrow\ x_{n+1} > x_n$

$\therefore$ By mathematical induction, $x_{n+1} > x_n \quad \forall\, n$

$\Rightarrow$ $\{x_n\}$ is monotonically increasing.

Now $x_1 = 1 < 2.$

Suppose $x_n < 2$, then $2 + x_n < 4 \ \Rightarrow\ \sqrt{2 + x_n} < 2 \ \Rightarrow\ x_{n+1} < 2.$

$\therefore$ By mathematical induction, $x_n < 2 \ \forall\, n$

$\Rightarrow$ $\{x_n\}$ is bounded above.

Since $\{x_n\}$ is monotonically increasing and bounded above, it is convergent.

Let $\underset{n \to \infty}{\text{Lt}}\ x_n = l$

Now $x_n = \sqrt{2 + x_{n-1}} \quad \Rightarrow\ x_n^{\,2} = 2 + x_{n-1}$

$\Rightarrow$ $\underset{n \to \infty}{\text{Lt}}\ x_n^{\,2} = \underset{n \to \infty}{\text{Lt}}\ (2 + x_{n-1}) \ \Rightarrow\ l^2 = 2 + l \ \Rightarrow\ l^2 - l - 2 = 0$

$\Rightarrow$ $(l - 2)(l + 1) = 0 \ \Rightarrow\ l = 2$ or -1

Since $x_n > 0 \ \ \forall\, n$, $l = \underset{n \to \infty}{\text{Lt}}\ x_n$ can't be – ve.

$\therefore$ $l \neq -1$. Hence $l = 2.$

Thus the sequence $\{x_n\}$ converges to the +ve root of the equation $x^2 - x - 2 = 0.$

Example 17. *Prove that the sequence $\{u_n\}$ defined by $u_1 = \sqrt{7}$, $u_{n+1} = \sqrt{7 + u_n}$ converges to the positive root of the equation $x^2 - x - 7 = 0$.*

Sol. $u_1 = \sqrt{7},\ u_{n+1} = \sqrt{7 + u_n}$

$\therefore$ $u_2 = \sqrt{7 + u_1} = \sqrt{7 + \sqrt{7}} > \sqrt{7} = u_1 \ \Rightarrow\ u_2 > u_1$

Suppose $u_n > u_{n-1}$

$\Rightarrow$ $7 + u_n > 7 + u_{n-1} \ \Rightarrow\ \sqrt{7 + u_n} > \sqrt{7 + u_{n-1}} \ \Rightarrow\ u_{n+1} > u_n$

$\therefore$ By mathematical induction, $u_{n+1} > u_n \ \forall\, n$

$\Rightarrow$ $\{u_n\}$ is monotonically increasing.

Now $u_1 = \sqrt{7} < 7$

Suppose $u_n < 7$

then $7 + u_n < 7 + 7 \ \Rightarrow\ \sqrt{7 + u_n} < \sqrt{14} < \sqrt{49} = 7 \ \Rightarrow\ u_{n+1} < 7$

$\therefore$ By mathematical induction, $u_n < 7 \ \forall\, n$

$\Rightarrow$ $\{u_n\}$ is bounded above.

Since $\{u_n\}$ is monotonically increasing and bounded above, it is convergent.

Let $\underset{n \to \infty}{\text{Lt}}\ u_n = l$

Now $u_{n+1} = \sqrt{7 + u_n} \quad \Rightarrow\ (u_{n+1})^2 = 7 + u_n$

$\Rightarrow$ $\underset{n \to \infty}{\text{Lt}}\ (u_{n+1})^2 = \underset{n \to \infty}{\text{Lt}}\ (7 + u_n) \quad \Rightarrow\ l^2 = 7 + l$

$\Rightarrow$ $l^2 - l - 7 = 0 \quad \Rightarrow\ l = \dfrac{1 \pm \sqrt{29}}{2}$

But $\dfrac{1 - \sqrt{29}}{2} < 0$ whereas $u_n > 0 \ \forall\, n$

$\therefore$ $l \neq \dfrac{1 - \sqrt{29}}{2}$

Hence the sequence $\{u_n\}$ converges to $\dfrac{1 + \sqrt{29}}{2}$ which is the +ve root of the equation $x^2 - x - 7 = 0.$

Example 18. *Prove that the sequence $\{x_n\}$ defined by $x_1 = \sqrt{2}$, $x_{n+1} = \sqrt{2 + x_n}$ converges to the positive root of the equation $x^2 - x - 2 = 0$.*

Sol. *Please try yourself.*

Example 19. *A sequence $< S_n >$ is defined as follows :*

$$S_1 = a > 0,\ S_{n+1} = \sqrt{\frac{ab^2 + S_n^2}{a+1}},\ b > a,\ n \geq 1.$$

Show that $< S_n >$ is convergent and find its limit.

Sol. $a < b$ (given) $\Rightarrow S_1 < b$ ...(1)

Also $S^2_{n+1} - b^2 = \dfrac{ab^2 + S_n^2}{a+1} - b^2 = \dfrac{S_n^2 - b^2}{a+1} < 0$ whenever $S_n^2 - b^2 < 0$

$\Rightarrow$ $S^2_{n+1} < b^2$ whenever $S_n^2 < b^2$

$\Rightarrow$ $S_{n+1} < b$ whenever $S_n < b$ ...(2)

From (1) and (2), by mathematical induction, we have $S_n < b\ \forall\ n$

$\therefore$ $< S_n >$ is bounded above.

Now $S^2_{n+1} - S_n^2 = \dfrac{ab^2 + S_n^2}{a+1} - S_n^2 = \dfrac{a(b^2 - S_n^2)}{a+1} > 0$ $[\because a > 0, b > S_n]$

$\Rightarrow$ $S^2_{n+1} > S_n^2 \Rightarrow S_{n+1} > S_n\ \forall\ n$

$\therefore$ $< S_n >$ is monotonical increasing.

Since the sequence $< S_n >$ is monotonically increasing and bounded above, it is convergent.

Let $\lim\limits_{n \to \infty} S_n = l$

Now $S^2_{n+1} = \dfrac{ab^2 + S_n^2}{a+1}$

When $n \to \infty$, $S_n \to l$ and also $S_{n+1} \to l$

$\therefore$ $l^2 = \dfrac{ab^2 + l^2}{a+1}$

$\Rightarrow$ $al^2 + l^2 = ab^2 + l^2 \Rightarrow l^2 = b^2 \therefore l = \pm b$

But l cannot be negative $(\because S_n > 0\ \forall\ n)$

$\therefore$ $l = b.$

Hence the sequence $< S_n >$ converges to b.

Example 20. *If a_1, b_1 are two positive unequal numbers and a_n, b_n are defined as*

$$a_n = \frac{1}{2}(a_{n-1} + b_{n-1}),\ b_n = \sqrt{a_{n-1}\, b_{n-1}} \text{ for } n \geq 2,$$

prove that two sequences $\{a_n\}$ and $\{b_n\}$ are monotonic, one increasing and the other decreasing and that they tend to the same limit.

Sol. Let $a_1 > b_1$.

For any two +ve numbers, the arithmetic mean is greater than the geometric mean.

$\therefore$ $a_n > b_n$, for all n. ...(i)

Also $a_{n+1} = \dfrac{1}{2}(a_n + b_n) < \dfrac{1}{2}(a_n + a_n) = a_n$ $[\because b_n < a_n]$

$\therefore$ $\{a_n\}$ is monotonic decreasing

$\Rightarrow$ $a_1 > a_2 > a_3 > a_4$ ....(ii)

Again $b_{n+1} = \sqrt{a_n . b_n} > \sqrt{b_n . b_n} = b_n$ $[\because a_n > b_n]$

$\therefore$ $\{b_n\}$ is monotonic increasing *i.e.,* $b_1 < b_2 < b_3 < b_4$ ....(iii)

Now $a_n = \dfrac{1}{2}(a_{n-1} + b_{n-1}) > \dfrac{1}{2}(b_{n-1} + b_{n-1}) = b_{n-1}$ $[\because$ of (i)$]$

i.e., $a_n > b_{n-1} > b_{n-2} > \ldots\ldots > b_2 > b_1$ [from (iii)]

$\therefore$ $a_n > b_1$ for all n.

This implies that the sequence $\{a_n\}$ is bounded below and being monotonic decreasing is convergent.

Again, $b_n = \sqrt{a_{n-1} . b_{n-1}} < \sqrt{a_{n-1} . a_{n-1}} = a_{n-1}$ $[\because$ of (i)$]$

i.e., $b_n < a_{n-1} < a_{n-2} < \ldots\ldots < a_2 < a_1$ [from (ii)]

or $b_n < a_1$ for all n.

$\therefore$ $\{b_n\}$ is bounded above and being monotonic increasing is convergent.

Now let $\text{Lt}\, a_n = l$ and $\text{Lt}\, b_n = l'$.

Since $a_n = \dfrac{1}{2}(a_{n-1} + b_{n-1})$ or $2a_n = a_{n-1} + b_{n-1}$

$\therefore$ $2a_{n+1} = a_n + b_n$

or $\text{Lt}\,(2a_{n+1}) = \text{Lt}\, a_n + \text{Lt}\, b_n$ or $2l = l + l'$ or $l = l'$.

$\therefore$ $\{a_n\}$ and $\{b_n\}$ converge to the same limit.

Example 21. *If $a_1 > 0, b_1 > 0$ and $a_n = \sqrt{a_{n-1} . b_{n-1}}$ and $b_n = \dfrac{2a_{n-1}\, b_{n-1}}{a_{n-1} + b_{n-1}}$, prove that (i) $\{a_n\}$ and $\{b_n\}$ are monotonic, the one increasing and the other decreasing, (ii) $\{a_n\}$ and $\{b_n\}$ both converge to a common limit.*

Sol. Suppose $a_1 > b_1$ ...(1)

$\because$ for any two +ve numbers, G.M. > H.M.

$\therefore$ $a_n > b_n\ \forall\ n$ ...(2)

Also $a_{n+1} = \sqrt{a_n . b_n} < \sqrt{a_n . a_n} = a_n$ $[\because b_n < a_n]$

$\therefore$ $\{a_n\}$ is monotonic decreasing, *i.e.,* $a_1 > a_2 > a_3 > \ldots\ldots > a_n > \ldots\ldots$

or $a_1 \geq a_n$ for all n. ...(3)

Again, $b_{n+1} = \dfrac{2a_n b_n}{a_n + b_n} > \dfrac{2b_n . b_n}{b_n + b_n} = b_n$ $[\because a_n > b_n]$

$\therefore$ $\{b_n\}$ is monotonic increasing, *i.e.,* $b_1 < b_2 < b_3 < \ldots\ldots < b_n < \ldots\ldots$

or $b_n \geq b_1$ for all n. ...(4)

From (1), (2), (3) and (4), we get $a_1 \geq a_n > b_n \geq b_1$ for all n. ...(5)

Now since $a_n > b_1\ \forall\ n$, $\therefore$ $\{a_n\}$ is bounded below and being monotonic decreasing converges to a limit say a, *i.e.,* $\text{Lt}\, a_n = a$.

Again from (5), since $b_n < a_1$ for all n, $\{b_n\}$ is bounded above and being monotonic increasing converges to a limit say b. *i.e.,* $\text{Lt}\, b_n = b$

Now $a_{n+1} = \sqrt{a_n . b_n}$ or $a^2_{n+1} = a_n b_n$.

$\therefore$ We have $\text{Lt}\, a^2_{n+1} = \text{Lt}\, a_n \,.\, \text{Lt}\, b_n$ or $a^2 = ab$ or $a = b$, which proves the result.

Example 22. *A sequence $<a_n>$ is defined as $a_1 = 1$, $a_{n+1} = \dfrac{4+3a_n}{3+2a_n}$, $n \geq 1$. Show that $<a_n>$ converges and find its limit.*

Sol. $a_1 = 1, \quad a_2 = \dfrac{4+3a_1}{3+2a_1} = \dfrac{7}{5} > 1 \Rightarrow a_2 > a_1.$

Let us assume that $a_{n+1} > a_n$

Then $$a_{n+2} - a_{n+1} = \frac{4+3a_{n+1}}{3+2a_{n+1}} - \frac{4+3a_n}{3+2a_n} = \frac{a_{n+1} - a_n}{(3+2a_{n+1})(3+2a_n)} > 0$$
$[\because a_{n+1} > a_n$ and $a_n > 0\ \forall\ n]$

$\Rightarrow a_{n+2} > a_{n+1}$.

$\therefore$ By mathematical induction, $<a_n>$ is monotonically increasing.

Also $$a_{n+1} = \frac{4+3a_n}{3+2a_n} = \frac{3}{2} - \frac{1}{2(3+2a_n)}$$
$$= \frac{3}{2} - (\text{a positive quantity less than } 1)$$
$[\because a_n > a_1 = 1\ \forall\ n]$
$$< \frac{3}{2} \Rightarrow a_{n+1} < \frac{3}{2}\ \forall\ n.$$

$\therefore$ The sequence $<a_n>$ is bounded above.

Since the sequence $<a_n>$ is monotonically increasing and bounded above, it is convergent.

Let the sequence $<a_n>$ converge to l, then $\lim\limits_{n\to\infty} a_n = l$

Now $a_{n+1} = \dfrac{4+3a_n}{3+2a_n} \Rightarrow \lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} \dfrac{4+3a_n}{3+2a_n}$

$\Rightarrow l = \dfrac{4+3l}{3+2l} \Rightarrow 3l + 2l^2 = 4 + 3l$

$\Rightarrow l^2 = 2 \quad \therefore l = \pm\sqrt{2}$

But l cannot be negative. $(\because a_n \geq 1\ \forall\ n, \ \therefore l = \lim\limits_{n\to\infty} a_n \geq 1)$.

$\therefore$ Rejecting $l = -\sqrt{2}$, we have $l = \sqrt{2}$.

Example 23. (*a*) *Prove that the sequence $<u_n>$ defined by $u_{n+1} = \dfrac{4u_n + 6}{3u_n + 4}$, $u_1 = 1$ converges. Find its limit.*

(*b*) *Let $<a_n>$ be a sequence defined by $a_1 = 1$, $a_{n+1} = \dfrac{3+2a_n}{2+a_n}$, $n \geq 1$. Show that $<a_n>$ is convergent and find its limit.*

Sol. (*a*) *Please try yourself.* **[Ans. $\sqrt{2}$]**

(*b*) *Please try yourself.* **[Ans. $\sqrt{3}$]**

Example 24. *Prove that a sequence $<a_n>$ is a null sequence if and only if the sequence $<|a_n|>$ is a null sequence.*

Sol. Suppose $<a_n>$ is a null sequence, then $\lim\limits_{n\to\infty} a = 0$

$\therefore$ Given $\varepsilon > 0$, $\exists$ a +ve integer m s.t. $|a_n - 0| < \varepsilon \quad \forall\ n \geq m$

$\Rightarrow |a_n| < \varepsilon \quad \forall\ n \geq m$

$\Rightarrow ||a_n| - 0| = ||a_n|| = |a_n| < \varepsilon \quad \forall\ n \geq m$

$\Rightarrow \lim\limits_{n\to\infty} |a_n| = 0.$

$\Rightarrow <|a_n|>$ is a null sequence.

$\therefore <a_n>$ is a null sequence $\Rightarrow <|a_n|>$ is also a null sequence.

Now, **suppose $<|a_n|>$ is a null sequence**, then $\lim\limits_{n\to\infty} |a_n| = 0$

$\therefore$ Given $\varepsilon > 0$, $\exists$ a +ve integer m s.t. $||a_n| - 0| < \varepsilon \quad \forall\ n \geq m$

$\Rightarrow ||a_n|| < \varepsilon \quad \forall\ n \geq m$

$\Rightarrow |a_n| < \varepsilon \quad \forall\ n \geq m$

$\Rightarrow |a_n - 0| < \varepsilon \quad \forall\ n \geq m$

$\Rightarrow \lim\limits_{n\to\infty} a_n = 0$

$\Rightarrow <a_n>$ is a null sequence.

$\therefore <|a_n|>$ is a null sequence $\Rightarrow <|a_n|>$ is also a null sequence.

Hence $<a_n>$ is a null sequence iff $<a_n>$ is a null sequence.

Example 25. *Prove that a sequence $<a_n>$ converges to l if and only if the sequence $<a_n - l>$ is a null sequence.*

Sol. Suppose $<a_n>$ converges to l, then given $\varepsilon > 0$, $\exists$ a +ve integer m s.t.

$|a_n - l| < \varepsilon \quad \forall\ n \geq m$

$\Rightarrow |(a_n - l) - 0| = |a_n - l| < \varepsilon \quad \forall\ n \geq m$

$\Rightarrow \lim\limits_{n\to\infty} (a_n - l) = 0$

$\Rightarrow <a_n - l>$ is a null sequence.

$\therefore <a_n>$ converges to $l \Rightarrow <a_n - l>$ is a null sequence.

Now, **suppose $<a_n - l>$ is a null sequence, then** $\lim\limits_{n\to\infty} (a_n - l) = 0$.

$\Rightarrow$ Given $\varepsilon > 0$, $\exists$ a +ve integer m $|(a_n - l) - 0| < \varepsilon \quad \forall\ n \geq m$

$\Rightarrow |a_n - l| < \varepsilon \quad \forall\ n \geq m$

$\Rightarrow \lim\limits_{n\to\infty} a_n = l$

$\Rightarrow <a_n>$ converges to l.

$\therefore <a_n - l>$ is a null sequence $\Rightarrow <a_n>$ converges to l.

Hence $<a_n>$ converges to l iff $<a_n - l>$ is a null sequence.

Example 26. *Prove that if $|a_n| \leq |b_n| \quad \forall\ n$ and $<b_n>$ is a null sequence, then $<a_n>$ is also a null sequence.*

Sol. $< b_n >$ is a null sequence

$\Rightarrow$ Given $\varepsilon > 0$, $\exists$ a +ve integer m s.t. $|b_n| < \varepsilon \quad \forall n \geq m$

Since $\quad |a_n| \leq |b_n| \quad \forall n$

$\therefore \quad |a_n| < \varepsilon \quad \forall n \geq m$

$\Rightarrow$ $< a_n >$ is a null sequence.

Example 27. *If $< a_n >$ and $< b_n >$ are null sequences, show that $< a_n + b_n >$ is also a null sequence.*

Sol. Let $\varepsilon > 0$ be given.

Since $a_n \to 0$ and $b_n \to 0$.

$\therefore$ $\exists$ positive integers m_1 and m_2 s.t. $\quad |a_n| < \dfrac{\varepsilon}{2} \quad \forall n \geq m_1$

and $\quad |b_n| < \dfrac{\varepsilon}{2} \quad \forall n \geq m_2$

Let $m = \max.\ \{m_1, m_2\}$, then $|a_n| < \dfrac{\varepsilon}{2}$ and $|b_n| < \dfrac{\varepsilon}{2} \quad \forall n \geq m$

Now $\quad |a_n + b_n| \leq |a_n| + |b_n| < \dfrac{\varepsilon}{2} + \dfrac{\varepsilon}{2} = \varepsilon \quad \forall n \geq m$

$\Rightarrow \quad a_n + b_n \to 0. \quad \Rightarrow \quad < a_n + b_n >$ is a null sequence

Example 28. *If $< a_n >$ is a null sequence and $< b_n >$ is a bounded sequence, then $< a_n b_n >$ is a null sequence.*

Sol. $< b_n >$ is a bounded sequence.

$\Rightarrow$ $\exists$ a real number $k > 0$ s.t. $\quad |b_n| < k \quad \forall n$

Also $< a_n >$ is a null sequence.

$\Rightarrow$ Given $\varepsilon > 0$, $\exists$ a +ve integer m s.t. $\quad |a_n| < \dfrac{\varepsilon}{k} \quad \forall n \geq m.$

Now $\quad |a_n b_n| = |a_n||b_n| < \dfrac{\varepsilon}{k} \cdot k = \varepsilon \quad \forall n \geq m$

$\Rightarrow \quad a_n b_n \to 0$

$\Rightarrow$ $< a_n b_n >$ is a null sequence.

Example 29. *If a sequence $\{a_n\}$ oscillates finitely and $\lim\limits_{n \to \infty} b_n = 0$, then $\lim\limits_{n \to \infty} (a_n b_n) = 0.$*

Sol. Since the sequence $\{a_n\}$ oscillates finitely, it is bounded.

Now proceed as in Example 28.

Example 30. *If $< a_n >$ is a null sequence and c is a constant, then $< ca_n >$ is a null sequence.*

Sol. $< a_n >$ is a null sequence.

$\Rightarrow$ Given $\varepsilon > 0$, $\exists$ a +ve integer m s.t. $|a_n| < \dfrac{\varepsilon}{|c|+1} \quad \forall n \geq m.$

Now $\quad |ca_n| = |c||a_n| < \left(\dfrac{|c|}{|c|+1}\right)\varepsilon < \varepsilon \quad \forall n \geq m$

$\Rightarrow \quad ca_n \to 0 \quad \Rightarrow \quad < ca_n >$ is a null sequence.

Example 31. *Give an example of a monotonically increasing sequence which is (i) convergent, (ii) divergent.*

Sol. (*i*) Consider the sequence $< a_n >$ where $\quad a_n = \dfrac{n}{n+1}$

$$a_{n+1} = \frac{n+1}{n+2}$$

$$a_{n+1} - a_n = \frac{n+1}{n+2} - \frac{n}{n+1} = \frac{(n+1)^2 - n(n+2)}{(n+2)(n+1)} = \frac{1}{(n+2)(n+1)} > 0 \quad \forall n$$

$\Rightarrow \quad a_{n+1} > a_n \quad \forall n$

$\Rightarrow$ $< a_n >$ is monotonically increasing.

Also $\quad n < n+1 \quad \forall n$

$\Rightarrow \quad \dfrac{n}{n+1} < 1 \quad \forall n \quad \Rightarrow \quad a_n < 1 \quad \forall n$

$\Rightarrow$ $< a_n >$ is bounded above.

Since $< a_n >$ is monotonically increasing and bounded above, it is convergent.

(*ii*) Consider the sequence $< a_n >$ where $a_n = n$

$$a_{n+1} - a_n = (n+1) - n = 1 > 0 \quad \forall n$$

$\Rightarrow \quad a_{n+1} > a_n \ \forall n$

$\Rightarrow$ $< a_n >$ is monotonically increasing.

Also, given any positive real number K, however large, there exists a +ve integer m s.t.

$$a_n > \text{K} \ \forall n \geq m$$

$\Rightarrow$ $< a_n >$ is not bounded above.

Since $< a_n >$ is monotonically increasing and not bounded above, it is divergent.

Example 32. *Give an example of a monotonically decreasing sequence which is (i) convergent (ii) divergent.*

Sol. (*i*) Consider the sequence $< a_n >$ where $\quad a_n = \dfrac{1}{n}.$

$$a_{n+1} - a_n = \frac{1}{n+1} - \frac{1}{n} = \frac{-1}{n(n+1)} < 0 \quad \forall n$$

$\Rightarrow \quad a_{n+1} < a_n \quad \forall n$

$\Rightarrow$ $< a_n >$ is monotonically decreasing.

Also $\quad a_n > 0 \quad \forall n.$

$\Rightarrow$ $< a_n >$ is bounded below.

Since $< a_n >$ is monotonically decreasing and bounded below, it is convergent.

(*ii*) Consider the sequence $< a_n >$, where $\quad a_n = -n$

$$a_{n+1} - a_n = -(n+1) - (-n) = -1 < 0 \quad \forall n$$

$\Rightarrow \quad a_{n+1} < a_n \ \forall n$

$\Rightarrow$ $< a_n >$ is monotonically decreasing.

Also, given any positive real number K, however large, there exists a +ve integer m s.t.

$$a_n < -\text{K} \quad \forall n \geq m$$

$\Rightarrow \quad <a_n>$ is not bounded below.

Since $<a_n>$ is monotonically decreasing and not bounded below, it is divergent.

Example 33. *Prove that the sequence whose nth term is $a_n = \sqrt{n+1} - \sqrt{n}$ is*

(i) monotonic *(ii) bounded* *(iii) convergent.*

Sol.
$$a_n = \sqrt{n+1} - \sqrt{n}$$
$$= (\sqrt{n+1} - \sqrt{n}) \times \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \frac{(n+1)-n}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}}$$

(i)
$$a_{n+1} = \frac{1}{\sqrt{n+2}+\sqrt{n+1}} < \frac{1}{\sqrt{n+1}+\sqrt{n}} = a_n \quad \forall\, n$$

$\Rightarrow \quad <a_n>$ is monotonically decreasing and hence monotonic.

(ii)
$$a_n = \frac{1}{\sqrt{n+1}+\sqrt{n}} > 0 \quad \forall\, n$$

Also
$$a_n = \frac{1}{\sqrt{n+1}+\sqrt{n}} < \frac{1}{\sqrt{n}+\sqrt{n}} = \frac{1}{2\sqrt{n}} \le \frac{1}{2} \;\forall\, n \;\Rightarrow\; 0 < a_n \le \frac{1}{2} \;\forall\, n$$

$\Rightarrow \quad <a_n>$ is bounded sequence.

(iii) Since $<a_n>$ is monotonically decreasing and bounded below, it is convergent.
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} = 0$$

$\therefore \quad <a_n>$ converges to 0.

Example 34. *Show that the sequence $a_n = \sqrt{n^2+4n} - n$ is convergent.*

Sol. $a_n = \sqrt{n^2+4n} - n$
$$= (\sqrt{n^2+4n} - n) \times \frac{\sqrt{n^2+4n}+n}{\sqrt{n^2+4n}+n} = \frac{(n^2+4n)-n^2}{\sqrt{n^2+4n}+n} = \frac{4n}{\sqrt{n^2+4n}+n} = \frac{4}{\sqrt{1+\frac{4}{n}}+1}$$

$$\operatorname*{Lt}_{n\to\infty} a_n = \operatorname*{Lt}_{n\to\infty} \frac{4}{\sqrt{1+\frac{4}{n}}+1} = \frac{4}{1+1} = 2$$

$\Rightarrow \quad \{a_n\}$ converges to 2.

Example 35. *What is a monotonic sequence ? Give an example of each of the following :*

(i) a monotonically increasing sequence. (ii) a monotonically decreasing sequence.

(iii) a sequence which is not monotonic.

(iv) a monotonically increasing sequence which is not bounded above.

(v) a monotonically decreasing sequence which is not bounded below.

(vi) a sequence which is bounded but not monotonic.

Sol. A sequence $<a_n>$ is said to be monotonically if it is either monotonically increasing or monotonically decreasing.

Thus $<a_n>$ is monotonic if either $a_{n+1} \ge a_n \;\forall\, n \in \mathbb{N}$ or $a_{n+1} \le a_n \;\forall\, n \in \mathbb{N}$.

(i) The sequence $<a_n>$ where $a_n = n$ is a monotonically increasing sequence because
$$a_{n+1} - a_n = (n+1) - n = 1 > 0 \quad \forall\, n$$
$$\Rightarrow \quad a_{n+1} > a_n \quad \forall\, n.$$

(ii) The sequence $<a_n>$ defined by $a_n = -n$ is a monotonically decreasing sequence because
$$a_{n+1} - a_n = -(n+1) - (-n) = -1 < 0 \quad \forall\, n$$
$$\Rightarrow \quad a_{n+1} < a_n \;\forall\, n.$$

(iii) The sequence $<a_n>$ defined by $a_n = (-1)^n \;\forall\, n \in \mathbb{N}$ is not monotonic, because it is neither monotonically increasing nor monotonically decreasing.

(iv) The sequence $<a_n>$ defined by $a_n = n$ is a monotonically increasing sequence which is not bounded above.

See Example 31 (ii).

(v) The sequence $<a_n>$ defined by $a_n = -n$ is a monotonically decreasing sequence which is not bounded below.

See Example 32 (ii).

(vi) The sequence $<a_n>$ defined by $a_n = (-1)^n$ is bounded because its range $= \{-1, 1\}$ is a finite set.

Since $<a_n>$ is neither monotonically increasing nor monotonically decreasing, it is not monotonic.

Example 36. *Give an example of a sequence $<a_n>$ which is not bounded but for which $\lim_{n\to\infty} \frac{a_m}{n} = 0$.*

Sol. Consider the sequence $<a_n>$ where $a_n = \sqrt{n}$.

It is bounded below but not above.

$\Rightarrow \quad <a_n>$ is not bounded.
$$\lim_{n\to\infty} \frac{a_n}{n} = \lim_{n\to\infty} \frac{\sqrt{n}}{n} = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0.$$

Example 37. *Prove that the sequence $<a_n>$ where $a_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots\ldots + \frac{1}{2n}$ is convergent and its limit lies between $\frac{1}{2}$ and 1.*

Sol.
$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots\ldots + \frac{1}{2n}$$
$$a_{n+1} = \frac{1}{n+2} + \frac{1}{n+3} + \ldots\ldots + \frac{1}{2n} + \frac{1}{2n+1} + \frac{1}{2n+2}$$
$$\therefore \quad a_{n+1} - a_n = \frac{1}{2n+2} + \frac{1}{2n+1} - \frac{1}{n+1}$$
$$> \frac{1}{2n+2} + \frac{1}{2n+2} - \frac{1}{n+1} = \frac{2}{2n+2} - \frac{1}{n+1} = \frac{1}{n+1} - \frac{1}{n+1} = 0$$
$$\Rightarrow \quad a_{n+1} - a_n > 0 \quad \forall\, n$$
$$\Rightarrow \quad a_{n+1} > a_n \quad \forall\, n$$

$\Rightarrow \quad <a_n>$ is monotonically increasing.

Also $a_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots\ldots + \frac{1}{n+n}$

$< \frac{1}{n+1} + \frac{1}{n+1} + \ldots\ldots + \frac{1}{n+1} = \frac{n}{n+1} < 1 \ \forall n$

$\Rightarrow$ $< a_n >$ is bounded above.

Since $< a_n >$ is monotonically increasing and bounded above, it is convergent.

Now $a_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots\ldots + \frac{1}{2n}$

$> \frac{1}{2n} + \frac{1}{2n} + \ldots\ldots + \frac{1}{2n} = \frac{n}{2n} = \frac{1}{2} \ \forall n$

$\therefore$ $\frac{1}{2} < a_n < 1 \ \forall n$

Hence $\frac{1}{2} \le \lim_{n\to\infty} a_n \le 1.$

4.23. ALGEBRA OF LIMITS

Theorem. *If* $\lim_{n\to\infty} a_n = a$ *and* $\lim_{n\to\infty} b_n = b$, *then*

(*i*) $\lim_{n\to\infty} ka_n = ka$ *where k is any constant* (*ii*) $\lim_{n\to\infty} | a_n | = | a |$

(*iii*) $\lim_{n\to\infty} (a_n + b_n) = a + b$ (*iv*) $\lim_{n\to\infty} (a_n - b_n) = a - b$

(*v*) $\lim_{n\to\infty} (a_n b_n) = ab$ (*vi*) $\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{a}$ *provided* $a \neq 0$

(*vii*) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$ *provided* $b \neq 0$.

Proof. (*i*) Let $\varepsilon > 0$ be given.

$$| ka_n - ka | = | k(a_n - a) | = | k || a_n - a |$$

Since $\lim_{n\to\infty} a_n = a$, $\exists$ a +ve integer m s.t. $| a_n - a | < \frac{\varepsilon}{|k|+1}$ $\forall n \ge m$

$\therefore$ $| ka_n - ka | = | k || a_n - a |$

$< \frac{|k|\varepsilon}{|k|+1} < \varepsilon$ $\forall n \ge m$

$\Rightarrow$ $| ka_n - ka | < \varepsilon$ $\forall n \ge m$

$\Rightarrow$ $\lim_{n\to\infty} ka_n = ka.$

(*ii*) Let $\varepsilon > 0$ be given.

We know that for any two real numbers x and y,

$$|| x | - | y || \le | x - y |.$$

$\therefore$ $|| a_n | - | a || \le | a_n - a |$

Since $\lim_{n\to\infty} a_n = a$, $\exists$ +ve integer m s.t. $| a_n - a | < \varepsilon$ $\forall n > m$

$\therefore$ $|| a_n | - | a || \le | a_n - a | < \varepsilon$ $\forall n \ge m$

$\Rightarrow$ $|| a_n | - | a || < \varepsilon$ $\forall n \ge m$

$\Rightarrow$ $\lim_{n\to\infty} | a_n | = | a |.$

(*iii*) Let $\varepsilon > 0$ be given.

Since $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$, $\exists$ positive integers m_1 and m_2 such that

$$| a_n - a | < \frac{\varepsilon}{2} \ \forall n \ge m_1 \quad \text{and} \quad | b_n - b | < \frac{\varepsilon}{2} \ \forall n \ge m_2$$

Let $m = \max. \{m_1, m_2\}$, then $| a_n - a | < \frac{\varepsilon}{2}$ and $| b_n - b | < \frac{\varepsilon}{2}$ $\forall n \ge m$

$\therefore$ $| (a_n + b_n) - (a + b) | = | (a_n - a) + (b_n - b) |$

$\le | a_n - a | + | b_n - b | < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ $\forall n \ge m$

$\Rightarrow$ $| (a_n + b_n) - (a + b) | < \varepsilon$ $\forall n \ge m$

$\Rightarrow$ $\lim_{n\to\infty} (a_n + b_n) = a + b$

Note. The converse of the above theorem is not necessarily true. *i.e.*, the existence of $\lim_{n\to\infty} (a_n + b_n)$ does not necessarily imply that the two limits $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ also exist.

For example, consider $a_n = n$ and $b_n = -n$.

Then both the sequences are divergent, but $a_n + b_n = n + (-n) = 0$ $\forall n$

so that $\lim_{n\to\infty} (a_n + b_n) = 0$ exists.

(*iv*) Let $\varepsilon > 0$ be given.

Since $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$, $\exists$ positive integers m_1 and m_2 such that

$$| a_n - a | < \frac{\varepsilon}{2} \ \forall n \ge m_1 \quad \text{and} \quad | b_n - b | < \frac{\varepsilon}{2} \ \forall n \ge m_2$$

Let $m = \max. \{m_1, m_2\}$, then $| a_n - a | < \frac{\varepsilon}{2}$ and $| b_n - b | < \frac{\varepsilon}{2}$ $\forall n \ge m$.

$\therefore$ $| (a_n - b_n) - (a - b) | = | (a_n - a) - (b_n - b) |$

$\le | a_n - a | + | b_n - b |$ $[\because \ | x - y | \le | x | + | y |]$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ $\forall n \ge m$

$\Rightarrow$ $| (a_n - b_n) - (a - b) | < \varepsilon$ $\forall n \ge m$

$\Rightarrow$ $\lim_{n\to\infty} (a_n - b_n) = a - b.$

Note. The converse of the above theorem is not necessarily true *i.e.*, the existence of $\lim_{n\to\infty} (a_n - b_n)$ does not necessarily imply that the two limits $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ also exist.

For example, consider $a_n = n$ and $b_n = n$.

Then both the sequences are divergent, but $a_n - b_n = n - n = 0 \quad \forall\, n$

so that $\lim_{n\to\infty} (a_n - b_n) = 0$ exists.

(*v*) Let $\varepsilon > 0$ be given.

$$\begin{aligned} |a_nb_n - ab| &= |a_nb_n - ab_n + ab_n - ab| = |b_n(a_n - a) + a(b_n - b)| \\ &\le |b_n(a_n - a)| + |a(b_n - b)| = |b_n|\,|a_n - a| + |a|\,|b_n - b| \end{aligned} \quad \text{...(1)}$$

The sequence $<b_n>$ being convergent is bounded.

$\therefore\ \exists$ a +ve real number k s.t. $|b_n| < k \quad \forall\, n$ ...(2)

Since $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$, $\exists$ positive integers m_1 and m_2 such that

$$|a_n - a| < \frac{\varepsilon}{2k} \ \forall\, n \ge m_1 \quad \text{and} \quad |b_n - b| < \frac{\varepsilon}{2(|a|+1)} \quad \forall\, n \ge m_2$$

Let $m = \max.\{m_1, m_2\}$, then $|a_n - a| < \frac{\varepsilon}{2k}$

and $$|b_n - b| < \frac{\varepsilon}{2(|a|+1)} \quad \forall\, n \ge m$$

From (1), (2) and (3), we have

$$\begin{aligned} |a_nb_n - ab| &\le |b_n|\,|a_n - a| + |a|\,|b_n - b| && \forall\, n \\ &< k\cdot\frac{\varepsilon}{2k} + |a|\cdot\frac{\varepsilon}{2(|a|+1)} && \forall\, n \ge m \\ &= \frac{\varepsilon}{2} + \frac{|a|}{|a|+1}\cdot\frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon && \forall\, n \ge m \end{aligned}$$

$\Rightarrow \quad |a_nb_n - ab| < \varepsilon \quad \forall\, n \ge m$

$\Rightarrow \quad \lim_{n\to\infty} a_nb_n = ab.$

Note. The converse of the above theorem is not necessarily true. *i.e.*, the existence of $\lim_{n\to\infty} (a_nb_n)$ does not necessarily imply that the two limits. $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ also exist.

For example, consider $a_n = b_n = (-1)^n$, then both the limits $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ do not exist.

But $a_nb_n = (-1)^n \cdot (-1)^n = (-1)^{2n} = 1 \quad \forall\, n$ so that $\lim_{n\to\infty} (a_nb_n) = 1$ exists.

(*vi*) Let $\varepsilon > 0$ be given.

$$\left|\frac{1}{a_n} - \frac{1}{a}\right| = \left|\frac{a - a_n}{a_na}\right| = \frac{|a - a_n|}{|a_na|} = \frac{|a_n - a|}{|a_n|\,|a|} \quad \text{...(1)}$$

Since $\lim_{n\to\infty} a_n = a$, $\therefore \lim_{n\to\infty} |a_n| = |a|$

Taking $\varepsilon = \frac{|a|}{2} > 0 \qquad (\because a \ne 0)$

$\exists$ a +ve integer m_1 s.t. $\big||a_n| - |a|\big| < \frac{|a|}{2} \quad \forall\, n \ge m_1$

$\Rightarrow \quad |a| - \frac{|a|}{2} < |a_n| < |a| + \frac{|a|}{2} \quad \forall\, n \ge m_1$

$\Rightarrow \quad |a_n| > \frac{|a|}{2} \quad \forall\, n \ge m_1$

$\Rightarrow \quad \frac{1}{|a_n|} < \frac{2}{|a|} \quad \forall\, n \ge m_1.$

Also, $\lim_{n\to\infty} a_n = a$, therefore, there exists a positive integer m_2, such that

$$|a_n - a| < \frac{|a|^2\varepsilon}{2} \quad \forall\, n \ge m_2$$

Let $m = \max.\{m_1, m_2\}$, then $\frac{1}{|a_n|} < \frac{2}{|a|}$

and $$|a_n - a| < \frac{|a|^2\varepsilon}{2} \quad \forall\, n \ge m \quad \text{...(2)}$$

$\therefore$ From (1) and (2), we have $\left|\frac{1}{a_n} - \frac{1}{a}\right| = \frac{|a_n - a|}{|a_n|\,|a|}$

$$< \frac{2}{|a|}\cdot\frac{1}{|a|}\cdot\frac{|a|^2\varepsilon}{2} = \varepsilon \quad \forall\, n \ge m$$

$\Rightarrow \quad \left|\frac{1}{a_n} - \frac{1}{a}\right| < \varepsilon \quad \forall\, n \ge m$

$\Rightarrow \quad \lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{a}.$

(*vii*) Let $\varepsilon > 0$ be given.

$$\begin{aligned} \left|\frac{a_n}{b_n} - \frac{a}{b}\right| &= \left|\frac{a_nb - ab_n}{b_nb}\right| = \left|\frac{a_nb - ab + ab - ab_n}{b_nb}\right| \\ &= \frac{|b(a_n - a) + a(b - b_n)|}{|b_nb|} \le \frac{|b(a_n - a)| + |a(b - b_n)|}{|b_n|\,|b|} \\ &= \frac{|b|\,|a_n - a| + |a|\,|b_n - b|}{|b_n|\,|b|} = \frac{|a_n - a|}{|b_n|} + \frac{|a|\,|b_n - b|}{|b_n|\,|b|} \end{aligned} \quad \text{...(1)}$$

Since $\lim_{n\to\infty} b_n = b$, $\therefore \lim_{n\to\infty} |b_n| = |b|$

Taking $\varepsilon = \frac{|b|}{2} > 0 \qquad (\because b \ne 0)$

$\exists$ a +ve integer m_1 s.t. $\big||b_n| - |b|\big| < \frac{|b|}{2} \quad \forall\, n \ge m_1$

$\Rightarrow \quad |b| - \dfrac{|b|}{2} < |b_n| < |b| + \dfrac{|b|}{2} \quad \forall\, n \geq m_1$

$\Rightarrow \quad |b_n| > \dfrac{|b|}{2} \quad \forall\, n \geq m_1$

$\Rightarrow \quad \dfrac{1}{|b_n|} < \dfrac{2}{|b|} \quad \forall\, n \geq m_1 \qquad \ldots(2)$

Also $\quad \lim\limits_{n\to\infty} a_n = a$ and $\lim\limits_{n\to\infty} b_n = b.$

$\Rightarrow$ $\exists$ positive integers m_2 and m_3 such that

$$|a_n - a| < \frac{|b|\varepsilon}{4} \quad \forall\, n \geq m_2$$

and $$|b_n - b| < \frac{|b|^2 \varepsilon}{4(|a|+1)} \quad \forall\, n \geq m_3 \qquad \ldots(3)$$

Let $m = \max.\{m_1, m_2, m_3\}$, then from (1), (2) and (3), we have

$$\left|\frac{a_n}{b_n} - \frac{a}{b}\right| \leq \frac{|a_n - a|}{|b_n|} + \frac{|a||b_n - b|}{|b_n||b|}$$

$$< \frac{2}{|b|}\cdot\frac{|b|\varepsilon}{4} + \frac{|a|}{|b|}\cdot\frac{2}{|b|}\cdot\frac{|b|^2\varepsilon}{4(|a|+1)} \quad \forall\, n \geq m$$

$$= \frac{\varepsilon}{2} + \frac{|a|}{|a|+1}\cdot\frac{\varepsilon}{2}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \;\; \forall\, n \geq m$$

$\Rightarrow \quad \left|\dfrac{a_n}{b_n} - \dfrac{a}{b}\right| < \varepsilon \quad \forall\, n \geq m$

$\Rightarrow \quad \lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{a}{b}.$

Note. The converse of the above theorem is not necessarily true, *i.e.*, the existence of $\lim\limits_{n\to\infty}\left(\dfrac{a_n}{b_n}\right)$ does not necessarily imply that the two limits $\lim\limits_{n\to\infty} a_n$ and $\lim\limits_{n\to\infty} b_n$ also exist.

For example, consider $a_n = b_n = n$, then both the limits $\lim\limits_{n\to\infty} a_n$ and $\lim\limits_{n\to\infty} b_n$ do not exist.

But $\dfrac{a_n}{b_n} = \dfrac{n}{n} = 1 \;\; \forall\, n$ so that $\lim\limits_{n\to\infty}\left(\dfrac{a_n}{b_n}\right) = 1$ exists.

4.24. SOME THEOREMS ON LIMITS

Theorem I. *If $\lim\limits_{n\to\infty} a_n = a$ and $a \neq 0$, then there exists a positive number k and a positive integer m such that $|a_n| > k$ whenever $n \geq m$.*

Proof. Since $a \neq 0$, we may take $\varepsilon = \dfrac{1}{2}|a| > 0.$

Also, $a_n \to a$, therefore, there exists a positive integer m such that

$$|a_n - a| < \varepsilon \quad \forall\, n \geq m \qquad \ldots(1)$$

Now $\quad |a| = |(a - a_n) + a_n| \leq |a - a_n| + |a_n| = |a_n - a| + |a_n|$

$\quad < \varepsilon + |a_n| \quad \forall\, n \geq m \qquad$ [By (1)]

$\Rightarrow \quad |a| - \varepsilon < |a_n| \quad \forall\, n \geq m$

$\Rightarrow \quad |a| - \dfrac{1}{2}|a| < |a_n| \quad \forall\, n \geq m$

$\Rightarrow \quad |a_n| > \dfrac{1}{2}|a| \quad \forall\, n \geq m.$

Therefore, we have found a positive number $k \left(= \dfrac{1}{2}|a|\right)$ and a positive integer m such that

$$|a_n| > k \;\forall\, n \geq m.$$

Theorem II. *If a sequence $\{a_n\}$ converges to a and $a_n \geq 0 \;\forall\, n$, then $a \geq 0$.*

Proof. If possible, let $a < 0$.

Since the sequence $\{a_n\}$ converges to a, $\exists$ a +ve integer m s.t.

$$|a_n - a| < \varepsilon \;\text{ for }\; n \geq m \;\Rightarrow\; a - \varepsilon < a_n < a + \varepsilon \text{ for } n \geq m$$

Taking $\quad \varepsilon = -\dfrac{a}{2} > 0 \qquad (\because\; a < 0)$

We have $\quad a + \dfrac{a}{2} < a_n < a - \dfrac{a}{2} \quad$ for $\; n \geq m$

$\Rightarrow \quad a_n < \dfrac{a}{2} < 0 \quad$ for $\; n \geq m$

But this contradicts the fact that $a_n \geq 0 \;\; \forall\, n.$

Hence $\quad a \geq 0.$

Theorem III. *If $a_n \to a$, $b_n \to b$ and $a_n \leq b_n \;\; \forall\, n$, then $a \leq b$.*

Proof. Let $\quad c_n = b_n - a_n \quad \forall\, n$

Then $\quad c_n \geq 0 \quad \forall\, n \quad \because\; b_n \geq a_n \;\; \forall\, n$

Also $\quad \mathop{\text{Lt}}\limits_{n\to\infty} c_n = \mathop{\text{Lt}}\limits_{n\to\infty} b_n - \mathop{\text{Lt}}\limits_{n\to\infty} a_n = b - a$

i.e., $\quad c_n \to b - a$ and $c_n \geq 0 \;\; \forall\, n$

$\therefore \quad b - a \geq 0 \;\Rightarrow\; a \leq b.$

Corollary. *If $a_n \leq k \;\; \forall\, n$ and $\lim\limits_{n\to\infty} a_n = a$, then $a \leq k$.*

Let $\quad b_n = k - a_n$, then $b_n \geq 0 \;\; \forall\, n$

Also $\quad \lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} (k - a_n) = k - a$

$\therefore \quad k - a \geq 0 \;\Rightarrow\; a \leq k.$

Theorem IV. (*Squeeze Principle*)

If $< a_n >$, $< b_n >$ and $< c_n >$ are three sequences such that

(i) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = 1$

(ii) for some positive integer p, $a_n \leq c_n \leq b_n \;\forall\, n \geq p$

then $\quad \lim\limits_{n\to\infty} c_n = 1.$

Proof. Let $\varepsilon > 0$ be given.

Since $a_n \to l$ and $b_n \to l$,

$\therefore$ $\exists$ positive integers m_1 and m_2 s.t.

$$|a_n - l| < \varepsilon \qquad \forall\, n \ge m_1$$

and $$|b_n - l| < \varepsilon \qquad \forall\, n \ge m_2$$

$\Rightarrow$ $$l - \varepsilon < a_n < l + \varepsilon \qquad \forall\, n \ge m_1$$

and $$l - \varepsilon < b_n < l + \varepsilon \qquad \forall\, n \ge m_2$$

Also $$a_n \le c_n \le b_n \qquad \forall\, n \ge p$$

Let $m = \max.\{m_1, m_2, p\}$, then $l - \varepsilon < a_n < l + \varepsilon \quad \forall\, n \ge m$

$$l - \varepsilon < b_n < l + \varepsilon \qquad \forall\, n \ge m$$

and $$a_n \le c_n \le b_n \qquad \forall\, n \ge m$$

$\Rightarrow$ $$l - \varepsilon < a_n \le c_n \le b_n < l + \varepsilon \qquad \forall\, n \ge m$$

$\Rightarrow$ $$l - \varepsilon < c_n < l + \varepsilon \qquad \forall\, n \ge m$$

$\Rightarrow$ $$|c_n - l| < \varepsilon \qquad \forall\, n \ge m$$

$\Rightarrow$ $$\lim_{n \to \infty} c_n = l.$$

Theorem V. *If $< a_n >$ is a sequence such that $a_n \neq 0$ for any n and $\frac{a_{n+1}}{a_n} \to l$ where $|l| < 1$, then $a_n \to 0$.*

Proof. Since $|l| < 1$, therefore, we can choose $\varepsilon_0 > 0$ s.t. $|l| + \varepsilon_0 = h < 1$ ...(1)

Since $\frac{a_{n+1}}{a_n} \to l$, therefore, there exists a positive integer m such that

$$\left|\frac{a_{n+1}}{a_n} - l\right| < \varepsilon_0 \quad \forall\, n \ge m$$

Now $$\left|\frac{a_{n+1}}{a_n}\right| = \left|\left(\frac{a_{n+1}}{a_n} - l\right) + l\right| \le \left|\frac{a_{n+1}}{a_n} - l\right| + |l|$$

$$< \varepsilon_0 + |l| \quad \forall\, n \ge m$$

$\Rightarrow$ $$\left|\frac{a_{n+1}}{a_n}\right| < h \qquad \forall\, n \ge m \qquad \text{[By (1)]}$$

Replacing n by $m, m+1, m+2, \ldots\ldots, n-1$ and multiplying the corresponding sides of the resulting $(n-m)$ inequalities, we have

$$\left|\frac{a_{m+1}}{a_m}\right| \cdot \left|\frac{a_{m+2}}{a_{m+1}}\right| \cdot \left|\frac{a_{m+3}}{a_{m+2}}\right| \ldots\ldots \left|\frac{a_n}{a_{n-1}}\right| < h^{n-m}$$

$\Rightarrow$ $$\left|\frac{a_{m+1}}{a_m} \cdot \frac{a_{m+2}}{a_{m+1}} \cdot \frac{a_{m+3}}{a_{m+2}} \ldots\ldots \frac{a_n}{a_{n-1}}\right| < h^{n-m} \Rightarrow \left|\frac{a_n}{a_m}\right| < h^{n-m}$$

$\Rightarrow$ $$|a_n| < h^n\left(\frac{|a_m|}{h^m}\right) \quad \forall\, n \ge m \qquad \ldots(1)$$

Also, since $0 < h < 1$, therefore, $h^n \to 0$.

Thus, given $\varepsilon > 0$, $\exists$ a positive integer p such that

$$|h^n| < \frac{h^m \varepsilon}{|a_m|} \quad \forall\, n \ge p \qquad \ldots(2)$$

Let $q = \max.\{m, p\}$, then $|a_n| < h^n\left(\frac{|a_m|}{h^m}\right) < \frac{h^m \varepsilon}{|a_m|} \cdot \frac{|a_m|}{h^m} = \varepsilon \;\; \forall\, n \ge q$ [By (1) and (2)]

$\Rightarrow$ $$|a_n| < \varepsilon \quad \forall\, n \ge q$$

$\Rightarrow$ $$a_n \to 0.$$

Theorem VI. Cauchy's First Theorem on Limits.

Statement. *If $\{a_n\}$ converges to l, then the sequence $\{x_n\}$ where*

$$x_n = \frac{a_1 + a_2 + \ldots\ldots + a_n}{n} \text{ also converges to } l$$

or $$\mathop{Lt}_{n \to \infty} a_n = l \Rightarrow \mathop{Lt}_{n \to \infty} \frac{a_1 + a_2 + \ldots\ldots + a_n}{n} = l.$$

Proof. Let $b_n = a_n - l$, then $\mathop{\text{Lt}}_{n \to \infty} b_n = \mathop{\text{Lt}}_{n \to \infty} a_n - l = l - l = 0$

$\Rightarrow$ $$b_n \to 0 \qquad \ldots(i)$$

Also $\because$ $$a_n = b_n + l$$

$\therefore$ $$x_n = \frac{(b_1 + l) + (b_2 + l) + \ldots\ldots + (b_n + l)}{n}$$

$$= \frac{(b_1 + b_2 + \ldots\ldots + b_n) + nl}{n} = \frac{b_1 + b_2 + \ldots\ldots + b_n}{n} + l.$$

$\therefore$ **In order to prove that $x_n \to l$, it is enough to prove that**

$$\frac{b_1 + b_2 + \ldots\ldots + b_n}{n} \to 0 \qquad \ldots(ii)$$

From (*i*), $\because$ $b_n \to 0$

$\therefore$ Given $\varepsilon > 0$, $\exists$ a +ve integer m s.t. $|b_n| < \varepsilon/2$ for $n \ge m$ ...(*iii*)

Again $b_n \to 0 \Rightarrow$ the sequence $\{b_n\}$ is convergent

$\Rightarrow$ $\{b_n\}$ is bounded

$\Rightarrow$ $\exists$ a real number $M > 0$ s.t. $|b_n| < M \;\forall\, n$ ...(*iv*)

Now let us prove (*ii*),

$$\left|\frac{b_1 + b_2 + \ldots\ldots + b_n}{n} - 0\right| = \left|\frac{b_1 + b_2 + \ldots\ldots + b_n}{n}\right| = \frac{|b_1 + b_2 + \ldots\ldots + b_n|}{|n|}$$

$$= \frac{1}{n}\,|b_1 + b_2 + \ldots\ldots b_m + b_{m+1} + \ldots\ldots b_n| \qquad [\because\; n \in N \;\therefore\; |n| = n]$$

$$\le \frac{1}{n}\left[(|b_1| + |b_2| \ldots + |b_m|) + (|b_{m+1}| \ldots + \ldots + |b_n|)\right]$$

$$< \frac{1}{n}\left[(M + M \ldots m \text{ times}) + \left(\frac{\varepsilon}{2} + \frac{\varepsilon}{2} + \ldots (n-m) \text{ times}\right)\right] \qquad \text{[Using (iv) and (iii)]}$$

$\Rightarrow$ $$\left|\frac{b_1 + b_2 + \ldots + b_n}{n} - 0\right| < \frac{1}{n}\left[mM + (n-m)\frac{\varepsilon}{2}\right]$$

$\Rightarrow$ $$\left|\frac{b_1 + b_2 + \ldots + b_n}{n} - 0\right| < \frac{mM}{n} + \frac{(n+m)\varepsilon}{2n} < \frac{mM}{n} + \frac{\varepsilon}{2} \qquad \left[\because\; \frac{n-m}{n} = 1 - \frac{m}{n} < 1\right]$$

Now $\quad \dfrac{mM}{n} < \dfrac{\varepsilon}{2}$ if $\dfrac{n}{mM} > \dfrac{2}{\varepsilon}$ i.e., if $n > \dfrac{2mM}{\varepsilon}$

If p is a natural number $> \dfrac{2mM}{\varepsilon}$ and $q = \max. \{m, p\}$, then

$$\left| \frac{b_1 + b_2 + \ldots + b_n}{n} - 0 \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \forall\, n \geq q$$

$$\Rightarrow \quad \frac{b_1 + b_2 + \ldots + b_n}{n} \to 0$$

$\therefore$ From (ii) $x_n \to l$ i.e., $\dfrac{a_1 + a_2 + \ldots + a_n}{n} \to l.$

Theorem VII. *Cauchy's Second Theorem on Limits.*

If $\{a_n\}$ is a sequence of positive terms, then $\underset{n\to\infty}{Lt}\, (a_n)^{1/n} = \underset{n\to\infty}{Lt}\, \dfrac{a_{n+1}}{a_n}$ *provided the limit on the right hand side exists, whether finite or infinite.*

Proof. Case I. Let $\underset{n\to\infty}{\text{Lt}}\, \dfrac{a_{n+1}}{a_n} = l$ where l is finite. Then given $\varepsilon > 0$, $\exists$ a +ve integer m s.t.

$$\left| \frac{a_{n+1}}{a_n} - l \right| < \frac{\varepsilon}{2} \qquad \forall\, n \geq m$$

$$\Rightarrow \quad l - \frac{\varepsilon}{2} < \frac{a_{n+1}}{a_n} < l + \frac{\varepsilon}{2} \qquad \forall\, n \geq m.$$

Putting $n = m, m + 1, m + 2, \ldots, n - 1$, we have

$$l - \frac{\varepsilon}{2} < \frac{a_{m+1}}{a_m} < l + \frac{\varepsilon}{2}$$

$$l - \frac{\varepsilon}{2} < \frac{a_{m+2}}{a_{m+1}} < l + \frac{\varepsilon}{2}$$

$$l - \frac{\varepsilon}{2} < \frac{a_{m+3}}{a_{m+2}} < l + \frac{\varepsilon}{2}$$

....

....

$$l - \frac{\varepsilon}{2} < \frac{a_n}{a_{n-1}} < l + \frac{\varepsilon}{2}.$$

Multiplying the above $(n - m)$ inequalities, we have

$$\left(l - \frac{\varepsilon}{2}\right)^{n-m} < \frac{a_n}{a_m} < \left(l + \frac{\varepsilon}{2}\right)^{n-m}$$

$$\Rightarrow \quad \left(l - \frac{\varepsilon}{2}\right)^{1-\frac{m}{n}} < \left(\frac{a_n}{a_m}\right)^{\frac{1}{n}} < \left(l + \frac{\varepsilon}{2}\right)^{1-\frac{m}{n}}$$

$$\Rightarrow \quad (a_m)^{1/n}\left(l - \frac{\varepsilon}{2}\right)^{1-\frac{m}{n}} < (a_n)^{1/n} < (a_m)^{1/n}\left(l + \frac{\varepsilon}{2}\right)^{1-\frac{m}{n}} \qquad \ldots(i)$$

Now as $n \to \infty$, $(a_m)^{1/n}\left(l - \dfrac{\varepsilon}{2}\right)^{1-\frac{m}{n}} \to l - \dfrac{\varepsilon}{2}$ $\qquad [\because\ a^{1/n} \to 1 \text{ as } n \to \infty \text{ for } a > 0]$

and $\quad (a_m)^{1/n}\left(l + \dfrac{\varepsilon}{2}\right)^{1-\frac{m}{n}} \to l + \dfrac{\varepsilon}{2}$

$\Rightarrow$ For above $\varepsilon > 0$, $\exists$ +ve integers m_1 and m_2 s.t.

$$\left| (a_m)^{1/n}\left(l - \frac{\varepsilon}{2}\right)^{1-\frac{m}{n}} - \left(l - \frac{\varepsilon}{2}\right) \right| < \frac{\varepsilon}{2} \qquad \forall\, n \geq m_1$$

and $$\left| (a_m)^{1/n}\left(l + \frac{\varepsilon}{2}\right)^{1-\frac{m}{n}} - \left(l + \frac{\varepsilon}{2}\right) \right| < \frac{\varepsilon}{2} \qquad \forall\, n \geq m_2$$

$$\Rightarrow \quad \left(l - \frac{\varepsilon}{2}\right) - \frac{\varepsilon}{2} < (a_m)^{1/n}\left(l - \frac{\varepsilon}{2}\right)^{1-\frac{m}{n}} < \left(l - \frac{\varepsilon}{2}\right) + \frac{\varepsilon}{2} \qquad \forall\, n \geq m_1$$

and $$\left(l + \frac{\varepsilon}{2}\right) + \frac{\varepsilon}{2} < (a_m)^{1/n}\left(l + \frac{\varepsilon}{2}\right)^{1-\frac{m}{n}} < \left(l + \frac{\varepsilon}{2}\right) + \frac{\varepsilon}{2} \qquad \forall\, n \geq m_2$$

$$\Rightarrow \quad l - \varepsilon < (a_m)^{1/n}\left(l - \frac{\varepsilon}{2}\right)^{1-\frac{m}{n}} < l \qquad \forall\, n \geq m_1 \qquad \ldots(ii)$$

and $$l < (a_m)^{1/n}\left(l + \frac{\varepsilon}{2}\right)^{1-\frac{m}{n}} < l + \varepsilon \qquad \forall\, n \geq m_2 \qquad \ldots(iii)$$

If $p = \max. \{m_1, m_2\}$, then combining (i), (ii) and (iii), we have

$$l - \varepsilon < (a_m)^{1/n}\left(l - \frac{\varepsilon}{2}\right)^{1-\frac{m}{n}} < (a_n)^{1/n} < (a_m)^{1/n}\left(l + \frac{\varepsilon}{2}\right)^{1-\frac{m}{n}} < l + \varepsilon \qquad \forall\, n \geq p$$

$$\Rightarrow \quad l - \varepsilon < (a_n)^{1/n} < l + \varepsilon \qquad \forall\, n \geq p$$

$$\Rightarrow \quad \underset{n\to\infty}{\text{Lt}}\, (a_n)^{1/n} = l.$$

Case II. Let $\underset{n\to\infty}{\text{Lt}}\, \dfrac{a_{n+1}}{a_n} = +\infty$

$$\Rightarrow \quad \underset{n\to\infty}{\text{Lt}}\, \frac{a_n}{a_{n+1}} = 0 \Rightarrow \underset{n\to\infty}{\text{Lt}}\, \frac{\dfrac{1}{a_{n+1}}}{\dfrac{1}{a_n}} = 0 \text{ (finite)}$$

$$\Rightarrow \quad \underset{n\to\infty}{\text{Lt}}\, \left(\frac{1}{a_n}\right)^{1/n} = 0 \qquad \text{[by Case I]}$$

$$\Rightarrow \quad \underset{n\to\infty}{\text{Lt}}\, a_n^{\,1/n} = +\infty.$$

Note 1. The converse of Cauchy's First Theorem on Limits is not always true.

For example, let $a_n = (-1)^n$.

Then $\dfrac{a_1 + a_2 + \ldots\ldots + a_n}{n} = 0$ if n is even

$= -\dfrac{1}{n}$ if n is odd

so that $\dfrac{a_1 + a_2 + \ldots\ldots + a_n}{n} \to 0$ as $n \to \infty$.

But $a_n = (-1)^n$ does not converge.

Note 2. The converse of Cauchy's Second Theorem on Limits is not always true.

For example, let $a_n = 2^{-n+(-1)^n}$

Then $(a_n)^{1/n} = 2^{-1+\frac{(-1)^n}{n}} \to 2^{-1} = \dfrac{1}{2}$ as $n \to \infty$

But $\dfrac{a_{n+1}}{a_n} = \dfrac{2^{-(n+1)+(-1)^{n+1}}}{2^{-n+(-1)^n}} = 2^{-1+(-1)^{n+1}-(-1)^n}$

$= 2^{-1-1-1}$ i.e., $\dfrac{1}{8}$ if n is even

$= 2^{-1+1+1}$ i.e., 2 if n is odd

so that $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n}$ does not exist.

ILLUSTRATIVE EXAMPLES

Example 1. *Show that the sequence $\{n^{1/n}\}$, converges to the limit 1.*

Sol. Let $a_n = n$, then $\dfrac{a_{n+1}}{a_n} = \dfrac{n+1}{n} = 1 + \dfrac{1}{n}$

$\therefore \quad \underset{n\to\infty}{\text{Lt}} \dfrac{a_{n+1}}{a_n} = 1$

By Cauchy's second theorem on limits, $\underset{n\to\infty}{\text{Lt}} (a_n)^{1/n} = \underset{n\to\infty}{\text{Lt}} \dfrac{a_{n+1}}{a_n} = 1$

$\Rightarrow \quad \underset{n\to\infty}{\text{Lt}}\, n^{1/n} = 1$. Hence $\{n^{1/n}\}$ converges to 1.

Other forms of above example

(*i*) Prove that $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$.

(*ii*) Show that $\lim\limits_{n\to\infty} a_n = 1$, where $a_n = n^{1/n}$, $\forall\, n \in \mathbf{N}$.

Example 2. (*a*) *Show that*

(*i*) $\underset{n\to\infty}{Lt} \dfrac{1}{n}\left(1 + \dfrac{1}{2} + \dfrac{1}{3} + \ldots\ldots + \dfrac{1}{n}\right) = 0$ (*ii*) $\underset{n\to\infty}{Lt} \dfrac{1}{n}\left(\dfrac{2}{1} + \dfrac{3}{2} + \dfrac{4}{3} + \ldots\ldots + \dfrac{n+1}{n}\right) = 1$

(*iii*) $\underset{n\to\infty}{Lt} \dfrac{1}{n}(1 + 2^{1/2} + 3^{1/3} + \ldots\ldots + n^{1/n}) = 1$ (*iv*) $\underset{n\to\infty}{Lt} \dfrac{1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots + \frac{1}{n^2}}{n} = 0$.

(*b*) *Using Cauchy's first theorem on limits, show that*

$$\lim_{n\to\infty}\left[\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \ldots\ldots + \frac{1}{\sqrt{n^2+n}}\right] = 1.$$

Sol. (*a*) (*i*) Let $a_n = \dfrac{1}{n}$, then $\underset{n\to\infty}{\text{Lt}}\, a_n = 0$

$\therefore$ By Cauchy's first theorem on limits.

$$\underset{n\to\infty}{\text{Lt}} \frac{a_1 + a_2 + a_3 + \ldots\ldots + a_n}{n} = 0 \quad \Rightarrow \quad \underset{n\to\infty}{\text{Lt}} \frac{1}{n}\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots + \frac{1}{n}\right) = 0.$$

(*ii*) Let $a_n = \dfrac{n+1}{n}$, then $\underset{n\to\infty}{\text{Lt}}\, a_n = \underset{n\to\infty}{\text{Lt}}\left(1 + \dfrac{1}{n}\right) = 1$

$\therefore$ By Cauchy's first theorem on limits

$$\underset{n\to\infty}{\text{Lt}} \frac{a_1 + a_2 + a_3 + \ldots\ldots + a_n}{n} = 1 \quad \Rightarrow \quad \underset{n\to\infty}{\text{Lt}} \frac{1}{n}\left(\frac{2}{1} + \frac{3}{2} + \frac{4}{3} + \ldots\ldots + \frac{n+1}{n}\right) = 1.$$

(*iii*) Let $a_n = n^{1/n}$, then $\underset{n\to\infty}{\text{Lt}}\, a_n = \underset{n\to\infty}{\text{Lt}}\, n^{1/n} = 1$

$\therefore$ By Cauchy's first theorem on limits

$$\underset{n\to\infty}{\text{Lt}} \frac{a_1 + a_2 + a_3 + \ldots\ldots + a_n}{n} = 1 \quad \Rightarrow \quad \underset{n\to\infty}{\text{Lt}} \frac{1}{n}(1 + 2^{1/2} + 3^{1/3} + \ldots\ldots + n^{1/n}) = 1.$$

(*iv*) *Please try yourself.*

(*b*) Let $a_k = \dfrac{n}{\sqrt{n^2+k}}$, then $a_n = \dfrac{n}{\sqrt{n^2+n}}$

and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{1}{\sqrt{1+\frac{1}{n}}} = 1$

$\therefore$ By Cauchy's first theorem on limits

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \ldots\ldots + a_n}{n} = 1$$

$$\Rightarrow \quad \lim_{n\to\infty} \frac{1}{n}\left[\frac{n}{\sqrt{n^2+1}} + \frac{n}{\sqrt{n^2+2}} + \ldots\ldots + \frac{n}{\sqrt{n^2+n}}\right] = 1$$

$$\Rightarrow \quad \lim_{n\to\infty}\left[\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \ldots\ldots + \frac{1}{\sqrt{n^2+n}}\right] = 1.$$

Example 3. *Find* $\underset{n\to\infty}{Lt} (n\,!)^{1/n}$.

Sol. Let $a_n = n\,!$, then $a_{n+1} = (n+1)\,!$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)\,!}{n\,!} = n + 1$$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}} \frac{a_{n+1}}{a_n} = \underset{n\to\infty}{\text{Lt}} (n+1) = \infty$$

By Cauchy's second theorem on limits

$$\underset{n\to\infty}{\text{Lt}}\ (a_n)^{1/n} = \infty \Rightarrow \underset{n\to\infty}{\text{Lt}}\ (n\,!)^{1/n} = \infty.$$

Example 4. *If* $x_n = \dfrac{n!}{n^n}$, *show that* $\underset{n\to\infty}{Lt}\ x_n = 0$.

Sol. $x_n = \dfrac{(n!)}{n^n} \Rightarrow x_{n+1} = \dfrac{(n+1)!}{(n+1)^{n+1}}$

$$\frac{x_n}{x_{n+1}} = \frac{(n!)}{n^n} \times \frac{(n+1)^{n+1}}{(n+1)!} = \frac{(n!)}{n^n}\cdot\frac{(n+1)^{n+1}}{(n+1)\,n!} = \frac{(n+1)^n}{n^n} = \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n$$

$$\Rightarrow \underset{n\to\infty}{\text{Lt}}\ \frac{x_n}{x_{n+1}} = \underset{n\to\infty}{\text{Lt}}\left(1+\frac{1}{n}\right)^n = e \Rightarrow \underset{n\to\infty}{\text{Lt}}\ \frac{x_{n+1}}{x_n} = \frac{1}{e} < 1$$

Hence $\underset{n\to\infty}{\text{Lt}}\ x_n = 0$. [See Theorem V on limits]

Example 5. *If* $x_n = \left[\left(\frac{2}{1}\right)^1\left(\frac{3}{2}\right)^2\left(\frac{4}{3}\right)^3 \ldots\ldots \left(\frac{n+1}{n}\right)^n\right]^{1/n}$, *show that* $\underset{n\to\infty}{Lt}\ x_n = e$.

Sol. Let $a_n = \left(\frac{2}{1}\right)^1\left(\frac{3}{4}\right)^2\left(\frac{4}{3}\right)^3 \ldots\ldots \left(\frac{n+1}{n}\right)^n$

then $a_{n+1} = \left(\frac{2}{1}\right)^1\left(\frac{3}{4}\right)^2\left(\frac{4}{3}\right)^3 \ldots\ldots \left(\frac{n+1}{n}\right)^n\left(\frac{n+2}{n+1}\right)^{n+1}$

$$\frac{a_{n+1}}{a_n} = \left(\frac{n+2}{n+1}\right)^{n+1} = \left(1+\frac{1}{n+1}\right)^{n+1}$$

$$\underset{n\to\infty}{\text{Lt}}\ \frac{a_{n+1}}{a_n} = \underset{n\to\infty}{\text{Lt}}\left(1+\frac{1}{n+1}\right)^{n+1} = e.$$

$\therefore$ By Cauchy's second theorem on limits $\underset{n\to\infty}{\text{Lt}}\ (a_n)^{1/n} = \underset{n\to\infty}{\text{Lt}}\ \dfrac{a_{n+1}}{a_n} = e$

$\therefore$ $\underset{n\to\infty}{\text{Lt}}\ x_n = e.$ $\quad | \because x_n = (a_n)^{1/n}$

Example 6. *Prove that* $\underset{n\to\infty}{Lt}\left(\dfrac{n^n}{n!}\right)^{1/n} = e$.

Sol. Let $a_n = \dfrac{n^n}{n!}$, then $a_{n+1} = \dfrac{(n+1)^{n+1}}{(n+1)!}$

$$\therefore \frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{(n+1)!} \times \frac{n!}{n^n} = \frac{(n+1)^{n+1}}{n+1} \times \frac{1}{n^n} = \frac{(n+1)^n}{n^n} = \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n$$

$$\underset{n\to\infty}{\text{Lt}}\ \frac{a_{n+1}}{a_n} = \underset{n\to\infty}{\text{Lt}}\left(1+\frac{1}{n}\right)^n = e.$$

By Cauchy's second theorem on limits $\underset{n\to\infty}{\text{Lt}}\ (a_n)^{1/n} = e \Rightarrow \underset{n\to\infty}{\text{Lt}}\left(\dfrac{n^n}{n!}\right)^{1/n} = e.$

Example 7. *Show that* $\underset{n\to\infty}{lim}\ \dfrac{(n\,!)^{1/n}}{n} = \dfrac{1}{e}$.

Sol. *Please try yourself.*

Example 8. *Evaluate* $\underset{n\to\infty}{Lt}\left[\dfrac{(n+1)(n+2)\ldots(n+n)}{n^n}\right]^{1/n}$.

Sol. Let $a_n = \dfrac{(n+1)(n+2)\ldots(n+n)}{n^n}$

then $a_{n+1} = \dfrac{(n+2)(n+3)\ldots(2n+1)(2n+2)}{(n+1)^{n+1}}$

$$\therefore \frac{a_{n+1}}{a_n} = \frac{(2n+1)(2n+2)}{(n+1)^{n+1}} \times \frac{n^n}{n+1} = \frac{(2n+1)(2n+2)}{(n+1)^2}\cdot\frac{n^n}{(n+1)^n}$$

$$= \frac{2n\left(1+\frac{1}{2n}\right)2n\left(1+\frac{1}{n}\right)}{n^2\left(1+\frac{1}{n}\right)^2}\cdot\left(\frac{n+1}{n}\right)^{-n} = \frac{4\left(1+\frac{1}{2n}\right)\left(1+\frac{1}{n}\right)^{-n}}{1+\frac{1}{n}}$$

$$\underset{n\to\infty}{\text{Lt}}\ \frac{a_{n+1}}{a_n} = \underset{n\to\infty}{\text{Lt}}\ \frac{4\left(1+\frac{1}{2n}\right)}{1+\frac{1}{n}}\cdot\left(1+\frac{1}{n}\right)^{-n} = 4\,e^{-1} = \frac{4}{e}$$

$$\left| \because \underset{n\to\infty}{\text{Lt}}\left(1+\frac{1}{n}\right)^{-n} = e^{-1}\right.$$

$\therefore$ By Cauchy's second theorem on limits $\underset{n\to\infty}{\text{Lt}}\ (a_n)^{1/n} = \underset{n\to\infty}{\text{Lt}}\ \dfrac{a_{n+1}}{a_n} = \dfrac{4}{e}$

$$\therefore \underset{n\to\infty}{\text{Lt}}\left[\frac{(n+1)(n+2)\ldots(n+n)}{n^n}\right]^{1/n} = \frac{4}{e}.$$

Example 9. *If* $a_n > 0$ *for all n and* $\underset{n\to\infty}{Lt}\ a_n = l > 0$, *then* $\underset{n\to\infty}{Lt}\ (a_1 a_2 \ldots a_n)^{1/n} = l$.

Sol. $\underset{n\to\infty}{\text{Lt}}\ a_n = l \Rightarrow \underset{n\to\infty}{\text{Lt}}\ \log a_n = \log l$

By Cauchy's first theorem on limits

$$\underset{n\to\infty}{\text{Lt}}\ \frac{\log a_1 + \log a_2 + \ldots\ldots + \log a_n}{n} = \log l$$

$$\Rightarrow \underset{n\to\infty}{\text{Lt}}\ \frac{1}{n}\log(a_1 a_2 \ldots\ldots a_n) = \log l$$

$$\Rightarrow \underset{n\to\infty}{\text{Lt}}\ \log(a_1 a_2 \ldots\ldots a_n)^{1/n} = \log l \Rightarrow \underset{n\to\infty}{\text{Lt}}\ \log(a_1 a_2 \ldots\ldots a_n)^{1/n} = l.$$

Example 10. *Prove that* $\underset{n\to\infty}{Lt}\ \dfrac{x^n}{n!} = 0$, *where x in any real number.*

Sol. Let $a_n = \dfrac{x^n}{n!}$

Since x is any real number, three cases can arise

(i) $x = 0$ (ii) $x > 0$ (iii) $x < 0$

When x = 0

$$a_n = 0 \quad \text{for all } n$$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}}\ a_n = 0 \Rightarrow \underset{n\to\infty}{\text{Lt}}\ \frac{x^n}{n!} = 0$$

When x > 0 or x < 0

$$a_n = \frac{x^n}{n!} \Rightarrow a_{n+1} = \frac{x^{n+1}}{(n+1)!}$$

$$\therefore \quad \frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n} = \frac{x}{n+1}$$

$$\underset{n\to\infty}{\text{Lt}}\ \frac{a_{n+1}}{a_n} = \underset{n\to\infty}{\text{Lt}}\ \frac{x}{n+1} = 0 < 1 \Rightarrow \underset{n\to\infty}{\text{Lt}}\ a_n = 0$$

| See Theorem V on limits

$$\Rightarrow \quad \underset{n\to\infty}{\text{Lt}}\ \frac{x^n}{n!} = 0.$$

Example 11. *Prove that the sequence* $\left\{\left(\frac{(3n)!}{(n!)^3}\right)^{1/n}\right\}$ *is convergent.*

Sol. Let $\quad a_n = \frac{(3n)!}{(n!)^3}$, then $a_{n+1} = \frac{3(n+1)!}{(n+1!)^3}$

$$\therefore \quad \frac{a_{n+1}}{a_n} = \frac{(3n+3)!}{(n+1!)^3}\cdot\frac{(n!)^3}{(3n)!} = \frac{(3n+3)(3n+2)(3n+1)(3n)!}{[(n+1)\,n!]^3}\cdot\frac{(n!)^3}{(3n)!}$$

$$= \frac{3(n+1)(3n+2)(3n+1)}{(n+1)^3} = \frac{3(3n+2)(3n+1)}{(n+1)^2}$$

$$= \frac{3\cdot 3n\left(1+\frac{2}{3n}\right)3n\left(1+\frac{1}{3n}\right)}{n^2\left(1+\frac{1}{n}\right)^2} = \frac{27\left(1+\frac{2}{3n}\right)\left(1+\frac{1}{3n}\right)}{\left(1+\frac{1}{n}\right)^2}$$

$$\underset{n\to\infty}{\text{Lt}}\ \frac{a_{n+1}}{a_n} = 27.$$

$\therefore$ By Cauchy's second theorem on limits $\underset{n\to\infty}{\text{Lt}}\ (a_n)^{1/n} = \underset{n\to\infty}{\text{Lt}}\ \frac{a_{n+1}}{a_n} = 27$

$$\Rightarrow \quad \underset{n\to\infty}{\text{Lt}}\ x_n = 27, \quad \text{where } x_n = (a_n)^{1/n} = \left(\frac{(3n)!}{(n!)^3}\right)^{1/n}$$

$\Rightarrow$ the sequence $\{x_n\}$ is convergent

$\Rightarrow$ the sequence $\left\{\left(\frac{(3n)!}{(n!)^3}\right)^{1/n}\right\}$ is convergent.

Example 12. *If $\{a_n\}$ converges and $\{b_n\}$ diverges, show that $\{a_n + b_n\}$ diverges.*

Sol. Let $\{a_n\}$ converge to a i.e., let $\underset{n\to\infty}{\text{Lt}}\ a_n = a$

Then given $\varepsilon > 0$, $\exists$ a +ve integer m_1 s.t. $|a_n - a| < \varepsilon \quad \forall\, n \ge m_1$

$\Rightarrow \quad a - \varepsilon < a_n < a + \varepsilon \quad \forall\, n \ge m_1$ ...(i)

Also $\{b_n\}$ diverges $\Rightarrow \underset{n\to\infty}{\text{Lt}}\ b_n = +\infty$ or $-\infty$.

$\therefore$ Two cases arise.

Case I. If $b_n \to +\infty$, then for any real number $k > 0$, $\exists$ a +ve integer m_2 s.t.

$b_n > k - (a - \varepsilon) \quad \forall\, n \ge m_2$ ...(ii)

Let $m = \max.\{m_1, m_2\}$, then from (i) and (ii), we have

$a_n > a - \varepsilon$ and $b_n > k - (a - \varepsilon) \quad \forall\, n \ge m$

$\Rightarrow \quad a_n + b_n > k \quad \forall\, n \ge m$

$\Rightarrow \quad \{a_n + b_n\}$ diverges to $+\infty$.

Case II. If $b_n \to -\infty$, then for any real number $k > 0$, $\exists$ a +ve integer m_3 s.t.

$b_n < -k - (a + \varepsilon) \quad \forall\, n \ge m_3$ ...(iii)

Let $m = \max.\{m_1, m_3\}$, then from (i) and (iii), we have

$a_n < a + \varepsilon$ and $b_n < -k - (a + \varepsilon) \quad \forall\, n \ge m$

$\Rightarrow \quad a_n + b_n < -k \quad \forall\, n \ge m$

$\Rightarrow \quad \{a_n + b_n\}$ diverges to $-\infty$.

Example 13. *Give examples of sequences $\{a_n\}$ and $\{b_n\}$ such that*

(i) $a_n \to +\infty$, $b_n \to -\infty$ *but $\{a_n + b_n\}$ converges*

(ii) $a_n \to +\infty$, $b_n \to -\infty$ *but $\{a_n + b_n\}$ diverges to $-\infty$*

(iii) $a_n \to +\infty$, $b_n \to -\infty$ *but $\{a_n + b_n\}$ oscillates.*

Sol. (i) Let $a_n = n$ and $b_n = -n\ \forall\, n$ then $a_n \to +\infty$ and $b_n \to -\infty$

Also $\quad a_n + b_n = n - n = 0 \Rightarrow a_n + b_n \to 0$

$\therefore \quad \{a_n + b_n\}$ converges.

(ii) Let $a_n = n$ and $b_n = -2n\ \forall\, n$ then $a_n \to +\infty$ and $b_n \to -\infty$

Also $\quad a_n + b_n = n - 2n = -n \Rightarrow a_n + b_n \to -\infty$

$\therefore \quad \{a_n + b_n\}$ diverges to $-\infty$.

(iii) Consider $\quad x_n = n$ if n is odd

$\quad = -n$ if n is even.

Then $\quad \{a_n\} = \{x_{2n-1}\} = \{1, 3, 5, \ldots\}$ diverges to $+\infty$.

$\quad \{b_n\} = \{x_{2n}\} = \{-2, -4, -6 \ldots\}$ diverges to $-\infty$

But $\{a_n + b_n\} = \{x_n\}$ oscillates infinitely.

Example 14. *Give examples of sequences $\{a_n\}$ and $\{b_n\}$ such that*

(i) $a_n \to +\infty$ *and $\{b_n\}$ converges but $\{a_n b_n\}$ diverges to $+\infty$*

(ii) $a_n \to +\infty$ *and $\{b_n\}$ converges but $\{a_n b_n\}$ converges*

(iii) $a_n \to +\infty$ *and $\{b_n\}$ converges but $\{a_n b_n\}$ oscillates.*

Sol. (i) Let $a_n = n^2$ and $b_n = \frac{1}{n}$ then $a_n \to +\infty$ and $\{b_n\}$ converges to 0.

$$a_n b_n = n^2 \cdot \frac{1}{n} = n \quad \therefore \quad \{a_n b_n\} \text{ diverges to } +\infty.$$

(*ii*) Let $a_n = n$ and $b_n = \frac{1}{n^2}$ then $a_n \to +\infty$ and $\{b_n\}$ converges to 0.

$$a_n b_n = \frac{1}{n} \quad \therefore \quad \{a_n b_n\} \text{ also converges to } 0.$$

(*iii*) Let $a_n = n$ and $b_n = \frac{(-1)^n}{n}$ then $a_n \to +\infty$ and $\{b_n\}$ converges to 0

$$a_n b_n = (-1)^n$$
$$= -1 \text{ if } n \text{ is odd}$$
$$+1 \text{ if } n \text{ is even}$$

$\therefore$ $\{a_n b_n\}$ oscillates finitely.

Example 15. *Show that*

(*i*) $\lim_{n\to\infty} \dfrac{1+\frac{1}{3}+\frac{1}{5}+\ldots\ldots+\frac{1}{2n-1}}{n} = 0$ (*ii*) $\lim_{n\to\infty}\left(\frac{2}{1}\cdot\frac{3}{2}\cdot\frac{4}{3}\ldots\ldots\frac{n}{n-1}\right)^{1/n} = 1.$

Sol. (*i*) Let $a_n = \frac{1}{2n-1}$, then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{2n-1} = 0$

$\therefore$ By Cauchy's first theorem on limits

$$\lim_{n\to\infty} \frac{a_1+a_2+a_3+\ldots\ldots+a_n}{n} = 0$$

$$\Rightarrow \quad \lim_{n\to\infty} \frac{1+\frac{1}{3}+\frac{1}{5}+\ldots\ldots+\frac{1}{2n-1}}{n} = 0.$$

(*ii*) Let $a_n = \frac{2}{1}\cdot\frac{3}{2}\cdot\frac{4}{3}\ldots\ldots\frac{n}{n-1}$ then $a_{n+1} = \lim_{n\to\infty} \frac{2}{1}\cdot\frac{3}{2}\cdot\frac{4}{3}\ldots\ldots\frac{n}{n-1}\cdot\frac{n+1}{n}$

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{n} = 1+\frac{1}{n}$$

$$\therefore \quad \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1$$

By Cauchy's second theorem on limits

$$\lim_{n\to\infty} (a_n)^{1/n} = 1 \Rightarrow \lim_{n\to\infty}\left(\frac{2}{1}\cdot\frac{3}{2}\cdot\frac{4}{3}\ldots\ldots\frac{n}{n-1}\right)^{1/n} = 1.$$

Example 16. *Show that*

(*i*) $\lim_{n\to\infty} (n^2+n)^{1/n} = 1$ (*ii*) $\lim_{n\to\infty} \dfrac{\sin\frac{n\pi}{3}}{\sqrt{n}} = 0$

(*iii*) $\lim_{n\to\infty} \dfrac{\cos\frac{n\pi}{4}}{\sqrt{n^2}} = 0$ (*iv*) $\lim_{n\to\infty} \dfrac{\cos n\pi}{n} = 0$

(*v*) $\lim_{n\to\infty} (\sqrt{n+1}-\sqrt{n}) = 0.$

Sol. (*i*) $\lim_{n\to\infty} (n^2+n)^{1/n} = \lim_{n\to\infty}\left[n^2\left(1+\frac{1}{n}\right)\right]^{1/n}$

$$= \lim_{n\to\infty} (n^{1/n})^2 \cdot \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{1/n} = 1\times 1 = 1.$$

(*ii*) Let $a_n = \sin\frac{n\pi}{3}$ and $b_n = \frac{1}{\sqrt{n}}$

Now $\left|\sin\frac{n\pi}{3}\right| \le 1 \;\forall\, n$

$\Rightarrow$ $|a_n| \le 1 \;\forall\, n$

$\Rightarrow$ $<a_n>$ is bounded ...(1)

Also $\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{\sqrt{n}} = 0$

$\Rightarrow$ $<b_n>$ is a null sequence ...(2)

From (1) and (2), $<a_n b_n>$ is a null sequence.

Hence $\lim_{n\to\infty} a_n b_n = 0$ *i.e.*, $\lim_{n\to\infty}\dfrac{\sin\frac{n\pi}{3}}{\sqrt{n}} = 0$

(*iii*) *Please try yourself.*

(*iv*) *Please try yourself.*

(*v*) Let $a_n = \sqrt{n+1}-\sqrt{n}$

Since $\sqrt{n+1} > \sqrt{n} \;\forall\, n$

$\therefore$ $a_n > 0 \;\forall\, n$

Also $a_n = \dfrac{(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})}{\sqrt{n+1}+\sqrt{n}}$

$$= \frac{(n+1)-n}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}} < \frac{1}{\sqrt{n}+\sqrt{n}} = \frac{1}{2\sqrt{n}} < \frac{1}{\sqrt{n}}$$

Thus $0 < a_n < \frac{1}{\sqrt{n}} \;\forall\, n$

Since $\lim_{n\to\infty}\frac{1}{\sqrt{n}} = 0.$

$\therefore$ By squeeze principle, we have

$$\lim_{n\to\infty} a_n = 0 \quad\text{or}\quad \lim_{n\to\infty}(\sqrt{n+1}-\sqrt{n}) = 0.$$

Example 17. (*a*) *Show that*

(*i*) $\lim_{n\to\infty}\left[\frac{1}{n^2}+\frac{1}{(n+1)^2}+\ldots\ldots+\frac{1}{(2n)^2}\right] = 0$ (*ii*) $\lim_{n\to\infty} [(n^2+1)^{1/8}-(n+1)^{1/4}] = 0$

(*iii*) $\lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\ldots\ldots+\frac{1}{\sqrt{n^2+n}}\right) = 1$

(*iv*) $\lim\limits_{n\to\infty}\left(\dfrac{1}{\sqrt{n+1}}+\dfrac{1}{\sqrt{n+2}}+\ldots\ldots+\dfrac{1}{\sqrt{2n}}\right)=\infty.$

(*b*) *Show that the sequence* $<b_n>$ *where* $b_n=\dfrac{1}{(n+1)^2}+\dfrac{1}{(n+1)^2}+\ldots\ldots+\dfrac{1}{(n+n)^2}$ *converges to zero.*

Sol. (*a*) (*i*) Let $\quad a_n=\dfrac{1}{n^2}+\dfrac{1}{(n+1)^2}+\ldots\ldots+\dfrac{1}{(2n)^2}$

Then $\quad a_n=\dfrac{1}{n^2}+\dfrac{1}{(n+1)^2}+\ldots\ldots+\dfrac{1}{(2n)^2}$

$$>\frac{1}{(2n)^2}+\frac{1}{(2n)^2}+\ldots\ldots+\frac{1}{(2n)^2}=\frac{n+1}{(2n)^2}>\frac{n}{4n^2}=\frac{1}{4n}$$

Also $\quad a_n=\dfrac{1}{n^2}+\dfrac{1}{(n+1)^2}+\ldots\ldots+\dfrac{1}{(2n)^2}$

$$<\frac{1}{n^2}+\frac{1}{n^2}+\ldots\ldots+\frac{1}{n^2}=\frac{n+1}{n^2}=\frac{1}{n}+\frac{1}{n^2}$$

Thus $\quad \dfrac{1}{4n}<a_n<\dfrac{1}{n}+\dfrac{1}{n^2}$

Since $\quad \lim\limits_{n\to\infty}\dfrac{1}{4n}=0=\lim\limits_{n\to\infty}\left(\dfrac{1}{n}+\dfrac{1}{n^2}\right)$

$\therefore$ By squeeze principle, we have

$$\lim_{n\to\infty}a_n=0\quad i.e.,\quad \lim_{n\to\infty}\left[\frac{1}{n^2}+\frac{1}{(n+1)^2}+\ldots\ldots+\frac{1}{(2n)^2}\right]=0.$$

(*ii*) Let $\quad a_n=(n^2+1)^{1/8}-(n+1)^{1/4}$

$=[(n^2+1)^{1/8}-(n^2)^{1/8}]+[(n^2)^{1/8}-(n+1)^{1/4}]$ | **Note this step**

$=[(n^2+1)^{1/8}-(n^2)^{1/8}]-[(n+1)^{1/4}-n^{1/4}]$

$=b_n-c_n$

where $\quad b_n=(n^2+1)^{1/8}-(n^2)^{1/8}$ and $c_n=(n+1)^{1/4}-n^{1/4}$

Now $\quad b_n=(n^2+1)^{1/8}-(n^2)^{1/8}>0\quad\forall\ n$

Also $\quad b_n=(n^2+1)^{1/8}-(n^2)^{1/8}$

$$=\frac{[(n^2+1)^{1/8}-(n^2)^{1/8}]\,[(n^2+1)^{1/8}+(n^2)^{1/8}]}{(n^2+1)^{1/8}+(n^2)^{1/8}}$$

$$=\frac{(n^2+1)^{1/4}-(n^2)^{1/4}}{(n^2+1)^{1/8}+(n^2)^{1/8}}=\frac{[(n^2+1)^{1/4}-(n^2)^{1/4}]\,[(n^2+1)^{1/4}+(n^2)^{1/4}]}{[(n^2+1)^{1/8}+(n^2)^{1/8}]\,[(n^2+1)^{1/4}+(n^2)^{1/4}]}$$

$$=\frac{(n^2+1)^{1/2}-(n^2)^{1/2}}{[(n^2+1)^{1/8}+(n^2)^{1/8}]\,[(n^2+1)^{1/4}+(n^2)^{1/4}]}$$

$$=\frac{[(n^2+1)^{1/2}-(n^2)^{1/2}]\,[(n^2+1)^{1/2}+(n^2)^{1/2}]}{[(n^2+1)^{1/8}+(n^2)^{1/8}]\,[(n^2+1)^{1/4}+(n^2)^{1/4}]\,[(n^2+1)^{1/2}+(n^2)^{1/2}]}$$

$$=\frac{(n^2+1)-n^2}{[(n^2+1)^{1/8}+(n^2)^{1/8}]\,[(n^2+1)^{1/4}+(n^2)^{1/4}]\,[(n^2+1)^{1/2}+(n^2)^{1/2}]}$$

$$=\frac{1}{[(n^2+1)^{1/8}+(n^2)^{1/8}]\,[(n^2+1)^{1/4}+(n^2)^{1/4}]\,[(n^2+1)^{1/2}+(n^2)^{1/2}]}$$

$$<\frac{1}{[(n^2+1)^{1/8}+(n^2)^{1/8}]\,[(n^2+1)^{1/8}+(n^2)^{1/8}]\,[(n^2+1)^{1/8}+(n^2)^{1/8}]}$$

$$<\frac{1}{[(n^2)^{1/8}+(n^2)^{1/8}]\,[(n^2)^{1/8}+(n^2)^{1/8}]\,[(n^2)^{1/8}+(n^2)^{1/8}]}$$

| $\because\ (n^2+1)^{1/8}>(n^2)^{1/8}$

$$=\frac{1}{(2n^{1/4})\,(2n^{1/4})\,(2n^{1/4})}=\frac{1}{8n^{3/4}}$$

Thus $\quad 0<b_n<\dfrac{1}{8n^{3/4}}$

Since $\quad \lim\limits_{n\to\infty}\dfrac{1}{8n^{3/4}}=0\quad\therefore\quad \lim\limits_{n\to\infty}b_n=0$

Similarly, $\quad c_n=(n+1)^{1/4}-n^{1/4}>0\ \forall\ n$

Also $\quad c_n=(n+1)^{1/4}-n^{1/4}$

$$=\frac{[(n+1)^{1/4}-n^{1/4}]\,[(n+1)^{1/4}+n^{1/4}]}{(n+1)^{1/4}+n^{1/4}}=\frac{(n+1)^{1/2}-n^{1/2}}{(n+1)^{1/4}+n^{1/4}}$$

$$=\frac{[(n+1)^{1/2}-n^{1/2}]\,[(n+1)^{1/2}+n^{1/2}]}{[(n+1)^{1/4}+n^{1/4}]\,[(n+1)^{1/2}+n^{1/2}]}=\frac{(n+1)-n}{[(n+1)^{1/4}+n^{1/4}]\,[(n+1)^{1/2}+n^{1/2}]}$$

$$=\frac{1}{[(n+1)^{1/4}+n^{1/4}]\,[(n+1)^{1/2}+n^{1/2}]}<\frac{1}{[(n+1)^{1/4}+n^{1/4}]\,[(n+1)^{1/4}+n^{1/4}]}$$

$$<\frac{1}{(n^{1/4}+n^{1/4})\,(n^{1/4}+n^{1/4})}$$ | $\because\ (n+1)^{1/4}>n^{1/4}$

$$=\frac{1}{(2n^{1/4})\,(2n^{1/4})}=\frac{1}{4\sqrt{n}}$$

Thus $\quad 0<c_n<\dfrac{1}{4\sqrt{n}}$

Since $\quad \lim\limits_{n\to\infty}\dfrac{1}{4\sqrt{n}}=0\quad\therefore\quad \lim\limits_{n\to\infty}c_n=0$

$\therefore\quad \lim\limits_{n\to\infty}a_n=\lim\limits_{n\to\infty}(b_n-c_n)=\lim\limits_{n\to\infty}b_n-\lim\limits_{n\to\infty}c_n=0-0=0$

Hence $\quad \lim\limits_{n\to\infty}[(n^2+1)^{1/8}-(n+1)^{1/4}]=0.$

(*iii*) Let $\quad a_n=\dfrac{1}{\sqrt{n^2+1}}+\dfrac{1}{\sqrt{n^2+2}}+\ldots+\dfrac{1}{\sqrt{n^2+n}}$

Then $\quad a_n=\dfrac{1}{\sqrt{n^2+1}}+\dfrac{1}{\sqrt{n^2+2}}+\ldots+\dfrac{1}{\sqrt{n^2+n}}$

$$\geq\frac{1}{\sqrt{n^2+n}}+\frac{1}{\sqrt{n^2+n}}+\ldots+\frac{1}{\sqrt{n^2+n}}=\frac{n}{\sqrt{n^2+n}}=\frac{1}{\sqrt{1+\frac{1}{n}}}$$

Also $$a_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \dots + \frac{1}{\sqrt{n^2+n}}$$

$$\le \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+1}} + \dots + \frac{1}{\sqrt{n^2+1}} = \frac{n}{\sqrt{n^2+1}} = \frac{1}{\sqrt{1+\frac{1}{n^2}}}.$$

Then $$\frac{1}{\sqrt{1+\frac{1}{n}}} \le a_n \le \frac{1}{\sqrt{1+\frac{1}{n^2}}}.$$

Since $$\lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}} = 1 = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n^2}}}$$

$\therefore$ $$\lim_{n\to\infty} a_n = 1 \quad i.e., \lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \dots + \frac{1}{\sqrt{n^2+n}}\right) = 1.$$

(*iv*) *Please try yourself.*

(*b*) Proceeding as in example 17 (*a*) (*i*), show that $\lim_{n\to\infty} b_n = 0$.

Example 18. (*i*) *If $< a_n >$ diverges to ∞ and $b_n \ge a_n \ \forall\, n$, show that $< b_n >$ diverges to ∞.*
(*ii*) *If $< a_n >$ diverges to $-\infty$ and $b_n \le a_n \ \forall\, n$, show that $< b_n >$ diverges to $-\infty$.*

Sol. (*i*) $< a_n >$ diverges to ∞.

$\Rightarrow$ Given any positive real number K, however large.

$\exists$ a positive integer m such that $\quad a_n > K \quad \forall\, n \ge m$

Since $\quad b_n \ge a_n \quad \forall\, n$

$\therefore \quad b_n > K \quad \forall\, n \ge m$

$\Rightarrow$ $< b_n >$ diverges to ∞.

(*ii*) *Please try yourself.*

Example 19. *If $< y_n >$ is a sequence of positive terms such that $y_{n+1} = \dfrac{y_n^2 + a^2}{2y_n}$, $a > 0$; show that $< y_n >$ converges to a.*

Sol. $$y_{n+1} = \frac{y_n^2+a^2}{2y_n} \Rightarrow \frac{y_{n+1}}{a} = \frac{y_n^2+a^2}{2ay_n}$$

By componendo and dividendo

$$\frac{y_{n+1}-a}{y_{n+1}+a} = \frac{y_n^2+a^2-2ay_n}{y_n^2+a^2+2ay_n} = \left(\frac{y_n-a}{y_n+a}\right)^2 \qquad \dots(1)$$

Changing n to $n-1$, we have

$$\frac{y_n-a}{y_n+a} = \left(\frac{y_{n-1}-a}{y_{n-1}+a}\right)^2$$

$\therefore$ From (1), $$\frac{y_{n+1}-a}{y_{n+1}+a} = \left(\frac{y_{n-1}-a}{y_{n+1}+a}\right)^{2\times 2} \qquad \dots(2)$$

Generalising from (1) and (2), we have

$$\frac{y_{n+1}-a}{y_{n+1}+a} = \left(\frac{y_1-a}{y_1+a}\right)^{2n} \qquad \dots(3)$$

Now $\quad y_n > 0 \quad$ and $\quad a > 0$

$\therefore$ $$\left|\frac{y_n-a}{y_n+a}\right| < 1 \ \forall\, n \Rightarrow \left|\frac{y_1-a}{y_1+a}\right| < 1$$

$\Rightarrow$ $$\left(\frac{y_1-a}{y_1+a}\right)^{2n} \to 0 \text{ as } n\to\infty$$

$\therefore$ From (3), we get $$\lim_{n\to\infty}\left(\frac{y_{n+1}-a}{y_{n+1}+a}\right) = \lim_{n\to\infty}\left(\frac{y_1-a}{y_1+a}\right)^{2n} = 0$$

$\Rightarrow$ $$\lim_{n\to\infty}(y_{n+1}-a) = 0$$

$\Rightarrow$ $$\lim_{n\to\infty} y_{n+1} = a \Rightarrow \lim_{n\to\infty} y_n = a$$

$\Rightarrow$ $< y_n >$ converges to a.

Example 20. *Discuss the nature of the sequence $< a^n >$ for all $a \in R$.*

Sol. The behaviour of the sequence $< a^n >$ depends upon the value of a. Following cases arise:

Case 1. Let $a > 1$.

Then $\quad a = 1 + h \quad$ where $\quad h > 0$

$\Rightarrow$ $$a^n = (1+h)^n = 1 + nh + \frac{n(n-1)}{2!}h^2 + \dots + h^n$$
$$\ge 1 + nh \qquad \text{(since each term is positive)}$$

As $\quad n\to\infty,\ 1 + nh \to \infty$

$\therefore \quad a^n \to \infty$ as $n\to\infty$

$\Rightarrow$ $< a^n >$ diverges to ∞.

Case 2. Let $\quad a = 1$

Then $\quad a^n = 1 \ \forall\, n \quad \therefore \quad \lim_{n\to\infty} a^n = 1$

$\Rightarrow$ $< a^n >$ is a constant sequence and converges to 1.

Case 3. Let $\quad 0 < a < 1$.

Then $$\frac{1}{a} > 1$$

$\Rightarrow$ $$\frac{1}{a} = 1 + h \text{ for some } h > 0$$

$\Rightarrow$ $$\frac{1}{a^n} = (1+h)^n = 1 + nh + \frac{n(n-1)}{2!}h^2 + \dots + h^n$$
$$\ge 1 + nh > nh \qquad \forall\, n$$

$\Rightarrow$ $$0 < a^n < \frac{1}{nh} \qquad \forall\, n$$

Since $\lim_{n\to\infty} \frac{1}{nh} = 0$

$\therefore$ By squeeze principle, $\lim_{n\to\infty} a^n = 0 \Rightarrow \langle a^n \rangle$ converges to 0.

Case 4. Let $a = 0$

Then $a^n = 0 \quad \forall\, n$

$\therefore \quad \lim_{n\to\infty} a^n = 0 \Rightarrow \langle a^n \rangle$ converges to 0.

Case 5. Let $-1 < a < 0$.

Put $a = -b$, then $-1 < a < 0 \Rightarrow -1 < -b < 0 \Rightarrow 0 < b < 1$

By Case 3, $b^n \to 0$ as $n \to \infty$

$\therefore \quad \lim_{n\to\infty} a^n = \lim_{n\to\infty} (-b)^n = \lim_{n\to\infty} (-1)^n b^n = 0$

$\Rightarrow \langle a^n \rangle$ converges to 0.

Case 6. Let $a = -1$.

Then $$a^n = (-1)^n = \begin{cases} -1 & \text{if } n \text{ is odd} \\ +1 & \text{if } n \text{ is even} \end{cases}$$

$\therefore$ The sequence is $-1, 1, -1, 1, \ldots\ldots$

$\Rightarrow \langle a^n \rangle$ oscillates finitely.

Case 7. Let $a < -1$.

Put $a = -b$, then $a < -1 \Rightarrow -b < -1 \Rightarrow b > 1$

By Case 1, $b^n \to \infty$ as $n \to \infty$

Now $$a^n = (-b)^n = \begin{cases} -b^n & \text{if } n \text{ is odd} \\ b^n & \text{if } n \text{ is even.} \end{cases}$$

$\therefore \quad a^n \to -\infty$ as $n \to \infty$ when n is odd.

$a^n \to \infty$ as $n \to \infty$ when n is even.

$\Rightarrow \langle a^n \rangle$ oscillates infinitely.

Hence $\langle a^n \rangle$ converges when $-1 < a \le 1$.

Example 21. *If $a > 0$, show that $\lim_{n\to\infty} a^{1/n} = 1$.*

Sol. Following cases arise :

Case 1. Let $a > 1$.

Then $a^{1/n} > 1 \quad \forall\, n$

Let $a^{1/n} = 1 + h_n$ where $h_n > 0$

$$\therefore \quad a = (1 + h_n)^n = 1 + nh_n + \frac{n(n-1)}{2!} h_n^2 + \ldots\ldots + h_n^n$$

$$\ge 1 + nh_n \Rightarrow \frac{a-1}{n} \ge h_n$$

$$\Rightarrow \quad 0 < h_n \le \frac{a-1}{n} \quad \forall\, n$$

Since $\lim_{n\to\infty} \frac{a-1}{n} = 0$, by squeeze principle, we have $\lim_{n\to\infty} h_n = 0$

$$\therefore \quad \lim_{n\to\infty} a^{1/n} = \lim_{n\to\infty} (1 + h_n) = 1 + 0 = 1.$$

Case 2. Let $a = 1$

Then $a^{1/n} = 1 \quad \forall\, n$

$\therefore \quad \lim_{n\to\infty} a^{1/n} = 1$

Case 3. Let $0 < a < 1$

Then $\frac{1}{a} > 1$ and hence by case 1, we have $\lim_{n\to\infty} \left(\frac{1}{a}\right)^{1/n} = 1$

$$\Rightarrow \quad \lim_{n\to\infty} \frac{1}{a^{1/n}} = 1 \Rightarrow \lim_{n\to\infty} a^{1/n} = 1.$$

4.25. CLUSTER POINTS (OR LIMIT POINTS) OF A SEQUENCE

Def. 1. *A real number l is said to be a cluster point of a sequence $\langle a_n \rangle$ if every neighbourhood of l contains infinitely many terms of the sequence.*

Thus $l \in \mathbf{R}$ is a cluster point of the sequence $\langle a_n \rangle$.

$\Leftrightarrow$ every nbd. of l contains infinitely many terms of the sequence.

$\Leftrightarrow \quad \forall\, \varepsilon > 0, c_n \in (l - \varepsilon, l + \varepsilon)$ for infinitely many values of n.

$\Leftrightarrow \quad \forall\, \varepsilon > 0, |a_n - l| < \varepsilon$ for infinitely many values of n.

Note 1. A *cluster point* of a sequence is also called '*a limit point*' or '*a condensation point*' or '*an accumulation point*' or '*a subsequential limit*' of the sequence.

Note 2. Limit point of a sequence is different from limit of a sequence.

If $l \in \mathbf{R}$ is the limit of a sequence $\langle a_n \rangle$, then for $\varepsilon > 0$, $\exists\, m \in \mathbf{N}$ s.t. $|a_n - l| < \varepsilon\ \forall\, n \ge m$. Thus, every nbd. of l contains all except a finite number of terms of the sequence. Whereas, if $l \in \mathbf{R}$ is a limit point of the sequence $\langle a_n \rangle$, then every nbd of l containing infinitely many terms of the sequence does not exclude the possibility of an infinite number of terms of the sequence lying outside that nbd. Hence limit of a sequence is a limit point of the sequence but a limit point of a sequence need not be the limit of the sequence.

Note 3. If $a_n = l$ for infinitely many values of n then l is a limit point of $\langle a_n \rangle$.

Note 4. If for an $\varepsilon > 0$; $a_n \in (l - \varepsilon, l + \varepsilon)$ for finitely many values of n, then l cannot be a cluster point of $\langle a_n \rangle$.

Note 5. Limit point of a sequence need not be a term of the sequence.

Def. 2. A real number l is said to be a cluster point of a sequence $\langle a_n \rangle$ if given $\varepsilon > 0$, however small, and a positive integer m, there exists a positive integer $k > m$ such that

$$|a_k - l| < \varepsilon \quad \text{i.e.,} \quad l - \varepsilon < a_k < l + \varepsilon.$$

Thus every nbd $(l - \varepsilon, l + \varepsilon)$ of l contains a term of the sequence. This is equivalent to saying that every nbd of l contains infinitely many terms of the sequence. [Because if $(l - \varepsilon, l + \varepsilon)$ contains only finitely many terms of $\langle a_n \rangle$, say $a_{n_1}, a_{n_2}, \ldots\ldots, a_{n_k}$, then

let $$\delta = \min.\ \{\, |l - a_{n_1}|, |l - a_{n_2}|, \ldots, |l - a_{n_k}| \,\}.$$

Now $(l - \delta, l + \delta)$ contains no term of the sequence $\langle a_n \rangle$ which is a contradiction].

- **Note.** Clearly, the above two definitions are equivalent.

Def. 3. A real number l is called a cluster point of the sequence $\langle a_n \rangle$ if there exists a subsequence $\langle a_{n_k} \rangle$ of $\langle a_n \rangle$ converging to l.

Note. On account of the above definition, a cluster point is also called a subsequential limit.

Example 1. *'0' is a limit point of the sequence* $<\frac{1}{n}>$.

Sol. For $\varepsilon > 0$, $\exists\, m \in N$ s.t. $\frac{1}{m} < \varepsilon$

$\therefore$ For $n \geq m$, $\quad 0 < \frac{1}{n} \leq \frac{1}{m} < \varepsilon$

$\Rightarrow \quad -\varepsilon < 0 < \frac{1}{n} < \varepsilon \;\; \forall\, n \geq m$

$\Rightarrow \quad \frac{1}{n} \in (-\varepsilon, \varepsilon) \quad \forall\, n \geq m$

$\Rightarrow$ Every nbd of '0' contains infinitely terms of the sequence $<\frac{1}{n}>$.

$\Rightarrow$ '0' is a limit point of the sequence $<\frac{1}{n}>$.

Example 2. *The sequence* $<(-1)^n>$ *has two limit points.*

Sol. Let $a_n = (-1)^n$, then $a_n = -1$ when n is odd and $a_n = 1$ when n is even.

Thus every nbd of -1 contains all the odd terms (since each $= -1$) of the sequence.

$\therefore$ -1 is a limit point.

Also every nbd of 1 contains all the even terms (since each = 1) of the sequence.

$\therefore$ 1 is a limit point.

Example 3. *The sequence* $<n>$ *has no limit point.*

Sol. Let l be any real number, then the nbd $(l - \frac{1}{4}, l + \frac{1}{4})$ of l contains at most one term of the sequence $<n>$.

$\Rightarrow$ l is not a limit point of the sequence $<n>$.

4.26. THEOREM

Theorem I. *If l is a limit point of the range of a sequence $<a_n>$, then l is a limit point of the sequence $<a_n>$.*

Proof. Let S = range of the sequence $<a_n>$.

Since l is a limit point of S, every nbd of l contains infinitely many elements of S.

But each element of S is a term of the sequence $<a_n>$.

$\therefore$ Every nbd of l contains infinitely many terms of the sequence $<a_n>$.

$\Rightarrow$ l is a limit point of the sequence $<a_n>$.

Note 1. The converse of this theorem may not be true.

Consider $\quad a_n = 1 + (-1)^n = \begin{cases} 0 & \text{when } n \text{ is odd} \\ 2 & \text{when } n \text{ is even} \end{cases}$

$\therefore$ 0, 2 are the limit points of the sequence $<a_n>$.

But the range of the sequence = {0, 2} is a finite set.

Since a finite set has no limit point, the range of $<a_n>$ has no limit point.

Note 2. If the terms of a sequence are distinct, then the limit points of the sequence are the limit points of the range set.

Theorem II. *If a sequence $<a_n>$ converges to l, then l is the only limit point of the sequence.*

Proof. The sequence $<a_n>$ converges to l

$\Rightarrow$ Given $\varepsilon > 0, \exists$ a + e integer m s.t. $\quad |a_n - l| < \varepsilon \quad \forall\, n \geq m$

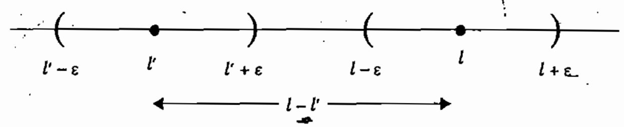
$\Rightarrow \quad l - \varepsilon < a_n < l + \varepsilon \quad \forall\, n \geq m \quad$ ...(1)

$\Rightarrow$ $a_n \in (l - \varepsilon, l + \varepsilon)$ for infinitely many values of n.

$\Rightarrow$ every nbd of l contains infinitely many terms of the sequence $<a_n>$.

$\Rightarrow$ l is a limit point of the sequence $<a_n>$.

If possible, let l' be another limit point of the sequence $<a_n>$.



Let $\quad \varepsilon = \frac{1}{3}(l - l') > 0$ where $l > l'$

Then $\quad (l' - \varepsilon, l' + \varepsilon) \cap (l - \varepsilon, l + \varepsilon) = \phi$.

From (1), $\quad a_n \in (l - \varepsilon, l + \varepsilon) \quad \forall\, n \geq m$

$\therefore$ $a_n \in (l' - \varepsilon, l' + \varepsilon)$ for almost $(m - 1)$ values of n.

$\Rightarrow$ Finitely many terms of $<a_n>$ lie in $(l' - \varepsilon, l' + \varepsilon)$

$\Rightarrow$ l' is not a limit point of the sequence.

Hence l is the only limit point of the sequence.

Theorem III. *(The sufficient condition for the existence of a limit point of a sequence).*

[Bolzano-Weierstrass Theorem for Sequences]

Every bounded sequence has at least one limit point.

Proof. Let $<a_n>$ be a bounded sequence and S be its range, *i.e.*, S = $\{a_n : n \in N\}$

Since $<a_n>$ is bounded, S is bounded.

Case 1. *Let S be a finite set.*

Then $\exists$ a real number l such that $a_n = l$ for an infinite number of values of $n \in$ N.

$\Rightarrow$ Given $\varepsilon > 0$, $a_n \in (l - \varepsilon, l + \varepsilon)$ for an infinite number of values of n.

$\Rightarrow$ Every nbd of l contains infinitely many terms of the sequence $<a_n>$.

$\therefore$ l is a limit point of the sequence $<a_n>$.

Case 2. *Let S be an infinite set.*

Since S is an infinite bounded set, by Bolzano-Weierstrass Theorem, S has at least one limit point, say l.

Now, l is a limit point of S.

$\Rightarrow$ Every nbd of l contains an infinite number of elements of S.

But each element of S is a term of the sequence $<a_n>$.

$\therefore$ Every nbd of l contains an infinite number of terms of the sequence $<a_n>$.

$\Rightarrow$ l is a limit point of the sequence $<a_n>$.

Corollary 1. *If S is a closed and bounded (i.e., compact) set, then every sequence in S has a limit point in S.*

Proof. [A sequence $<a_n>$ is in S if $a_n \in$ S $\forall\, n \in$ N].

Let $<a_n>$ be a sequence in S, then $a_n \in S \; \forall \; n$.

$\therefore$ Since S is bounded, the sequence $<a_n>$ is bounded and consequently it has a limit point, say l. (By Bolzano-Weierstrass Theorem).

We shall show that $l \in S$.

Suppose $l \in S^c$, then S being closed, S^c is open.

$\therefore$ S^c is a nbd of l.

But S^c contains no term of $<a_n>$. This contradicts the fact that l is a limit point of $<a_n>$.

$\therefore$ $l \notin S^c$. Hence $l \in S$.

Corollary 2. *If I is a closed interval, then every sequence in I has a limit point in I.*

Proof. I is a closed interval.

$\Rightarrow$ I is a closed and bounded set.

The result not follows from Cor. 1.

Theorem IV. *The set of limit points of a bounded sequence is bounded.*

Proof. Let $<a_n>$ be a bounded sequence. Then there exist real numbers k and K($k \le$ K) such that

$$k \le a_n \le K \quad \forall \; n \in N$$

$$\therefore \quad a_n \notin (-\infty, k) \text{ and } a_n \notin (K, \infty) \text{ for any } n.$$

Let l be any real number.

If $l \in (-\infty, k)$, then $(-\infty, k)$ contains no term of the sequence $<a_n>$ and consequently l is not a limit point of $<a_n>$.

If $l \in (K, \infty)$, then (K, ∞) contains no term of the sequence $<a_n>$ and consequently l is not a limit point of $<a_n>$.

Thus, no point outside $[k, K]$ is a limit point of $<a_n>$.

$\Rightarrow$ The limit points of $<a_n>$ lie in $[k, K]$.

$\Rightarrow$ The set of all the limit points of a bounded sequence is bounded.

Note 1. The bounds of the set of all the limit points of a bounded sequence are the same as the bounds of the sequence.

Note 2. The set of limit points of an unbounded sequence may or may not be bounded.

For example, (*i*) the sequence $< 1, \frac{1}{2}, 2, \frac{1}{3}, 3, \ldots\ldots >$ is unbounded but the set of its limit points is {0}, which is bounded.

(*ii*) the sequence $< 2, 1 + \frac{1}{2}, 2 + \frac{1}{2}, 1 + \frac{1}{3}, 2 + \frac{1}{3}, 3 + \frac{1}{3}, 1 + \frac{1}{4}, 2 + \frac{1}{4}, 3 + \frac{1}{4}, 4 + \frac{1}{4}, \ldots\ldots >$ is unbounded and the set of its limit points is N which is unbounded.

Theorem V. *Every bounded sequence has the greatest and the least limit points.*

Proof. Let $<a_n>$ be a bounded sequence. Then the set S of the limit points of $<a_n>$ is also bounded and $S \ne \phi$. (Bolzano-Weierstrass Theorem).

By the completeness axiom, S has infimum and supermum.

Let $\quad \inf. S = l \quad$ and $\quad \sup. S = u$.

We shall show that $l, u \in S$.

For $\varepsilon > 0$, let $(u - \varepsilon, u + \varepsilon)$ be a nbd of u.

$$\sup. S = u \Rightarrow \exists \text{ some } x \in S \text{ s.t.}$$

$$u - \varepsilon < x \le u < u + \varepsilon \Rightarrow x \in (u - \varepsilon, u + \varepsilon)$$

$\Rightarrow$ $(u - \varepsilon, u + \varepsilon)$ is a nbd of x.

Since $x \in S$, x is a limit point of $<a_n>$.

$\therefore$ Every nbd of x contains infinitely many terms of $<a_n>$.

$\Rightarrow$ $(u - \varepsilon, u + \varepsilon)$ contains infinitely many terms of $<a_n>$.

This is true for every $\varepsilon > 0$.

$\therefore$ Every nbd of u contains infinitely many terms of $<a_n>$.

$\Rightarrow$ u is a limit point of $<a_n>$. $\Rightarrow$ $u \in S$

Similarly, we can prove that $l \in S$.

Theorem VI. *The set of limit points of a sequence is a closed set.*

Proof. Let S be the set of all limit points of a sequence $<a_n>$.

If $S' = \phi$, then $S' \subset S$.

$\Rightarrow$ S is closed.

If $S' \ne \phi$, let $x \in S'$.

$x \in S' \Rightarrow x$ is a limit point of S.

$\Rightarrow$ every nbd of x contains infinitely many element of S.

$\Rightarrow$ given $\varepsilon > 0$, $(x - \varepsilon, x + \varepsilon) \cap S$ is an infinite set.

Let $y \in (x - \varepsilon, x + \varepsilon) \cap S$, then $y \in (x - \varepsilon, x + \varepsilon)$ and $y \in S$

$\Rightarrow$ $(x - \varepsilon, x + \varepsilon)$ is a nbd of y and y is a limit point of the sequence $<a_n>$.

$\Rightarrow$ $a_n \in (x - \varepsilon, x + \varepsilon)$ for infinitely many n.

$\Rightarrow$ x is a limit point of $<a_n>$.

$\Rightarrow$ $\quad x \in S \quad \therefore \quad x \in S' \Rightarrow x \in S$

$\Rightarrow$ $\quad S' \subset S$

$\Rightarrow$ S is closed.

Theorem VII. *The set of limit points of a bounded sequence is a compact set.*

Proof. Let S be the set of all limit points of a bounded sequence $<a_n>$.

Then S is closed. [Theorem 6]

and S is bounded. [Theorem 4]

$\Rightarrow$ S is a compact set.

4.27. LIMIT SUPERIOR AND LIMIT INFERIOR OF A SEQUENCE

Let $<a_n>$ be a bounded sequence, then the sequence has the least and the greatest limit points. (See Theorem 5 with 4.26)

The least limit point of $<a_n>$ is called the *limit inferior* (or *inferior limit* or *lower limit*) of $<a_n>$ and is denoted by $\lim\limits_{n \to \infty} \inf. a_n$ or $\varliminf\limits_{n \to \infty} a_n$.

The greatest limit point of $<a_n>$ is called the *limit superior* (or *superior limit* or *upper limit*) of $<a_n>$ and is denoted by $\lim\limits_{n \to \infty} \sup. a_n$ or $\varlimsup\limits_{n \to \infty} a_n$.

Note 1. If $<a_n>$ is unbounded above, we write $\lim\limits_{n \to \infty} \sup. a_n = \infty$

If $<a_n>$ is unbounded below, we write $\lim\limits_{n \to \infty} \inf. a_n = -\infty$.

Note 2. Since the greatest limit point of a sequence $\langle a_n \rangle \ge$ the least limit point

$$\therefore \quad \lim_{n \to \infty} \sup. a_n \ge \lim_{n \to \infty} \inf. a_n.$$

Examples. (*i*) For the sequence $< a_n >$ defined by $a_n = (-1)^n \ \forall\ n$, the only limit points are -1 and 1.

The set of limit points $= \{-1, 1\}$ which is bounded.

$\therefore \lim\limits_{n\to\infty} \inf. a_n = -1$ and $\lim\limits_{n\to\infty} \sup. a_n = 1$.

(*ii*) For the sequence $< a_n >$ defined by $a_n = \dfrac{1}{n} \ \forall\ n$, the only limit point is 0.

The set of limit points $= \{0\}$ which is bounded.

$$\therefore \quad \lim_{n\to\infty} \inf. a_n = 0 = \lim_{n\to\infty} \sup. a_n.$$

(*iii*) For the constant sequence $\ a_n = k \ \ \forall\ n$,

$$\lim_{n\to\infty} \inf. a_n = k = \lim_{n\to\infty} \sup. a_n.$$

(*iv*) If $a_n = \begin{cases} 2 & \text{when } n \text{ is odd} \\ -n & \text{when } n \text{ is even} \end{cases}$ then 2 is a limit point of $< a_n >$ which is unbounded below.

$$\therefore \quad \lim_{n\to\infty} \inf. a_n = -\infty \quad \text{and} \quad \lim_{n\to\infty} \sup. a_n = 2.$$

(*v*) For the sequence $\ a_n = (-1)^n n \ \ \forall\ n \in \mathbb{N}$,

$$\lim_{n\to\infty} \inf. a_n = -\infty \text{ and } \lim_{n\to\infty} \sup. a_n = \infty.$$

4.28. THEOREM

Theorem I. *A real number u is the limit superior of a bounded sequence $< a_n >$ if and only if the following two conditions are satisfied :*

(*i*) *for each $\varepsilon > 0$, $a_n > u - \varepsilon$ for infinity many values of n*

(*ii*) *for each $\varepsilon > 0$, $a_n < u + \varepsilon$ for all except finitely many values of n.*

Proof. Necessity

Let u be the limit superior of a bounded sequence $< a_n >$ and let $\varepsilon > 0$ be given.

Since u is a limit point of $< a_n >$, we have $u - \varepsilon < a_n < u + \varepsilon$ for infinitely many values of n.

In particular $a_n > u - \varepsilon$ for infinitely many values of n.

Again, since u is the greatest limit point, $u + \varepsilon$ is not a limit point and, therefore, $a_n \geq u + \varepsilon$ for only finitely many values of n. (If for some $\varepsilon > 0$, $a_n \geq u + \varepsilon$ for infinitely many values of n, then $< a_n >$ will have a limit point $p \geq u + \varepsilon$).

$\therefore \ a_n < u + \varepsilon$ for all except finitely many values of n.

Sufficiency

Let us assume that u satisfies both the conditions.

Given any $\varepsilon > 0$, $u - \varepsilon < a_n$ for infinitely many values of n and $a_n < u + \varepsilon$ for all except finitely many values of n.

$\Rightarrow \ u - \varepsilon < a_n < u + \varepsilon$ for infinitely many values of n.

$\therefore \ u$ is a limit point of $< a_n >$.

Now we shall show that no number greater than u can be a limit point of $< a_n >$.

Let u' be any number greater than u. Let p and q be two numbers such that

$$u < p < u' < q.$$

By the second condition, for each $\varepsilon > 0$, $a_n < u + \varepsilon$ for all except finitely many value of n.

Choosing $\varepsilon = p - u > 0$, we have $a_n < p$ for all except finitely many values of n and therefore, (p, q) is a nbd of u' containing a_n for finitely many values of n. This implies that u' is not a limit point of $< a_n >$ so that u is the greatest limit point of $< a_n >$.

Hence u is the limit superior of the sequence $< a_n >$.

Theorem II. *A real number l is the limit inferior of a bounded sequence $< a_n >$ if and only if the following two conditions are satisfied.*

(*i*) *for each $\varepsilon > 0$, $a_n < l + \varepsilon$ for infinitely many values of n.*

(*ii*) *for each $\varepsilon > 0$, $a_n > l - \varepsilon$ for all except finitely many values of n.*

Proof. Necessity.

Let l be the limit inferior of a bounded sequence $< a_n >$ and let $\varepsilon > 0$ be given.

Since l is a limit point of $< a_n >$, we have $l - \varepsilon < a_n < l + \varepsilon$ for infinitely many values of n.

In particular, $a_n < l + \varepsilon$ for infinitely many values of n.

Again, since l is the least limit point, $l - \varepsilon$ is not a limit point and, therefore, $a_n \leq l - \varepsilon$ for only finitely many values of n. (If for some $\varepsilon > 0$, $a_n \leq l - \varepsilon$ for infinitely many values of n, then $< a_n >$ will have a limit point $p \leq l - \varepsilon$).

$\therefore \ a_n > l - \varepsilon$ for all except finitely many values of n.

Sufficiency

Let us assume that l satisfies both the conditions.

Given any $\varepsilon > 0$, $a_n < l + \varepsilon$ for infinitely many values of n and $a_n > l - \varepsilon$ for all except finitely many value of n.

$\Rightarrow \ l - \varepsilon < a_n < l + \varepsilon$ for infinitely many values of n.

$\therefore \ l$ is a limit point of $< a_n >$.

Now we shall show that no number less than l can be a limit point of $< a_n >$.

Let l' be any number less than l. Let p and q be two numbers such that $p < l' < q < l$.

By the second condition, for each $\varepsilon > 0$, $a_n > l - \varepsilon$ for all except finitely many values of n.

Choosing $\varepsilon = l - q > 0$, we have $a_n > q$ for all except finitely many values of n and, therefore, (p, q) is a nbd of l' containing a_n for finitely many values of n. This implies that l' is not a limit point of $< a_n >$ so that l is the least limit point of $< a_n >$. Hence l is the limit inferior of the sequence $< a_n >$.

Theorem III. *A sequence $< a_n >$ converges to l if and only if*

$$\lim_{n\to\infty} \sup. a_n = \lim_{n\to\infty} \inf. a_n = l.$$

Proof. Let the sequence $< a_n >$ converge to l.

Then given $\varepsilon > 0$, $\exists$ a positive integer m, such that

$$| a_n - l | < \varepsilon \qquad \forall\ n \geq m$$

$$\Rightarrow \qquad l - \varepsilon < a_n < l + \varepsilon \ \ \forall\ n \geq m$$

Since the nbd $(l - \varepsilon, l + \varepsilon)$ of l contains a_n for infinitely many values of n and since ε is arbitrary, therefore, every nbd of l contains infinitely terms of the sequence $< a_n >$.

$\therefore \ l$ is a limit point of $< a_n >$.

Now we shall show that l is the only limit point of the sequence $< a_n >$.

Let l' be any number other than l. Two cases arise :

(*i*) $l < l'$ (*ii*) $l' < l$

Suppose $l < l'$. Let p, q, r be three numbers such that $p < l < q < l' < r$.

Since $a_n \to l$, therefore, every nbd of l contains a_n for all except finitely many values of n. In particular, $a_n \in (p, q)$ for all except finitely many values of n.

$\Rightarrow$ The nbd (q, r) of l' contains a_n for almost finitely many values of n.

$\Rightarrow$ l' cannot be a limit point of $< a_n >$.

Similarly, when $l' < l$, l' is not a limit point of $< a_n >$.

Thus l is the only limit point of $< a_n >$.

Hence $\lim\limits_{n \to \infty} \sup. a_n = \lim\limits_{n \to \infty} \inf. a_n = l.$

Conversely, suppose that $\lim\limits_{n \to \infty} \sup. a_n = \lim\limits_{n \to \infty} \inf. a_n = l$

Let $\varepsilon > 0$ be given.

Since l is a limit superior of $< a_n >$, therefore, $a_n < l + \varepsilon$ for all except finitely many values of n.

$\Rightarrow$ $\exists$ a positive integer m, such that $a_n < l + \varepsilon \quad \forall\, n \geq m_1$ ...(1)

Again, since l is a limit inferior of $< a_n >$, therefore, $a_n > l - \varepsilon$ for all except finitely many values of n.

$\Rightarrow$ $\exists$ a positive integer m_2 such that $a_n > l - \varepsilon \quad \forall\, n \geq m_2$ ...(2)

Let $m = \max. \{n_1, m_2\}$, then from (1) and (2), we have

$$l - \varepsilon < a_n < l + \varepsilon \quad \forall\, n \geq m$$

$$\Rightarrow \quad | a_n - l | < \varepsilon \quad \forall\, n \geq m$$

$$\Rightarrow \quad a_n \to l.$$

4.29. LIMIT SUPERIOR AND LIMIT INFERIOR OF A BOUNDED SEQUENCE (*Second Definition*)

Let $< a_n >$ be a bounded sequence. Let $< a_n >$ be bounded above by K. Then, for each $n \in$ N, the set $S_n = \{a_n, a_{n+1}, \ldots\}$ is bounded above by K. By completeness axiom, S_n has the l.u.b. (or supermum) M_n (say). Clearly $M_n \geq M_{n+1} \quad \forall\, n \in$ N. Thus, the sequence $< M_n >$ being a decreasing sequence is either convergent or diverges to $-\infty$. If $< M_n >$ is convergent, then $\lim\limits_{n \to \infty} M_n$ is called the limit superior of $< a_n >$. If $< M_n >$ is divergent, then the limit superior of $< a_n >$ is taken as $-\infty$.

Thus $\lim\limits_{n \to \infty} \sup. a_n = \lim\limits_{n \to \infty} M_n = \lim\limits_{n \to \infty} \sup. S_n = \lim\limits_{n \to \infty} \sup. \{a_n, a_{n+1}, \ldots\}$

Let $< a_n >$ be a bounded sequence. Let $< a_n >$ be bounded below by k. Then, for each $n \in$ N, the set $S_n = \{a_n, a_{n+1}, \ldots\}$ is bounded below by k. By completeness axiom, S_n has the g.l.b. (or infinium) m_n (say). Clearly $m_n \leq m_{n+1} \; \forall\, n \in$ N. Thus, the sequence $< m_n >$ being an increasing sequence is either convergent or diverges to ∞. If $< m_n >$ is convergent, then $\lim\limits_{n \to \infty}$ is called the limit inferior of $< a_n >$. If $< m_n >$ is divergent, then the limit inferior of $< a_n >$ is taken as ∞.

Thus $\lim\limits_{n \to \infty} \inf. a_n = \lim\limits_{n \to \infty} m_n = \lim\limits_{n \to \infty} \inf. S_n = \lim\limits_{n \to \infty} \inf. \{a_n, a_{n+1} \ldots\}$

4.30. THEOREM

(*i*) *If a sequence* $< a_n >$ *is such that* $\overline{\lim\limits_{n \to \infty}}\, a_n = \underline{\lim\limits_{n \to \infty}}\, a_n = +\infty$ *then* $< a_n >$ *diverges to* ∞.

(*ii*) *If a sequence* $< a_n >$ *is such that* $\overline{\lim\limits_{n \to \infty}}\, a_n = \underline{\lim\limits_{n \to \infty}}\, a_n = -\infty$ *then* $< a_n >$ *diverges to* $-\infty$.

Proof. (*i*) Let $M_n = \sup \{a_n, a_{n+1}, \ldots\}$ and $m_n = \inf. \{a_n, a_{n+1}, \ldots\} \; \forall\, n \in$ N.

Then $m_n \leq a_n \leq M_n \; \forall\, n$ ...(1)

Since $\lim\limits_{n \to \infty} m_n = \underline{\lim\limits_{n \to \infty}}\, a_n = \infty$ and $\lim\limits_{n \to \infty} M_n = \overline{\lim\limits_{n \to \infty}}\, a_n = \infty$

$\therefore$ From (1), $\lim\limits_{n \to \infty} a_n = \infty$

$\Rightarrow$ The sequence $< a_n >$ diverges to ∞.

(*ii*) *Please try yourself.*

ILLUSTRATIVE EXAMPLES

Example 1. *Give examples of sequences having*

(*i*) *no cluster point* (*ii*) *one cluster point*

(*iii*) *two cluster points* (*iv*) *infinitely many cluster points.*

Sol. (*i*) The sequence $< n >$ has no cluster point.

(*ii*) The sequence $< \frac{1}{n} >$ has one cluster point, namely 0.

(*iii*) The sequence $< (-1)^n >$ has two cluster points, namely -1 and 1.

(*iv*) The sequence $< 2, 1 + \frac{1}{2}, 2 + \frac{1}{2}, 1 + \frac{1}{3}, 2 + \frac{1}{3}, 3 + \frac{1}{3}, 1 + \frac{1}{4}, 2 + \frac{1}{4}, 3 + \frac{1}{4}, 4 + \frac{1}{4}, \ldots >$ has infinitely many cluster points. Every natural number is a cluster point.

Example 2. *Examine the sequences whose nth terms are given below, for cluster points :*

(*i*) $(-1)^n$ (*ii*) 5 (*iii*) $\frac{(-1)^n}{n}$

(*iv*) n (*v*) $(-1)^n \left(1 + \frac{1}{n}\right)$ (*vi*) $\left(1 + \frac{1}{n}\right)^{n+1}$

(*vii*) $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!}$

Sol. (*i*) Here $a_n = (-1)^n = \begin{cases} -1 & \text{when } n \text{ is odd} \\ 1 & \text{when } n \text{ is even} \end{cases}$

$\therefore$ The sequence $< a_n >$ has two cluster points, namely -1 and 1.

(*ii*) Here $a_n = 5$ is a constant sequence converging to 5.

$\therefore$ The sequence $< a_n >$ has only one cluster point, namely 5.

(*iii*) Here $a_n = \frac{(-1)^n}{n} = \begin{cases} -\frac{1}{n} & \text{when } n \text{ is odd.} \\ \frac{1}{n} & \text{when } n \text{ is even} \end{cases}$

$\lim\limits_{n\to\infty} a_n = 0$ so that the sequence $< a_n >$ converges to 0.

$\therefore$ The sequence $< a_n >$ has only one cluster point, 0.

(*iv*) For any $l \in$ R, the *nbd* $\left(l - \frac{1}{4}, l + \frac{1}{4}\right)$ of l contains at most one term of the sequence $< n >$.

$\therefore$ $l \in$ R is not a limit point of $< n >$.

$\Rightarrow$ The sequence $< n >$ has no limit point.

(*v*) Here
$$a_n = (-1)^n \left(1+\frac{1}{n}\right) = \begin{cases} -\left(1+\frac{1}{n}\right) & \text{when } n \text{ is odd} \\ \left(1+\frac{1}{n}\right) & \text{when } n \text{ is even} \end{cases}$$

The sequence $< a_n >$ has two cluster points, -1 and 1.

(*vi*) Here
$$a_n = \left(1+\frac{1}{n}\right)^{n+1}$$
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n \left(1+\frac{1}{n}\right) = e \times 1 = e$$

$\Rightarrow$ The sequence $< a_n >$ converges to e.

$\Rightarrow$ The sequence has only one cluster point e.

(*vii*) Here
$$a_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots\ldots + \frac{1}{n!}$$

The sequence $< a_n >$ converges to e.

$\therefore$ The sequence has only one cluster point e.

Example 3. *Find the limit superior and limit inferior of each of the following sequences :*

(*i*) $< 1, 3, 5, 1, 3, 5, 1, 3, 5, \ldots\ldots >$ (*ii*) $< 1, 5, 17, 19, 1, 5, 17, 19, 1, 5, 17, 19, \ldots\ldots >$

(*iii*) $< a_n >$ *where* $a_n = \sin \frac{n\pi}{3}$ (*iv*) $< a_n >$ *where* $a_n\ (-2)^{-n}\left(1+\frac{1}{n}\right)$

(*v*) $< a_n >$ *where* $a_n = (-10)^n \left(1+\frac{1}{n}\right)^2$ (*vi*) $< a_n >$ *where* $a_n = (-1)^n \left(1-\frac{1}{n}\right)$

(*vii*) $< a_n >$ *where* $a_n = \left(1+\frac{1}{n}\right)^{n+1}$ (*viii*) $< a_n >$ *where* $a_n = (-1)^n (2^n + 3^n)$.

Sol. (*i*) Here
$$a_n = \begin{cases} 1 & \text{if } n = 3m-2 \\ 3 & \text{if } n = 3m-1, m \in N \\ 5 & \text{if } n = 3m \end{cases}$$

The set of cluster points of $< a_n >$ is E = {1, 3, 5}.

$\therefore$
$$\overline{\lim_{n\to\infty}}\ a_n = \max.\ \{1, 3, 5\} = 5$$
$$\underline{\lim}_{n\to\infty}\ a_n = \min.\ \{1, 3, 5\} = 1.$$

(*ii*) *Please try yourself.* **[Ans.** $\overline{\lim\limits_{n\to\infty}}\ a_n = 19,\ \underline{\lim}_{n\to\infty}\ a_n = 1$]

(*iii*) Here
$$a_n = \sin\frac{n\pi}{3} = \begin{cases} 0 & \text{if } n = 3m \\ \frac{\sqrt{3}}{2} & \text{if } n = 6m-5 \text{ or } 6m-4 \quad \text{where } m \in N \\ -\frac{\sqrt{3}}{2} & \text{if } n = 6m-2 \text{ or } 6m-1 \end{cases}$$

The set of cluster points of $< a_n >$ is $E = \left\{0, \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}\right\}$

$\therefore$
$$\overline{\lim_{n\to\infty}}\ a_n = \max.\ \left\{0, \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}\right\} = \frac{\sqrt{3}}{2}$$
$$\underline{\lim}_{n\to\infty}\ a_n = \min.\ \left\{0, \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}\right\} = -\frac{\sqrt{3}}{2}.$$

(*iv*) Here
$$a_n = (-2)^{-n}\left(1+\frac{1}{n}\right)$$
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-2)^{-n}\left(1+\frac{1}{n}\right) = 0 \times 1 = 0$$

$\Rightarrow$ The sequence $< a_n >$ converges to 0.

$\Rightarrow$ The set of cluster points of $< a_n >$ is E = {0}

$\therefore$
$$\overline{\lim_{n\to\infty}}\ a_n = 0 = \underline{\lim}_{n\to\infty}\ a_n.$$

(*v*) Here
$$a_n = (-10)^n\left(1+\frac{1}{n}\right)^2 = \begin{cases} -(10)^n\left(1+\frac{1}{n}\right)^2 & \text{when } n \text{ is odd} \\ (10)^n\left(1+\frac{1}{n}\right)^2 & \text{when } n \text{ is even} \end{cases}$$

Since $\lim\limits_{n\to\infty} (10)^n\left(1+\frac{1}{n}\right)^2 = \infty$, the set of cluster points of $< a_n >$ is E = $\{-\infty, \infty\}$

$\therefore$
$$\overline{\lim_{n\to\infty}}\ a_n = \infty \quad \text{and} \quad \underline{\lim}_{n\to\infty}\ a_n = -\infty.$$

(*vi*) *Please try yourself.* **[Ans.** $\overline{\lim\limits_{n\to\infty}}\ a_n = 1,\ \underline{\lim}_{n\to\infty}\ a_n = -1$]

(*vii*) *Please try yourself.* **[Ans.** $\overline{\lim\limits_{n\to\infty}}\ a_n = e = \underline{\lim}_{n\to\infty}\ a_n$]

(*viii*) Here
$$a_n = (-1)^n (2^n + 3^n) = \begin{cases} -(2^n + 3^n) & \text{when } n \text{ is odd} \\ (2^n + 3^n) & \text{when } n \text{ is even} \end{cases}$$

Since $\lim\limits_{n\to\infty} (2^n + 3^n) = \infty$, the set of cluster points of $< a_n >$ is E = $\{-\infty, \infty\}$

$\therefore$
$$\overline{\lim_{n\to\infty}}\ a_n = \infty \quad \text{and} \quad \underline{\lim}_{n\to\infty}\ a_n = -\infty.$$

Note. E is the set of all the cluster points of $< a_n >$, including $+\infty$ and $-\infty$.

Example 4. *For a sequence* $<a_n>$, $\underline{\lim}_{n\to\infty} a_n \le \overline{\lim}_{n\to\infty} a_n$.

Sol. If $<a_n>$ is unbounded, then either $\overline{\lim}_{n\to\infty} a_n = \infty$ or $\underline{\lim}_{n\to\infty} a_n = -\infty$

and hence there is nothing to prove.

Now, let $<a_n>$ be a bounded sequence.

Let $\quad m_n = \text{g.l.b.}\{a_n, a_{n+1}, \ldots\}$ and $M_n = \text{l.u.b.}\{a_n, a_{n+1}, \ldots\}\ \forall\, n \in N$

Then $\quad m_n \le M_n$

$$\Rightarrow \quad \lim_{n\to\infty} m_n \le \lim_{n\to\infty} M_n \quad \Rightarrow \quad \underline{\lim}_{n\to\infty} a_n \le \overline{\lim}_{n\to\infty} a_n.$$

Example 5. *If* $<a_n>$ *and* $<b_n>$ *are bounded sequences such that* $a_n \le b_n\ \forall\, n \in N$, *then*

(*i*) $\overline{\lim}_{n\to\infty} a_n \le \overline{\lim}_{n\to\infty} b_n$. $\qquad$ (*ii*) $\underline{\lim}_{n\to\infty} a_n \le \underline{\lim}_{n\to\infty} b_n$.

Sol. Let
$$M_n = \text{l.u.b.}\{a_n, a_{n+1}, \ldots\}$$
$$M_n' = \text{l.u.b.}\{b_n, b_{n+1}, \ldots\}$$
$$m_n = \text{g.l.b.}\{a_n, a_{n+1}, \ldots\}$$
and
$$m_n' = \text{g.l.b.}\{b_n, b_{n+1}, \ldots\}$$
Since $\quad a_n \le b_n \quad \forall\, n \in N$

$\therefore \quad M_n \le M_n'$ and $m_n \le m_n'\ \forall\, n$

$$\Rightarrow \quad \lim_{n\to\infty} M_n \le \lim_{n\to\infty} M_n' \quad \text{and} \quad \lim_{n\to\infty} m_n \le \lim_{n\to\infty} m_n'$$

$$\Rightarrow \quad \overline{\lim}_{n\to\infty} a_n \le \overline{\lim}_{n\to\infty} b_n \quad \text{and} \quad \underline{\lim}_{n\to\infty} a_n \le \underline{\lim}_{n\to\infty} b_n.$$

Example 6. *If* $<a_n>$ *and* $<b_n>$ *are bounded sequences, then show that*

(*i*) $\overline{\lim}_{n\to\infty} (a_n + b_n) \le \overline{\lim}_{n\to\infty} a_n + \overline{\lim}_{n\to\infty} b_n$

(*ii*) $\underline{\lim}_{n\to\infty} (a_n + b_n) \ge \underline{\lim}_{n\to\infty} a_n + \underline{\lim}_{n\to\infty} b_n$.

Sol. Let
$$M_n = \sup\{a_n, a_{n+1}, \ldots\}$$
$$M_n' = \sup\{b_n, b_{n+1}, \ldots\}$$
$$m_n = \inf\{a_n, a_{n+1}, \ldots\}$$
and
$$m_n' = \inf\{b_n, b_{n+1}, \ldots\}$$

Since $<a_n>$ and $<b_n>$ are bounded, so is $<a_n + b_n>$ and $\sup\{a_n + b_n, a_{n+1} + b_{n+1}, \ldots\}$
$$\le \sup\{a_n, a_{n+1}, \ldots\} + \sup\{b_n, b_{n+1}, \ldots\}$$
$$= M_n + M_n'$$
$$\inf\{a_n + b_n, a_{n+1} + b_{n+1}, \ldots\}$$
$$\ge \inf\{a_n, a_{n+1}, \ldots\} + \inf\{b_n, b_{n+1}, \ldots\} = m_n + m_n'.$$

(*i*)
$$\overline{\lim}_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} \sup\{a_n + b_n, a_{n+1} + b_{n+1}, \ldots\}$$
$$\le \lim_{n\to\infty} (M_n + M_n')$$
$$= \lim_{n\to\infty} M_n + \lim_{n\to\infty} M_n' = \overline{\lim}_{n\to\infty} a_n + \overline{\lim}_{n\to\infty} b_n$$

$$\therefore \quad \overline{\lim}_{n\to\infty} (a_n + b_n) \le \overline{\lim}_{n\to\infty} a_n + \overline{\lim}_{n\to\infty} b_n.$$

(*ii*)
$$\underline{\lim}_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} \inf\{a_n + b_n, a_{n+1} + b_{n+1}, \ldots\}$$
$$\ge \lim_{n\to\infty} (m_n + m_n') = \lim_{n\to\infty} m_n + \lim_{n\to\infty} m_n' = \underline{\lim}_{n\to\infty} a_n + \underline{\lim}_{n\to\infty} b_n.$$

$$\therefore \quad \underline{\lim}_{n\to\infty} (a_n + b_n) \ge \underline{\lim}_{n\to\infty} a_n + \underline{\lim}_{n\to\infty} b_n.$$

Note 1. Since $\underline{\lim}_{n\to\infty} (a_n + b_n) \le \overline{\lim}_{n\to\infty} (a_n + b_n)$

we have, by combining the above two parts,

$$\boxed{\underline{\lim}_{n\to\infty} a_n + \underline{\lim}_{n\to\infty} b_n \le \underline{\lim}_{n\to\infty} (a_n + b_n) \le \overline{\lim}_{n\to\infty} (a_n + b_n) \le \overline{\lim}_{n\to\infty} a_n + \overline{\lim}_{n\to\infty} b_n.}$$

Note 2. In certain cases, strict inequalities may hold.

For example, consider $a_n = (-1)^n$ and $b_n = (-1)^{n+1}$ then $\overline{\lim}_{n\to\infty} a_n = 1$, $\overline{\lim}_{n\to\infty} b_n = 1$

Since $\quad a_n + b_n = 0\ \forall\, n$
$$\overline{\lim}_{n\to\infty} (a_n + b_n) = 0$$

Clearly $\quad \overline{\lim}_{n\to\infty} (a_n + b_n) \ne \overline{\lim}_{n\to\infty} a_n + \overline{\lim}_{n\to\infty} b_n$

rather $\quad \overline{\lim}_{n\to\infty} (a_n + b_n) < \overline{\lim}_{n\to\infty} a_n + \overline{\lim}_{n\to\infty} b_n$

Similarly $\quad \underline{\lim}_{n\to\infty} (a_n + b_n) \ne \underline{\lim}_{n\to\infty} a_n + \underline{\lim}_{n\to\infty} b_n$

rather $\quad \underline{\lim}_{n\to\infty} (a_n + b_n) > \underline{\lim}_{n\to\infty} a_n + \underline{\lim}_{n\to\infty} b_n$.

Example 7. *If* $<a_n>$ *is a bounded sequence, show that*

(*i*) $\overline{\lim}_{n\to\infty} (-a_n) = -\underline{\lim}_{n\to\infty} a_n$ $\qquad$ (*ii*) $\underline{\lim}_{n\to\infty} (-a_n) = -\overline{\lim}_{n\to\infty} a_n$

(*iii*) $\overline{\lim}_{n\to\infty} (\lambda a_n) = \lambda\, \overline{\lim}_{n\to\infty} a_n, \lambda > 0$ $\qquad$ (*iv*) $\underline{\lim}_{n\to\infty} (\lambda a_n) = \lambda\, \underline{\lim}_{n\to\infty} a_n, \lambda > 0$

(*v*) $\overline{\lim}_{n\to\infty} (\lambda a_n) = \lambda\, \underline{\lim}_{n\to\infty} a_n, \lambda < 0$.

Sol. Since $<a_n>$ is a bounded sequence, so are $<-a_n>$ and $<\lambda a_n>$.

(*i*)
$$\overline{\lim}_{n\to\infty} (-a_n) = \lim_{n\to\infty} \sup\{-a_n, -a_{n+1}, \ldots\} = \lim_{n\to\infty} -\inf\{a_n, a_{n+1}, \ldots\}$$
$$= -\lim_{n\to\infty} \inf\{a_n, a_{n+1}, \ldots\} = -\underline{\lim}_{n\to\infty} a_n.$$

(*ii*)
$$\underline{\lim}_{n\to\infty} (-a_n) = \lim_{n\to\infty} \inf\{-a_n, -a_{n+1}, \ldots\} = \lim_{n\to\infty} -\sup\{a_n, a_{n+1}, \ldots\}$$
$$= -\lim_{n\to\infty} \sup\{a_n, a_{n+1}, \ldots\} = -\overline{\lim}_{n\to\infty} a_n.$$

(*iii*)
$$\overline{\lim}_{n\to\infty} (\lambda a_n) = \lim_{n\to\infty} \sup\{\lambda a_n, \lambda a_{n+1}, \ldots\}$$

$$= \lim_{n\to\infty} \lambda \sup \{a_n, a_{n+1}, \ldots\ldots\} \qquad (\because \lambda > 0)$$

$$= \lambda \lim_{n\to\infty} \sup \{a_n, a_{n+1}, \ldots\ldots\} = \lambda \overline{\lim_{n\to\infty}}\, a_n.$$

(iv) *Please try yourself.*

(v) $$\overline{\lim_{n\to\infty}}\, (\lambda a_n) = \lim_{n\to\infty} \sup \{\lambda a_n, \lambda a_{n+1}, \ldots\ldots\}$$

$$= \lim_{n\to\infty} \lambda \inf \{a_n, a_{n+1}, \ldots\ldots\} \qquad (\because \lambda < 0)$$

$$= \lambda \lim_{n\to\infty} \inf \{a_n, a_{n+1}, \ldots\ldots\} = \lambda \underline{\lim_{n\to\infty}}\, a_n.$$

4.31. SUBSEQUENCES

In a sequence $< a_n >$ if we keep only the terms whose suffixes are $n_1, n_2, n_3, \ldots$ maintaining the same order as in the sequence, we get another sequence $< a_{n_k} >$. It is apt to call $< a_{n_k} >$ a subsequence of $< a_n >$. In the subsequence $< a_{n_k} >$, the suffixes $n_1, n_2, n_3, \ldots\ldots$ form a strictly increasing sequence of positive integers. Sometimes it is convenient to determine the nature of a sequence by using its subsequences.

Def. *Let $< a_n >$ be a given sequence. If $< n_k >$ is a strictly increasing sequence of natural numbers (i.e., $n_1 < n_2 < n_3 < \ldots\ldots$), then $< a_{n_k} >$ is called a subsequence of $< a_n >$.*

For example (i) $< a_{2n} >, < a_{2n-1} >, < a_n{}^2 >$ are all subsequences of $< a_n >$.

(ii) The sequences $< 2, 4, 6, \ldots\ldots >, < 1, 3, 5, \ldots\ldots >, < 1, 4, 9, 16, \ldots\ldots >$ are all subsequences of the sequence $< n >$.

Note 1. The terms of a subsequence occur in the same order in which they occur in the original sequence.

Note 2. If $< u_n >$ is a subsequence of $< a_n >$ and $< v_n >$ is a subsequence of $< u_n >$, then $< v_n >$ is a subsequence of $< a_n >$.

Note 3. Every sequence is a subsequence of itself.

Note 4. The interval in the various terms of a subsequence need not be regular.

Note 5. Given a term a_m of the sequence $< a_n >$, there is a term of the subsequence following it.

4.32. THEOREM

Theorem I. *If a sequence $< a_n >$ converges to l, then every subsequence of $< a_n >$ also converges to l.*

Proof. Let $< a_{n_k} >$ be a subsequence of $< a_n >$.

Since $< a_n >$ converges to l.

$\therefore$ Given $\varepsilon > 0$, $\exists$ a positive integer m such that $\quad | a_n - l | < \varepsilon \quad \forall\, n \geq m \qquad \ldots(1)$

We can find a natural number $n_{k_0} \geq m$

If $n_k \geq n_{k_0}$, then $n_k \geq m$.

$\therefore$ From (1), we have $\quad | a_{n_k} - l | < \varepsilon \quad \forall\, n_k \geq m$

$\Rightarrow \quad < a_{n_k} >$ converges to l.

Note 1. *The converse of the above theorem is not true.*

That is if a subsequence or even if infinitely many subsequences of a given sequence converge, the original sequence may not converge.

For example, let $a_n = (-1)^n$.

The sequence $< a_n >$ does not converge. However, the two subsequences.

$$< a_1, a_3, a_5, \ldots\ldots > = < a_{2n-1} > \quad \text{and} \quad < a_2, a_4, a_6, \ldots\ldots > = < a_{2n} >$$

converge to -1 and 1 respectively.

Note 2. If all subsequences of a sequence $< a_n >$ converge to the same limit l, only then $< a_n >$ converges to l.

To prove that a given sequence is not convergent, it is sufficient to show that two of its subsequences converge to different limits. (See example with Note 1).

Theorem II. *If the subsequences $< a_{2n-1} >$ and $< a_{2n} >$ of a sequence $< a_n >$ converge to the same limit l, then the sequence $< a_n >$ converges to l.*

Proof. Let $\varepsilon > 0$ be given

$< a_{2n-1} >$ converges to l

$\Rightarrow$ For $\varepsilon > 0$, $\exists$ a positive integer m_1 such that $\quad | a_{2n-1} - l | < \varepsilon \quad \forall\, n \geq m_1$

$< a_{2n} >$ converges to l

$\Rightarrow$ For $\varepsilon > 0$, $\exists$ a positive integer m_2 such that $\quad | a_{2n} - l | < \varepsilon \quad \forall\, n \geq m_2$

Let $m = \max. \{m_1, m_2\}$, then $\quad | a_{2n-1} - l | < \varepsilon$ and $| a_{2n} - l | < \varepsilon \quad \forall\, n \geq m$

$\Rightarrow \qquad | a_n - l | < \varepsilon \quad \forall\, n \geq m$

$\Rightarrow \quad < a_n >$ converges to l.

Theorem III. (i) *If a sequence $< a_n >$ diverges to ∞, then every subsequence of $< a_n >$ also diverges to ∞.*

(ii) *If a sequence $< a_n >$ diverges to $-\infty$, then every subsequence of $< a_n >$ also diverges to $-\infty$.*

Proof. (i) Let $< a_{n_k} >$ be a subsequence of $< a_n >$.

Since $< a_n >$ diverges to ∞.

$\therefore$ For every positive real number K, however large, $\exists$ a positive integer m such that

$$a_n > K \quad \forall\, n \geq m \qquad \ldots(1)$$

We can find a natural number $\quad n_{k_0} \geq m$.

If $n_k \geq n_{k_0}$, then $n_k \geq m$.

$\because$ From (1), we have $\quad a_{n_k} > K \quad \forall\, n_k \geq m$

$\Rightarrow \quad < a_{n_k} >$ diverges to ∞.

(ii) *Please try yourself.*

Note 1. *The converse of the above theorem is not true.*

That is if a subsequence of a given sequence diverges to $+\infty$ (or $-\infty$), the sequence need not diverges to $+\infty$ (or $-\infty$).

For example, let $a_n = (-1)^n\, n = \begin{cases} -n & \text{if } n \text{ is odd} \\ n & \text{if } n \text{ is even} \end{cases}$

Then the subsequence $< a_{2n-1} >$ diverges to $-\infty$ and the subsequence $< a_{2n} >$ diverges to $+\infty$, but the sequence $< a_n >$ does not diverge either to $+\infty$ or $-\infty$. $< a_n >$ is an oscillatory sequence.

Note 2. If all the subsequences of a sequence $< a_n >$ diverge to ∞ (or $-\infty$), only then the sequence $< a_n >$ diverges to ∞ (or $-\infty$).

4.33. PEAK POINT OF A SEQUENCE

Def. A natural number m is called a **peak point** of the sequence $< a_n >$ if $a_n < a_m \ \forall \ n > m$.

For example (*i*) The sequence $< (-1)^n >$ has no peak point.

(*ii*) If $a_n = \frac{1}{n}$ when $n \leq 5$

$= -n$ when $n > 5$

then 1, 2, 3, 4, 5 are 5 peak points.

(*iii*) If $a_n = 1$ when $n = 1, 2, \ldots, m$

$= -1$ when $n > m$

then m is the only peak point.

(*iv*) If $a_n = \frac{1}{n}$, then every natural number is a peak point.

For, let m be any natural number, then for $n > m$, we have

$$\frac{1}{n} < \frac{1}{m} \quad i.e., \quad a_n < a_m \quad \forall \ n > m$$

Thus a sequence may have no peak point, a finite number of peak points or an infinite number of peak points.

Note. For a strictly monotonically decreasing sequence, every natural number is a peak point.

4.34. THEOREM

Theorem I. *Every sequence contains a monotonic subsequence.*

Proof. Let $< a_n >$ be any sequence. Then three cases arise according as it has no peak point, finitely many peak points or infinitely many peak points.

Case (*i*) **The sequence has no peak point.**

Since 1 is not a peak point, $\exists$ a natural number $n_2 > 1$ such that $a_{n_2} > a_1$.

Again, since n_2 is not a peak point, $\exists$ a natural number $n_3 > n_2$ such that $a_{n_3} > a_{n_2}$.

Repeating the above argument, we get a subsequence

$$< a_{n_k} > \text{so that } a_{n_1} < a_{n_2} < a_{n_3} < \ldots\ldots \text{ where } n_1 = 1.$$

Thus the sequence $< a_n >$ contains a monotonically increasing subsequence $< a_{n_k} >$.

Case (*ii*) **The sequence has a finite number of peak points.**

Let m be the largest peak point. Let n_1 be a natural number such that $n_1 > m$, then n_1 is not a peak point.

$\therefore$ $\exists$ a natural number $n_2 > n_1$ such that $a_{n_2} > a_{n_1}$.

Again n_2 is not a peak point.

$\therefore$ $\exists$ a natural number $n_3 > n_2$ such that $a_{n_3} > a_{n_2}$

Repeating the above argument, we get a subsequence $< a_{n_k} >$ so that $a_{n_1} < a_{n_2} < a_{n_3} < \ldots\ldots$

Thus the sequence $< a_n >$ contains a monotonically increasing sequence $< a_{n_k} >$.

Case (*iii*) **The sequence has an infinite number of peak points.**

Let the peak points be $n_1, n_2, n_3, \ldots\ldots$ such that $n_1 < n_2 < n_3 < \ldots\ldots$

$\because$ n_1 is a peak point and $n_2 > n_1$ therefore, $a_{n_2} < a_{n_1}$.

$\because$ n_2 is a peak point and $n_3 > n_2$ therefore, $a_{n_3} < a_{n_2}$.

Repeating the above argument, we get a subsequence $< a_{n_k} >$ so that $a_{n_1} > a_{n_2} > a_{n_3} > \ldots\ldots$

Thus the sequence $< a_n >$ contains a monotonically decreasing sequence $< a_{n_k} >$.

Theorem 2. *Every bounded sequence in R contains a convergent subsequence.*

Proof. Let $< a_n >$ be a bounded sequence.

Since every sequence has a monotonic subsequence, therefore, $< a_n >$ has a monotonic subsequence, say $< a_{n_k} >$.

Since $< a_n >$ is bounded, therefore, the subsequence, $< a_{n_k} >$ is also bounded.

Now $< a_{n_k} >$ is a bounded monotonic sequence, therefore, $< a_{n_k} >$ is convergent.

Hence $< a_n >$ has a convergent subsequence.

4.35. SUBSEQUENTIAL LIMIT

Def. Let $< a_n >$ be a sequence. A real number l is called a *subsequential limit* of the sequence $< a_n >$ if there exists a subsequence of $< a_n >$ converging to l.

A subsequential limit of a sequence $< a_n >$ is also a *cluster point* (or *limit point*) of the sequence $< a_n >$.

The following theorem establishes the equivalence between the cluster points of a sequence and its subsequential limits.

4.36. THEOREM

A real number l is a subsequential limit of the sequence $< a_n >$ if and only if each neighbourhood $(l - \varepsilon, l + \varepsilon)$, $\varepsilon > 0$, of l contains infinitely many terms of $< a_n >$ (i.e., if and only if l is a cluster point of $< a_n >$).

Proof. Let l be a subsequential limit of $< a_n >$, then $\exists$ a subsequence $< a_{n_k} >$ of $< a_n >$ converging to l.

$\therefore$ Given $\varepsilon > 0$, $\exists$ a positive integer k_0 such that $| a_{n_k} - l | < \varepsilon \quad \forall \ k \geq k_0$

$$\Rightarrow \quad a_{n_k} \in (l - \varepsilon, l + \varepsilon) \quad \forall \ k \geq k_0$$

$\Rightarrow$ Infinitely many terms of the subsequence $< a_{n_k} >$ and hence of the sequence $< a_n >$ lie in $(l - \varepsilon, l + \varepsilon)$.

Conversely. Let each nbd. $(l - \varepsilon, l + \varepsilon)$ of l contains infinitely many terms of $< a_n >$.

Then $a_n \in (l - \varepsilon, l + \varepsilon)$ for infinitely many values of n.

In particular, $a_n = \left(l - \frac{1}{n}, l + \frac{1}{n}\right) = I_n$ for infinitely many values of n | Taking $\varepsilon = \frac{1}{n}$

Choose $a_{n_1} \in (l-1, l+1) = I_1$. Then $\exists\, n_2 > n_1$ such that $a_{n_2} \in (l - \frac{1}{2}, l + \frac{1}{2}) = I_2$.

Continuing like this, $\exists$ a natural number n_{k_0} such that

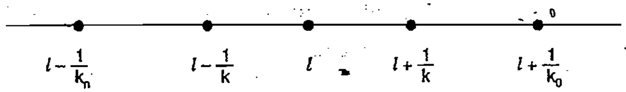
$$n_{k_0} > \ldots\ldots > n_2 > n_1 \quad \text{and} \quad a_{n_{k_0}} \in \left(l - \frac{1}{k_0}, l + \frac{1}{k_0}\right) = I_{k_0}$$

Again, continuing like this, we get a subsequence $< a_{n_k} >$ of $< a_n >$.

Now for all $n_k \geq n_{k_0}$, we have $k \geq k_0$

$$\therefore \quad \frac{1}{k} \leq \frac{1}{k_0} \quad \text{and} \quad -\frac{1}{k} \geq -\frac{1}{k_0}$$

$$\Rightarrow \quad l + \frac{1}{k} \leq l + \frac{1}{k_0} \quad \text{and} \quad l - \frac{1}{k} \geq l - \frac{1}{k_0}$$



$$\Rightarrow \quad \left(l - \frac{1}{k}, l + \frac{1}{k}\right) \subset \left(l - \frac{1}{k_0}, l + \frac{1}{k_0}\right)$$

$$\Rightarrow \quad I_k \subset I_{k_0} \quad \forall\, k \geq k_0 \quad i.e., \quad \forall\, n_k \geq n_{k_0}$$

$$\Rightarrow \quad \forall\, n_k \geq n_{k_0}, a_{n_k} \in I_k \ \Rightarrow\ a_{n_k} \in I_{k_0}$$

$$\Rightarrow \quad a_{n_k} \in \left(l - \frac{1}{k_0}, l + \frac{1}{k_0}\right) \quad \forall\, n_k \geq n_{k_0}$$

$$\Rightarrow \quad \left|a_{n_k} - l\right| < \frac{1}{k_0} = \varepsilon \quad \forall\, n_k \geq n_{k_0}$$

$\Rightarrow$ $< a_{n_k} >$ converges to l.

$\Rightarrow$ l is a subsequential limit of the sequence $< a_n >$.

4.37. CAUCHY SEQUENCES

Def. 1. A sequence $< a_n >$ is said to be a Cauchy sequence if given $\varepsilon > 0$, however small, there exists a positive integer m (depending on ε) such that $|\, a_n - a_m \,| < \varepsilon \quad \forall\, n \geq m$.

Def. 2. A sequence $< a_n >$ is said to be a Cauchy sequence if given $\varepsilon > 0$, however small, there exists a positive integer m (depending on ε) such that $|\, a_{m+p} - a_m \,| < \varepsilon \quad \forall\, p > 0, p \in N$.

Note. $p > 0$ can also be replaced by $p \geq 0$.

Def. 3. A sequence $< a_n >$ is said to be a Cauchy sequence if given $\varepsilon > 0$, however small, there exists a positive integer m (depending on ε) such that $|\, a_p - a_q \,| < \varepsilon \quad \forall\, p, q \geq m$.

Note. Clearly all the definitions are equivalent.

4.38. THEOREM

Theorem I. *Every Cauchy sequence is bounded.*

Proof. Let $< a_n >$ be a Cauchy sequence.

Taking $\varepsilon = 1$, by def., there exists a positive integer m such that

$$|\, a_n - a_m \,| < 1 \qquad \forall\, n \geq m$$

$$\Rightarrow \quad a_m - 1 < a_n < a_m + 1 \qquad \forall\, n \geq m$$

Let $\quad k = \min.\, \{a_1, a_2, \ldots\ldots, a_{m-1}, a_m - 1\}$

and $\quad K = \max.\, \{a_1, a_2, \ldots\ldots a_{m-1}, a_m + 1\}$

then $\quad k \leq a_n \leq K \qquad \forall\, n$

Hence the sequence $< a_n >$ is bounded.

Another Proof.

Let $< a_n >$ be a Cauchy sequence.

$\therefore$ Given $\varepsilon > 0$, $\exists$ a positive integer m such that $|\, a_n - a_m \,| < \varepsilon \quad \forall\, n \geq m$ ...(1)

Now $\quad |\, a_n \,| = |\, (a_n - a_m) + a_m \,| \leq |\, a_n - a_m \,| + |\, a_m \,|$

$\qquad < \varepsilon + |\, a_m \,| \quad \forall\, n \geq m$ by (1)

Let $\quad M = \max.\, \{\, |\, a_1 \,|, |\, a_2 \,|, \ldots\ldots, |\, a_{m-1} \,|, \varepsilon + |\, a_m \,|\, \}$

then $\quad |\, a_n \,| \leq M \quad \forall\, n$

Hence the sequence $< a_n >$ is bounded.

Note. *The converse of the above theorem is not true, i.e.,* every bounded sequence need not be a Cauchy sequence.

For example, the sequence $< (-1)^n >$ is bounded but it is not a Cauchy sequence.

Theorem II. *(Cauchy's General Principle of Convergence)*

A sequence is convergent if and only if it is a Cauchy sequence.

Proof. First, let $< a_n >$ be a convergent sequence, converging to l.

We shall show that it is a Cauchy sequence.

Let $\varepsilon > 0$ be given. Then there exists a positive integer m such that

$$|\, a_n - l \,| < \frac{\varepsilon}{2} \quad \forall\, n \geq m \qquad \ldots(1)$$

In particular, for $n = m$, we have $|\, a_m - l \,| < \frac{\varepsilon}{2}$ ...(2)

Now, $\quad |\, a_n - a_m \,| = |\, (a_n - l) - (a_m - l) \,| \leq |\, a_n - l \,| + |\, a_m - l \,|$

$\qquad < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \qquad \forall\, n \geq m$ [Using (1) and (2)]

Thus $\quad |\, a_n - a_m \,| < \varepsilon \qquad \forall\, n \geq m$

$\Rightarrow$ $< a_n >$ is a Cauchy sequence.

Conversely, let $< a_n >$ be a Cauchy sequence.

Since every Cauchy sequence is bounded, therefore, $< a_n >$ is bounded.

Since every bounded sequence has a cluster point (Bolzano- Weierstrass Theorem), $< a_n >$ has a cluster point l (say).

We shall show that $< a_n >$ converges to l.

Let $\varepsilon > 0$ be given. Since $< a_n >$ is a Cauchy sequence, $\exists$ a positive integer m such that

$$|\, a_n - a_m \,| < \frac{\varepsilon}{3} \quad \forall\, n \geq m \qquad \ldots(3)$$

Since l is a cluster point of $<a_n>$, every nbd of l contains infinitely many terms of $<a_n>$.

$\Rightarrow \quad a_n \in \left(l - \frac{\varepsilon}{3}, l + \frac{\varepsilon}{3}\right)$ for infinitely many values of n.

In particular, we can find a positive integer $k > m$ such that

$$a_k \in \left(l - \frac{\varepsilon}{3}, l + \frac{\varepsilon}{3}\right) \qquad \text{...(3)}$$

i.e., $$| a_k - l | < \frac{\varepsilon}{3} \qquad \text{...(4)}$$

Also, since $k > m$, therefore, from (3), we have $| a_k - a_m | < \frac{\varepsilon}{3}$ ...(5)

Now $$\begin{aligned} | a_n - l | &= |(a_n - a_m) + (a_m - a_k) + (a_k - l)| \\ &\le | a_n - a_m | + | a_m - a_k | + | a_k - l | \\ &= | a_n - a_m | + | a_k - a_m | + | a_k - l | \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \qquad \text{[Using (3), (4), (5)]} \end{aligned}$$

Thus $| a_n - l | < \varepsilon \;\; \forall\, n \ge m$

$\Rightarrow \; <a_n>$ converges to l.

Note 1. The importance of Cauchy's General Principle of Convergence lies in the fact that it decides the convergence or otherwise of a sequence without any idea of the limit of a sequence and involves only the terms of the sequence.

Note 2. Cauchy's General Principle of Convergence can also be stated as follows :

A sequence $<a_n>$ is convergent if and only if for each $\varepsilon > 0$, $\exists$ a positive integer m such that

$$| a_n - a_m | < \varepsilon \quad \forall\, n > m$$

i.e., $$| a_{m+1} + a_{m+2} + \dots + a_n | < \varepsilon \quad \forall\, n > m.$$

ILLUSTRATIVE EXAMPLES

Example 1. *Prove, by definition, that the sequences whose nth terms are given below are Cauchy sequences :*

(i) $\frac{1}{n}$ (ii) $\frac{1}{n^2}$ (iii) $\frac{n}{n+1}$

(iv) $\frac{n+1}{n}$ (v) $\frac{(-1)^n}{n}$ (vi) $8 + \frac{1}{n^3}$

(vii) $\frac{1}{2^n}$.

Sol. (i) Here $a_n = \frac{1}{n}$

Let $\varepsilon > 0$ be given and let $n > m$.

Now $$| a_n - a_m | = \left| \frac{1}{n} - \frac{1}{m} \right| = \frac{1}{m} - \frac{1}{n} \qquad \left[\because n > m \Rightarrow \frac{1}{n} < \frac{1}{m}\right.$$

$$< \frac{1}{m} < \varepsilon \text{ whenever } m > \frac{1}{\varepsilon}$$

$\therefore$ For each $\varepsilon > 0$, $\exists$ a +ve integer m such that $| a_n - a_m | < \varepsilon \;\; \forall\, n > m$

$\Rightarrow \; <a_n>$ is a Cauchy sequence.

(ii) Here $a_n = \frac{1}{n^2}$

Let $\varepsilon > 0$ be given and let $n > m$.

Now $$\begin{aligned} | a_n - a_m | &= \left| \frac{1}{n^2} - \frac{1}{m^2} \right| \\ &= \frac{1}{m^2} - \frac{1}{n^2} \qquad \left[\because n > m \Rightarrow n^2 > m^2 \Rightarrow \frac{1}{n^2} < \frac{1}{m^2}\right. \\ &< \frac{1}{m^2} < \varepsilon \text{ whenever } m^2 > \frac{1}{\varepsilon} \text{ i.e., whenever } m > \frac{1}{\sqrt{\varepsilon}}. \end{aligned}$$

$\therefore$ For each $\varepsilon > 0$, $\exists$ a +ve integer m such that $| a_n - a_m | < \varepsilon \;\; \forall\, n > m$

$\Rightarrow \; <a_n>$ is a Cauchy sequence.

(iii) Here $a_n = \frac{n}{n+1}$

Let $\varepsilon > 0$ be given and let $n > m$.

Now $$\begin{aligned} | a_n - a_m | &= \left| \frac{n}{n+1} - \frac{m}{m+1} \right| \\ &= \left| \left(1 - \frac{1}{n+1}\right) - \left(1 - \frac{1}{m+1}\right) \right| = \left| \frac{1}{m+1} - \frac{1}{n+1} \right| \\ &= \frac{1}{m+1} - \frac{1}{n+1} \qquad \left[\because n > m \Rightarrow n+1 > m+1 \Rightarrow \frac{1}{n+1} < \frac{1}{m+1}\right. \\ &< \frac{1}{m+1} < \frac{1}{m} < \varepsilon \text{ whenever } m > \frac{1}{\varepsilon} \end{aligned}$$

$\therefore$ For each $\varepsilon > 0$, $\exists$ a +ve integer m such that $| a_n - a_m | < \varepsilon \;\; \forall\, n > m$

$\Rightarrow \; <a_n>$ is a Cauchy sequence.

(iv) *Please try yourself.* $\left[\textbf{Hint.}\; a_n = 1 + \frac{1}{n}.\right]$

(v) Here $a_n = \frac{(-1)^n}{n}$

Let $\varepsilon > 0$ be given and let $n > m$.

Now $$\begin{aligned} | a_n - a_m | &= \left| \frac{(-1)^n}{n} - \frac{(-1)^m}{m} \right| \le \left| \frac{(-1)^n}{n} \right| + \left| \frac{(-1)^m}{m} \right| = \frac{1}{n} + \frac{1}{m} < \frac{1}{m} + \frac{1}{m} \\ &= \frac{2}{m} < \varepsilon \text{ whenever } m > \frac{2}{\varepsilon} \end{aligned}$$

$\therefore$ For each $\varepsilon > 0$, $\exists$ a +ve integer m such that $| a_n - a_m | < \varepsilon \;\; \forall\, n > m$

$\Rightarrow \; <a_n>$ is a Cauchy sequence.

(*vi*) Here $a_n = 8 + \frac{1}{n^3}$

Let $\varepsilon > 0$ be given and let $n > m$.

Now $$| a_n - a_m | = \left|\left(8+\frac{1}{n^3}\right)-\left(8+\frac{1}{m^3}\right)\right| = \left|\frac{1}{n^3}-\frac{1}{m^3}\right|$$

$$= \frac{1}{m^3}-\frac{1}{n^3} \qquad \left[\because n>m \Rightarrow n^3>m^3 \Rightarrow \frac{1}{n^3}<\frac{1}{m^3}\right]$$

$$< \frac{1}{m^3} < \varepsilon \text{ whenever } m^3 > \frac{1}{\varepsilon} \quad i.e., \text{ whenever } m > \frac{1}{(\varepsilon)^{1/3}}$$

$\therefore$ For each $\varepsilon > 0$, $\exists$ a +ve integer m such that $| a_n - a_m | < \varepsilon \quad \forall\, n > m$.

$\Rightarrow$ $< a_n >$ is a Cauchy sequence.

(*vii*) Here $a_n = \frac{1}{2^n}$

Let $\varepsilon > 0$ be given and let $n > m$.

Now $$| a_n - a_m | = \left|\frac{1}{2^n}-\frac{1}{2^m}\right|$$

$$= \frac{1}{2^m}-\frac{1}{2^n} \qquad \left[\because n>m \Rightarrow 2^n>2^m \Rightarrow \frac{1}{2^n}<\frac{1}{2^m}\right]$$

$$< \frac{1}{2^m} < \varepsilon \text{ whenever } 2^m > \frac{1}{\varepsilon}$$

i.e., whenever $m \log 2 > \log\frac{1}{\varepsilon}$ *i.e.*, whenever $m > \dfrac{\log\frac{1}{\varepsilon}}{\log 2}$.

$\therefore$ For each $\varepsilon > 0$, $\exists$ a +ve integer m such that $| a_n - a_m | < \varepsilon \quad \forall\, n > m$.

$\Rightarrow$ $< a_n >$ is a Cauchy sequence.

Example 2. *Show that the sequences whose nth terms are given below are not Cauchy sequences :*

(*i*) $(-1)^n$ (*ii*) n

(*iii*) n^2 (*iv*) $(-1)^n n$.

Sol. (*i*) Here $a_n = (-1)^n$

Let $p = 2m+1$ and $q = 2m$ so that $p, q > m$

Now $| a_p - a_q | = | (-1)^{2m+1} - (-1)^{2m} | = | (-1) - (1) | = 2$.

$\therefore$ If $\varepsilon = \frac{1}{2}$, it is not possible to find any positive integer m such that

$$| a_p - a_q | < \varepsilon \quad \forall\, p, q > m$$

$\Rightarrow$ $< a_n >$ is not a Cauchy sequence.

(*ii*) Here $a_n = n$

Let $p = 2m+1$ and $q = 2m$ so that $p, q > m$.

Now $| a_p - a_q | = | p - q | = | (2m+1) - 2m | = 1$.

$\therefore$ If $\varepsilon = \frac{1}{2}$, it is not possible to find any positive integer m such that

$$| a_p - a_q | < \varepsilon \quad \forall\, p, q > m$$

$\Rightarrow$ $< a_n >$ is not a Cauchy sequence.

(*iii*) Here $a_n = n^2$

Let $p = 2m+1$ and $q = 2m$ so that $p, q > m$

Now $$| a_p - a_q | = | p^2 - q^2 | = | (p+q)(p-q) |$$
$$= | 4m+1 | = 4m+1 > 1 \quad \forall\, m \in \mathbb{N}.$$

$\therefore$ If $\varepsilon = \frac{1}{2}$, it is not possible to find any positive integer m such that

$$| a_p - a_q | < \varepsilon \quad \forall\, p, q > m$$

$\Rightarrow$ $< a_n >$ is not a Cauchy sequence.

(*iv*) Here $a_n = (-1)^n n$

Let $p = 2m+1$ and $q = 2m$ so that $p, q > m$.

Now $$| a_p - a_q | = | (-1)^p . p - (-1)^q . q |$$
$$= | (-1)^{2m+1}(2m+1) - (-1)^{2m} . 2m |$$
$$= | -(2m+1) - 2m | = | -(4m+1) |$$
$$= 4m+1 > 1 \quad \forall\, m \in \mathbb{N}.$$

$\therefore$ If $\varepsilon = \frac{1}{2}$, it is not possible to find any positive integer m such that

$$| a_p - a_q | < \varepsilon \quad \forall\, p, q > m$$

$\Rightarrow$ $< a_n >$ is not a Cauchy sequence.

Example 3. *Using Cauchy's general principle of convergence, show that the sequence* $< a_n >$ *where*

$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots + \frac{1}{n} \textit{ is not convergent.}$$

Sol. For $n > m$,

$$| a_n - a_m | = \left|\frac{1}{m+1}+\frac{1}{m+2}+\ldots\ldots+\frac{1}{n}\right|$$
$$= \frac{1}{m+1}+\frac{1}{m+2}+\ldots\ldots+\frac{1}{n} \qquad [\because |x| = x \text{ for } x > 0]$$

Taking $n = 2m$, we have $| a_{2m} - a_m | = \frac{1}{m+1}+\frac{1}{m+2}+\ldots\ldots+\frac{1}{2m}$

$$> \frac{1}{2m}+\frac{1}{2m}+\ldots\ldots+\frac{1}{2m} = \frac{m}{2m} = \frac{1}{2}$$

i.e., $$| a_{2m} - a_m | \geq \frac{1}{2} \qquad \ldots(1)$$

If $< a_n >$ is convergent, then by Cauchy's general principle of convergence, for each $\varepsilon > 0$, $\exists$ a +ve integer m such that $| a_n - a_m | < \varepsilon \;\forall\, n > m$

In particular, for $\varepsilon = \frac{1}{2}$ and $n = 2m$, we must have $| a_{2m} - a_m | < \frac{1}{2}$ which contradicts (1)

Hence the sequence $< a_n >$ cannot converge.

Note. $$a_{n+1} - a_n = \left(1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots+\frac{1}{n}+\frac{1}{n+1}\right)-\left(1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots+\frac{1}{n}\right)$$
$$= \frac{1}{n+1} > 0 \quad \forall\, n$$

$$\Rightarrow \qquad a_{n+1} > a_n \quad \forall\, n$$

$\Rightarrow$ $< a_n >$ is a monotonically increasing sequence.

Since it does not converge, it must diverge to $+\infty$.

Example 4. *Show by applying Cauchy's convergence criterion that the sequence $< a_n >$ where*

$$a_n = 1 + \frac{1}{3} + \frac{1}{5} + \ldots\ldots + \frac{1}{2n-1} \text{ is not convergent.}$$

Sol. For $n > m$,

$$| a_n - a_m | = \left| \frac{1}{2m+1} + \frac{1}{2m+3} + \ldots\ldots + \frac{1}{2n-1} \right| - \frac{1}{2m+1} + \frac{1}{2m+3} + \ldots \quad \frac{1}{2n-1}$$

Taking $n = 2m$, we have

$$| a_{2m} - a_m | = \frac{1}{2m+1} + \frac{1}{2m+3} + \ldots\ldots + \frac{1}{4m-1}$$

$$> \frac{1}{4m} + \frac{1}{4m} + \ldots\ldots + \frac{1}{4m} = \frac{m}{4m} = \frac{1}{4}$$

i.e., $$| a_{2m} - a_m | > \frac{1}{4} \qquad \ldots(1)$$

If $< a_n >$ is convergent, then by Cauchy's general principle of convergence, for each $\varepsilon > 0$, $\exists$ a +ve integer m such that $| a_n - a_m | < \varepsilon \quad \forall\, n > m$.

In particular, for $\varepsilon = \frac{1}{4}$ and $n = 2m$, we must have $| a_{2m} - a_m | < \frac{1}{4}$ which contradicts (1).

Hence the sequence $< a_n >$ cannot converge.

Note. $$a_{n+1} - a_n = \left(1 + \frac{1}{3} + \frac{1}{5} + \ldots\ldots + \frac{1}{2n-1} + \frac{1}{2n+1}\right) - \left(1 + \frac{1}{3} + \frac{1}{5} + \ldots\ldots + \frac{1}{2n-1}\right)$$

$$= \frac{1}{2n+1} > 0 \quad \forall\, n$$

$$\Rightarrow \qquad a_{n+1} > a_n \ \forall\, n$$

$\Rightarrow$ $< a_n >$ is a monotonically increasing sequence.

Since it does not converge, it must diverge to $+\infty$.

Example 5. *Apply Cauchy's general principle of convergence to prove that the sequence $< a_n >$ defined by $a_n = 1 + \frac{1}{4} + \frac{1}{7} + \ldots\ldots + \frac{1}{3n+2}$ is not convergent.*

Sol. *Please try yourself.*

Example 6. *If $< a_n >$ is a sequence of positive real numbers such that*

$$a_n = \frac{1}{2}(a_{n-1} + a_{n-2}) \ \forall\, n \geq 3,$$

then show that $< a_n >$ converges to $\frac{1}{3}(a_1 + 2a_2)$.

Sol. To show that $< a_n >$ is convergent, we shall show that it is a Cauchy sequence.

Now $$| a_n - a_{n-1} | = \left| \frac{a_{n-1} + a_{n-2}}{2} - a_{n-1} \right| = \frac{1}{2} | a_{n-2} - a_{n-1} |$$

$$= \frac{1}{2} | a_{n-1} - a_{n-2} | \qquad \ldots(1)$$

Changing n to $n - 1$, we have

$$| a_{n-1} - a_{n-2} | = \frac{1}{2} | a_{n-2} - a_{n-3} |$$

$\therefore$ From (1), $$| a_n - a_{n-1} | = \frac{1}{2^2} | a_{n-2} - a_{n-3} |$$

Continuing like this, we have

$$| a_n - a_{n-1} | = \frac{1}{2^{n-2}} | a_2 - a_1 |$$

$\therefore$ For $n > m$, $| a_n - a_m | = | (a_n - a_{n-1}) + (a_{n-1} - a_{n-2}) + \ldots\ldots + (a_{m+1} - a_m) |$

$$\leq | a_n - a_{n-1} | + | a_{n-1} - a_{n-2} | + \ldots\ldots + | a_{m+1} - a_m |$$

$$= \frac{1}{2^{n-2}} | a_2 - a_1 | + \frac{1}{2^{n-3}} | a_2 - a_1 | + \ldots\ldots + \frac{1}{2^{m-1}} | a_2 - a_1 |$$

$$= \left(\frac{1}{2^{n-2}} + \frac{1}{2^{n-3}} + \ldots\ldots + \frac{1}{2^{m-1}}\right) | a_2 - a_1 |$$

$$= \left(\frac{1}{2^{m-1}} + \frac{1}{2^{m}} + \ldots\ldots + \frac{1}{2^{n-2}}\right) | a_2 - a_1 |$$

$\left[\text{a G.P. of } (n-2) - (m-2) = n - m \text{ terms with first term} = \frac{1}{2^{m-1}} \text{ and C.R.} = \frac{1}{2}\right]$

$$= \frac{\frac{1}{2^{m-1}}\left(1 - \frac{1}{2^{n-m}}\right)}{1 - \frac{1}{2}} | a_2 - a_1 | = \frac{1}{2^{m-2}}\left(1 - \frac{1}{2^{n-m}}\right) | a_2 - a_1 |$$

$$< \frac{1}{2^{m-2}} | a_2 - a_1 | < \varepsilon \text{ whenever } 2^{m-2} > \frac{| a_2 - a_1 |}{\varepsilon}$$

i.e., $$(m-2)\log 2 > \log \frac{| a_2 - a_1 |}{\varepsilon} \quad i.e., \quad m > 2 + \frac{\log \frac{| a_2 - a_1 |}{\varepsilon}}{\log 2}$$

Thus, given $\varepsilon > 0$, we can choose a positive integer m such that

$$| a_n - a_m | < \varepsilon \quad \forall\, n > m$$

$\therefore$ $< a_n >$ is a Cauchy sequence.

Hence $< a_n >$ is a convergent sequence.

Let $< a_n >$ converge to l.

Since $$a_n = \frac{1}{2}(a_{n-1} + a_{n-2}) \ \forall\, n \geq 3$$

$\therefore$ $$a_3 = \frac{1}{2}(a_2 + a_1)$$

$$a_4 = \frac{1}{2}(a_3 + a_2)$$

$$a_5 = \frac{1}{2}(a_4 + a_3)$$

....

....

$$a_{n-1} = \frac{1}{2}(a_{n-2} + a_{n-3})$$

$$a_n = \frac{1}{2}(a_{n-1} + a_{n-2})$$

Adding these equations, we have

$$a_n + a_{n-1} = \frac{1}{2}a_1 + a_2 + \frac{1}{2}a_{n-1} \quad \Rightarrow \quad a_n + \frac{1}{2}a_{n-1} = \frac{1}{2}(a_1 + 2a_2)$$

Taking limit as $n \to \infty$, we have

$$\lim_{n\to\infty} a_n + \frac{1}{2}\lim_{n\to\infty} a_{n-1} = \frac{1}{2}(a_1 + 2a_2)$$

$$\Rightarrow \quad l + \frac{1}{2}l = \frac{1}{2}(a_1 + 2a_2) \quad \Rightarrow \quad l = \frac{1}{3}(a_1 + 2a_2)$$

Hence $< a_n >$ converges to $\frac{1}{3}(a_1 + 2a_2)$.

5
Infinite Series

5.1. INFINITE SERIES

If $\{u_n\}$ is a sequence of real numbers, then the expression

$$u_1 + u_2 + u_3 + \ldots + u_n \ldots\ldots$$

[i.e., the sum of the terms of the sequence, which are infinite in number] is called an infinite series.

The infinite series $u_1 + u_2 + \ldots + u_n + \ldots$ is usually denoted by $\sum_{n=1}^{\infty} u_n$ or more briefly, by Σu_n.

The numbers $u_1, u_2, u_3, \ldots, u_n, \ldots$ are called the first, second, third, ..., nth term (or general term), ... of the series.

5.2. SERIES OF POSITIVE TERMS

If all the terms of the series $\Sigma u_n = u_1 + u_2 + \ldots + u_n + \ldots$ are positive *i.e.*, if $u_n > 0, \forall n$, then the series Σu_n is called a series of positive terms.

5.3. ALTERNATING SERIES

A series in which the terms are alternately positive and negative is called an alternating series. Thus, the series

$$\Sigma(-1)^{n-1} u_n = u_1 - u_2 + u_3 - u_4 + \ldots + (-1)^{n-1} u_n + \ldots$$

where $u_n > 0 \; \forall \; n$ is an alternating series.

5.4. PARTIAL SUMS

If $\Sigma u_n = u_1 + u_2 + u_3 + \ldots + u_n + \ldots$ is an infinite series where the terms may be + ve or – ve, then $S_n = u_1 + u_2 + \ldots + u_n$ is called the nth partial sum of Σu_n. Thus, the nth partial sum of an infinite series is the sum of its first n terms.

$S_1, S_2, S_3, \ldots$, are the first, second, third ... partial sums of the series.

Since $n \in N$, $\{S_n\}$ is a sequence called the **sequence of partial sums** of the infinite series Σu_n.

$\therefore$ To every infinite series Σu_n, there corresponds a sequence $\{S_n\}$ of its partial sums.

5.5. BEHAVIOUR OF AN INFINITE SERIES

An infinite series Σu_n converges, diverges or oscillates (finitely or infinitely) according as the sequence $\{S_n\}$ of its partial sums converges, diverges or oscillates (finitely or infinitely).

(*i*) The series Σu_n converges (or is said to be convergent) if the sequence $\{S_n\}$ of its partial sums converges.

Thus, Σu_n is convergent if $\underset{n\to\infty}{\text{Lt}}\ S_n =$ finite.

(*ii*) The series Σu_n diverges (or is said to be divergent) if the sequence $\{S_n\}$ of its partial sums diverges.

Thus, Σu_n is divergent if $\underset{n\to\infty}{\text{Lt}}\ S_n = +\infty$ or $-\infty$.

(*iii*) The series Σu_n oscillates finitely if the sequence $\{S_n\}$ of its partial sums oscillates finitely.

Thus, Σu_n oscillates finitely if $\{S_n\}$ is bounded and neither converges nor diverges.

(*iv*) The series Σu_n oscillates infinitely if the sequence $\{S_n\}$ of its partial sums oscillates infinitely.

Thus, Σu_n oscillates infinitely if $\{S_n\}$ is unbounded and neither converges nor diverges.

Example 1. *Discuss the convergence or otherwise of the series*

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots\ldots+\frac{1}{n(n+1)}+\ldots\ldots \text{ to } \infty.$$

Sol. Here $\quad u_n = \dfrac{1}{n(n+1)} = \dfrac{1}{n} - \dfrac{1}{n+1}$

Putting $\quad n = 1, 2, 3, \ldots\ldots, n$

$$u_1 = \frac{1}{1} - \frac{1}{2}$$

$$u_2 = \frac{1}{2} - \frac{1}{3}$$

$$u_3 = \frac{1}{3} - \frac{1}{4}$$

....

....

$$u_n = \frac{1}{n} - \frac{1}{n+1}$$

Adding, $\quad S_n = 1 - \dfrac{1}{n+1}$

$$\underset{n\to\infty}{\text{Lt}}\ S_n = 1 - 0 = 1$$

$\Rightarrow$ $\{S_n\}$ converges to 1 $\Rightarrow$ Σu_n converges to 1.

(**Note.** For another method, see Comparison Test.)

Example 2. *Show that the series* $1^2 + 2^2 + 3^2 + \ldots\ldots + n^2 + \ldots\ldots$ *diverges to* $+\infty$

Sol. $\quad S_n = 1^2 + 2^2 + 3^2 + \ldots\ldots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

$$\underset{n\to\infty}{\text{Lt}}\ S_n = +\infty$$

$\Rightarrow$ $\{S_n\}$ diverges to $+\infty$

$\Rightarrow$ The given series diverges to $+\infty$.

Example 3. *~~Show~~ that the series* $-1-2-3-\ldots\ldots-n-\ldots\ldots$ *diverges to* $-\infty$

Sol. $\quad S_n = -1-2-3 \ldots\ldots -n$

$$= -(1+2+3 \ldots\ldots + n) = -\frac{n(n+1)}{2}$$

$$\underset{n\to\infty}{\text{Lt}}\ S_n = -\infty \quad \Rightarrow \quad \{S_n\} \text{ diverges to } -\infty$$

$\Rightarrow$ The given series diverges to $-\infty$.

Example 4. *Test the convergence or otherwise of* $\sum\limits_{n=0}^{\infty} (-1)^n$.

Sol. $\quad S_n = 1-1+1-1+1-1+\ldots\ldots$ to n terms

$\quad = 1$ or 0 according as n is odd or even.

The subsequence $\{S_{2n-1}\}$ converges to 1, while the subsequence $\{S_{2n}\}$ converges to 0.

$\Rightarrow$ $\{S_n\}$ is not convergent.

Since $\{S_n\}$ is bounded.

$\therefore$ $\{S_n\}$ oscillates finitely. $\Rightarrow$ $\sum\limits_{n=0}^{\infty} (-1)^n$ oscillates finitely.

Article 1. The geometric series $1 + x + x^2 + x^3 + \ldots\ldots$ to ∞:

(*i*) **converges if $-1 < x < 1$** *i.e.,* **$|x| < 1$** (*ii*) **diverges if $x \geq 1$**

(*iii*) **oscillates finitely if $x = -1$** (*iv*) **oscillates infinitely if $x < -1$.**

Proof. (*i*) **When $|x| < 1$.**

Since $\quad |x| < 1, x^n \to 0$ as $n \to \infty$

$$S_n = 1 + x + x^2 + \ldots\ldots \text{ to } n \text{ terms} = \frac{1(1-x^n)}{1-x} = \frac{1}{1-x} - \frac{x^n}{1-x}$$

$$\underset{n\to\infty}{\text{Lt}}\ S_n = \frac{1}{1-x}$$

$\Rightarrow$ the sequence $\{S_n\}$ is convergent

$\Rightarrow$ the given series is convergent.

(*ii*) **When $x \geq 1$.**

Sub-case I. When $x = 1$

$$S_n = 1 + 1 + 1 + \ldots\ldots \text{ to } n \text{ terms} = n$$

$$\underset{n\to\infty}{\text{Lt}}\ S_n = \infty$$

$\Rightarrow$ the sequence $\{S_n\}$ diverges to ∞

$\Rightarrow$ the given series diverges to ∞.

Sub-case II. When $x > 1$, $x^n \to \infty$ as $n \to \infty$

$$S_n = 1 + x + x^2 + \ldots\ldots \text{ to } n \text{ terms} = \frac{1(x^n-1)}{x-1}$$

$$\underset{n\to\infty}{\text{Lt}}\ S_n = \infty$$

$\Rightarrow$ the sequence $\{S_n\}$ diverges to ∞

$\Rightarrow$ the given series diverges to ∞.

(*iii*) **When $x = -1$**

$$S_n = 1-1+1-1+\ldots\ldots \text{ to } n \text{ terms}$$

$$= 1 \text{ or } 0 \text{ according as } n \text{ is odd or even.}$$

$\therefore \quad S_{2n-1} \to 1$ and $S_{2n} \to 0$

$\Rightarrow$ the sequence $\{S_n\}$ oscillates finitely.
$\Rightarrow$ the given series oscillates finitely.

(*iv*) **When x < – 1**

$$x < -1 \Rightarrow -x > 1$$

Let $r = -x$, then $r > 1$

$\therefore$ $r^n \to \infty$ as $n \to \infty$

$$S_n = 1 + x + x^2 + x^3 + \ldots \text{ to } n \text{ terms}$$

$$= \frac{1-x^n}{1-x} = \frac{1-(-r)^n}{1+r} \qquad [\because \ x = -r]$$

$$= \frac{1-r^n}{1+r} \text{ or } \frac{1+r^n}{1+r} \text{ according as } n \text{ is even or odd}$$

$\therefore$ $S_{2n} \to -\infty$ and $S_{2n-1} \to \infty$

$\Rightarrow$ the sequence $\{S_n\}$ oscillates infinitely
$\Rightarrow$ the given series oscillates infinitely.

☞**Remember.** A geometric series converges only when its common ratio is numerically less than 1.

Article 2. If a series Σu_n is convergent, then $\underset{n\to\infty}{\text{Lt}}\ u_n = 0$. Is the converse true ?

Proof. Let S_n denote the nth partial sum of the series Σu_n.

Then Σu_n is convergent $\Rightarrow$ $\{S_n\}$ is convergent

$\Rightarrow$ $\underset{n\to\infty}{\text{Lt}}\ S_n$ is finite and unique $= s$ (say).

$$\Rightarrow \quad \underset{n\to\infty}{\text{Lt}}\ S_{n-1} = s$$

Now $S_n = u_1 + u_2 + \ldots + u_{n-1} + u_n$

$S_{n-1} = u_1 + u_2 + \ldots + u_{n-1}$

$\therefore$ $S_n - S_{n-1} = u_n$

$\therefore$ $\underset{n\to\infty}{\text{Lt}}\ u_n = \underset{n\to\infty}{\text{Lt}}\ (S_n - S_{n-1}) = \underset{n\to\infty}{\text{Lt}}\ S_n - \underset{n\to\infty}{\text{Lt}}\ S_{n-1} = s - s = 0.$

☞**Hence Σu_n is convergent $\Rightarrow$ $\underset{n\to\infty}{\text{Lt}}\ u_n = 0$.**

The converse of the above theorem is not always true, *i.e.*, the nth term may tend to zero as $n \to \infty$ even if the series is not convergent.

For example, the series $1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} + \ldots$ diverges, through

$$\underset{n\to\infty}{\text{Lt}}\ u_n = \underset{n\to\infty}{\text{Lt}}\ \frac{1}{n} = 0.$$

☞**Note 1.** Σu_n is convergent $\Rightarrow$ $\underset{n\to\infty}{\text{Lt}}\ u_n = 0$.

Note 2. $\underset{n\to\infty}{\text{Lt}}\ u_n = 0$ $\Rightarrow$ Σu_n may or may not be convergent

Note 3. $\underset{n\to\infty}{\text{Lt}}\ u_n \neq 0$ $\Rightarrow$ Σu_n is not convergent.

Article 3. A positive term series either converges or diverges to + ∞

Proof. Let Σu_n be a positive term series and S_n be its nth partial sum.

Then $S_{n+1} = u_1 + u_2 + \ldots + u_n + u_{n+1} = S_n + u_{n+1}$

$\Rightarrow$ $S_{n+1} - S_n = u_{n+1} > 0 \quad \forall\, n \qquad [\because \ u_n > 0 \quad \forall\, n]$

$\Rightarrow$ $S_{n+1} > S_n \ \forall\, n$

$\Rightarrow$ $\{S_n\}$ is a monotonically increasing sequence.

Two cases arise. The sequence $\{S_n\}$ may be bounded or unbounded above.

Case I. When $\{S_n\}$ is bounded above.

Since $\{S_n\}$ is monotonically increasing and bounded above, it is convergent $\Rightarrow$ Σu_n is convergent.

Case II. When $\{S_n\}$ is not bounded above.

Since $\{S_n\}$ is monotonically increasing and not bounded above, it diverges to $+\infty$ $\Rightarrow$ Σu_n diverges to $+\infty$.

☞**Hence a positive term series either converges or diverges to + ∞.**

Cor. If $u_n > 0\ \forall\, n$ and $\underset{n\to\infty}{\text{Lt}}\ u_n \neq 0$, then the series Σu_n diverges to $+\infty$.

Proof. $u_n > 0\ \forall\, n \Rightarrow$ Σu_n is a series of +ve terms.

$\Rightarrow$ Σu_n either converges or diverges to $+\infty$

Since $\underset{n\to\infty}{\text{Lt}}\ u_n \neq 0$ (given)

$\therefore$ Σu_n does not converge.

Hence Σu_n diverges to $+\infty$.

Article 4. The necessary and sufficient condition for the convergence of a positive term series Σu_n is that the sequence $\{S_n\}$ of its partial sums is bounded above.

Proof. (*i*) Suppose the sequence $\{S_n\}$ is bounded above. Since the series Σu_n is of positive terms, the sequence $\{S_n\}$ is monotonically increasing. Since every monotonically increasing sequence which is bounded above, converges, therefore $\{S_n\}$ and hence Σu_n converges.

(*ii*) Conversely, suppose Σu_n coverges. Then the sequence $\{S_n\}$ of its partial sums also converges. Since every convergent sequence is bounded, $\{S_n\}$ is bounded. In particular, $\{S_n\}$ is bounded above.

Article 5. Cauchy's General Principle of Convergence of series

The necessary and sufficient condition for the infinite series Σu_n to converge is that given $\varepsilon > 0$, however small, there exists a positive integer m such that

$$|\, u_{m+1} + u_{m+2} + \ldots + u_n \,| < \varepsilon\ \forall\, n > m.$$

Proof. The series Σu_n is convergent iff the sequence $\{S_n\}$ of its partial sums is convergent. By Cauchy's general principle of convergence for sequences, the sequence $\{S_n\}$ is convergent iff for each given $\varepsilon > 0$, there exists a positive integer m such that

$$|\, S_n - S_m \,| < \varepsilon \quad \forall\, n > m$$

$$\Rightarrow \quad |\, u_{m+1} + u_{m+2} + \ldots + u_n \,| < \varepsilon \quad \forall\, n > m$$

Hence the result.

Example. *Prove with the help of Cauchy's general principle of convergence that the series*

$$\sum \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} + \ldots \textit{ does not converge.}$$

Sol. If possible, suppose the given series is convergent.

Take $\varepsilon = \frac{1}{2}$.

By Cauchy's general principle of convergence, there exists a positive integer m such that

$$\left|\frac{1}{m+1}+\frac{1}{m+2}+\dots+\frac{1}{n}\right|<\frac{1}{2} \quad \forall\, n>m$$

$$\Rightarrow \quad \frac{1}{m+1}+\frac{1}{m+2}+\dots+\frac{1}{n}<\frac{1}{2} \quad \forall\, n>m \qquad \dots(1)$$

By taking $n = 2m$, we see that

$$\frac{1}{m+1}+\frac{1}{m+2}+\dots+\frac{1}{n}=\frac{1}{m+1}+\frac{1}{m+2}+\dots+\frac{1}{2m}$$
$$>\frac{1}{2m}+\frac{1}{2m}+\dots+\frac{1}{2m}=\frac{m}{2m}=\frac{1}{2}$$

i.e., $\quad \frac{1}{m+1}+\frac{1}{m+2}+\dots+\frac{1}{n}>\frac{1}{2}$ where $n = 2m > m$.

This contradicts (1).

$\Rightarrow$ Our supposition is wrong.

$\Rightarrow$ The given series does not converge.

Article 6. If m is a given positive integer, then the two series

$u_1 + u_2 + \dots + u_{m+1} + u_{m+2} + \dots$ and $u_{m+1} + u_{m+2} + \dots$ behave alike.

Proof. Let S_n and s_n denote the nth partial sums of the two series.

Then $\quad S_n = u_1 + u_2 + \dots u_n$

$$s_n = u_{m+1} + u_{m+2} + \dots + u_{m+n} = (u_1 + u_2 + \dots + u_{m+n}) - (u_1 + u_2 + \dots + u_m)$$
$$= S_{m+n} - S_m \Rightarrow s_n = S_{m+n} - S_m$$

S_m being the sum of a finite number of terms of Σu_n is a fixed finite quantity.

$\therefore$ The sequences $\{S_n\}$ and $\{s_n\}$ have same behaviour *i.e.*, converge or diverge or oscillate together.

$\Rightarrow$ The two given series behave alike. Hence the result.

Note 1. The above theorem shows that the convergence, divergence or oscillation of a series is not affected by addition or omission of a finite number of its terms.

Note 2. Another form of above theorem

If $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ be two series such that $v_n = u_{m+n}$, where m is a given natural number, then the two series behave alike.

Article 7. (*i*) **If c is a constant and Σu_n converges to u, then Σcu_n converges to cu**

(*ii*) **If Σu_n is divergent, then so is Σcu_n where $c \neq 0$.**

Proof. (*i*) Let $\quad S_n = u_1 + u_2 + \dots u_n$ and $s_n = cu_1 + cu_2 + \dots + cu_n$

Then $\quad s_n = c(u_1 + u_2 + \dots + u_n) = cS_n$.

Since Σu_n converges to u $\therefore$ $\underset{n\to\infty}{\text{Lt}}\, S_n = u$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}}\, s_n = \underset{n\to\infty}{\text{Lt}}\,(cS_n) = c\,\underset{n\to\infty}{\text{Lt}}\, S_n = cu$$

$\Rightarrow$ Σcu_n converges to cu.

(*ii*) If Σu_n diverges to $+\infty$, then $\underset{n\to\infty}{\text{Lt}}\, S_n = +\infty$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}}\, s_n = \underset{n\to\infty}{\text{Lt}}\,(cS_n) = c\,\underset{n\to\infty}{\text{Lt}}\, S_n = c\,(\infty)$$
$$= +\infty \quad \text{if } c > 0$$
$$-\infty \quad \text{if } c < 0$$

$\Rightarrow$ $\forall$ values of $c \neq 0$, Σcu_n is divergent.

Similarly, if Σu_n diverges to $-\infty$, Σcu_n is divergent for $c \neq 0$.

Article 8. If Σu_n and Σv_n converge to u and v respectively, then

(*i*) **$\Sigma(u_n + v_n)$ converges to (u + v)** (*ii*) **$\Sigma(u_n - v_n)$ converges to (u − v).**

Proof. (*i*) Let $\quad U_n = u_1 + u_2 + \dots + u_n$

$V_n = v_1 + v_2 + \dots + v_n$

and $\quad S_n = (u_1 + v_1) + (u_2 + v_2) + \dots + (u_n + v_n)$

Then $\quad S_n = (u_1 + u_2 + \dots + u_n) + (v_1 + v_2 + \dots + v_n) = U_n + V_n$.

Since Σu_n converges to u, $\underset{n\to\infty}{\text{Lt}}\, U_n = u$

Σv_n converges to v, $\quad \underset{n\to\infty}{\text{Lt}}\, V_n = v$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}}\, S_n = \underset{n\to\infty}{\text{Lt}}\,(U_n + V_n) = \underset{n\to\infty}{\text{Lt}}\, U_n + \underset{n\to\infty}{\text{Lt}}\, V_n$$
$$= u + v.$$

$\Rightarrow$ $\Sigma(u_n + v_n)$ converges to $(u + v)$.

(*ii*) *Please try yourself.*

Article 9. An infinite series of positive terms is divergent if each of its terms after a fixed stage is greater than some fixed positive quantity

Proof. Let Σu_n be a series of +ve terms and $u_n > k$, $\forall\, n > m$ where k is a fixed +ve quantity.

$\Rightarrow \quad u_{m+1}, u_{m+2}, \dots$ are all $> k$.

Now $u_1 + u_2 + \dots + u_m$ = sum of a finite number of +ve terms of Σu_n

= a fixed, finite, +ve quantity = M (*say*)

Let $\quad S_n = u_1 + u_2 + \dots + u_n$

then $\quad S_n = (u_1 + u_2 + \dots + u_m) + (u_{m+1} + u_{m+2} + \dots + u_n)$

$= M + (u_{m+1} + u_{m+2} + \dots + u_n) > M = [k + k + \dots (n-m) \text{ times}]$

$\Rightarrow \quad S_n > M + (n-m)\,k$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}}\, S_n \geq \underset{n\to\infty}{\text{Lt}}\,[M + (n-m)\,k] = \infty.$$

$\Rightarrow$ Σu_n is divergent.

Article 10. Comparison Tests

Test I. If Σu_n and Σv_n are series of positive terms and Σv_n is convergent and there is a positive constant k such that $u_n \leq kv_n$, $\forall$ n, then Σu_n is also convergent.

Proof. Let $\quad U_n = u_1 + u_2 + \dots + u_n$ and $V_n = v_1 + v_2 + \dots + v_n$.

Now $\quad U_n = u_1 + u_2 + \ldots\ldots + u_n$

$\quad\quad \le kv_1 + kv_2 + \ldots\ldots + kv_n \quad [\because \; u_n \le kv_n \; \forall n]$

$\quad\quad = k(v_1 + v_2 + \ldots\ldots + v_n) = kV_n$

$\Rightarrow \quad U_n \le kV_n \quad \ldots(1)$

Since Σv_n is convergent, the sequence $\{V_n\}$ is convergent and hence bounded.

$\therefore$ $\exists$ a positive number k_0 such that $V_n < k_0 \;\; \forall n$

From (1), $\quad U_n \le k\, V_n < kk_0 \;\; \forall n$

$\Rightarrow \quad U_n < K \;\; \forall n$ where $K = kk_0$

$\Rightarrow$ $\{U_n\}$ is bounded above.

Also Σu_n being a series of +ve terms, $\{U_n\}$ is monotonically increasing. [*See Article 3*]

$\because$ $\{U_n\}$ is a monotonic increasing sequence and is bounded above.

$\therefore$ $\{U_n\}$ is convergent $\Rightarrow$ Σu_n is convergent.

Test II. If Σu_n and Σv_n are series of positive terms and Σv_n is divergent and there is a positive constant k such that $u_n \ge kv_n \; \forall n$, then Σu_n is also divergent.

Proof. Let $\quad U_n = u_1 + u_2 + \ldots\ldots + u_n$ and $V_n = v_1 + v_2 + \ldots\ldots + v_n$

Now $\quad U_n = u_1 + u_2 + \ldots\ldots + u_n$

$\quad\quad \ge kv_1 + kv_2 + \ldots\ldots + kv_n \quad [\because \; u_n \ge kv_n, \; \forall n]$

$\quad\quad = k(v_1 + v_2 + \ldots\ldots + v_n) = kV_n$

$\Rightarrow \quad U_n \ge kV_n \;\; \forall n \quad \ldots(1)$

Since Σv_n is divergent, the sequence $\{V_n\}$ is divergent.

$\Rightarrow$ For each positive real number k_0, however large, there exists a +ve integer m such that

$$V_n > k_0 \quad \forall n > m.$$

From (1), $\quad U_n \ge kV_n > kk_0 \quad \forall n > m$

$\Rightarrow \quad U_n > K \quad \forall n > m$, where $K = kk_0 > 0$

$\Rightarrow$ The sequence $\{U_n\}$ is divergent

$\Rightarrow$ The series Σu_n is divergent.

Test III. If Σu_n and Σv_n are series of positive terms and Σv_n is convergent and there is a positive constant k such that $u_n \le kv_n$, $\forall n > m$, then Σu_n is also convergent.

Proof. Let $\quad U_n = u_1 + u_2 + \ldots\ldots + u_n$ and $V_n = v_1 + v_2 + \ldots\ldots + v_n$

Now $\quad u_n \le kv_n \; \forall n > m$

$\Rightarrow \quad u_{m+1} \le kv_{m+1}$

$\quad\quad u_{m+2} \le kv_{m+2}$

$\quad\quad \ldots\ldots\ldots\ldots\ldots$

Adding $\quad u_{m+1} + u_{m+2} + \ldots\ldots + u_n \le k(v_{m+1} + v_{m+2} + \ldots\ldots + v_n)$

$\Rightarrow \quad U_n - U_m \le k(V_n - V_m) \quad \forall n > m$

$\Rightarrow \quad U_n \le kV_n + (U_m - kV_m) \quad \forall n > m$

$\Rightarrow \quad U_n \le kV_n + k_0 \quad \forall n > m \quad \ldots(1)$

where $k_0 = U_m - kV_m$ is a fixed number. Since Σv_n is convergent, the sequence $\{V_n\}$ is convergent and hence bounded above.

$\therefore$ From (1), the sequence $\{U_n\}$ is bounded above.

$\because$ Σu_n is a series of +ve terms.

$\{U_n\}$ is monotonically increasing.

$\because$ $\{U_n\}$ is monotonically increasing sequence and is bounded above.

$\therefore$ It is convergent.

$\Rightarrow$ Σu_n is convergent.

Test IV. If Σu_n and Σv_n are two series of positive terms and Σv_n is divergent and there is a positive constant k such that $u_n > kv_n$, $\forall n > m$, then Σu_n is also divergent.

Proof. Let $\quad U_n = u_1 + u_2 + \ldots\ldots + u_n$ and $V_n = v_1 + v_2 + \ldots\ldots + v_n$

Now $\quad u_n > kv_n \quad \forall n > m$

$\Rightarrow \quad u_{m+1} > kv_{m+1}$

$\quad\quad u_{m+2} > kv_{m+2}$

$\quad\quad \ldots\ldots\ldots\ldots\ldots$

$\quad\quad \ldots\ldots\ldots\ldots\ldots$

Adding $\quad u_{m+1} + u_{m+2} + \ldots\ldots + u_n > k(v_{m+1} + v_{m+2} + \ldots\ldots + v_n)$

$\Rightarrow \quad U_n - U_m > k(V_n - V_m) \quad \forall n > m$

$\Rightarrow \quad U_n > kV_n + (U_m - kV_m) \quad \forall n > m$

$\Rightarrow \quad U_n > kV_n + k_0 \quad \forall n > m \quad \ldots(1)$

where $k_0 = U_m - kV_m$ is a fixed number.

Since Σv_n is divergent, the sequence $\{V_n\}$ is divergent.

$\Rightarrow$ For each positive real number k_1, however large, there exists a +ve integer m' such that

$$V_n > k_1 \;\; \forall n > m'$$

Let $\quad m^* = \max\{m, m'\}$, then $V_n > k_1 \;\; \forall n > m^*$

From (1), $\quad U_n > kk_1 + k_0 = K \quad \forall n > m^*$

$\Rightarrow$ $\{U_n\}$ is divergent

$\Rightarrow$ Σu_n is divergent.

Test V. If Σu_n and Σv_n are two positive term series and there exist two positive constants H and K (independent of n) and a positive integer m such that $H < \dfrac{u_n}{v_n} < K$ $\forall n > m$, then the two series Σu_n, and Σv_n converge or diverge together.

Proof. Since Σv_n is a series of +ve terms, $v_n > 0, \;\; \forall n$

$$\therefore \quad H < \frac{u_n}{v_n} < K \quad \forall n \ge m$$

$\Rightarrow \quad Hv_n < u_n < Kv_n \quad \forall n > m \quad \ldots(1)$

Case I. When Σv_n is convergent

From (1), $\quad u_n < Kv_n \quad \forall n > m$

and Σv_n is convergent. $\Rightarrow$ Σu_n is convergent. [See Test III]

Case II. When Σv_n is divergent

From (1), $\quad u_n > Hv_n \quad \forall n > m$ and Σv_n is divergent.

$\Rightarrow$ Σu_n is divergent. [See Test IV]

Case III. When Σu_n is convergent

From (1), $\quad Hv_n > u_n \quad \forall n > m$

$\Rightarrow \quad v_n < \frac{1}{H} u_n \quad \forall\, n > m \qquad (\because\ H > 0)$

Since $\sum u_n$ is convergent $\therefore$ $\sum v_n$ is convergent. [See Test III]

Case IV. When Σu_n is divergent

From (1), $\quad Kv_n > u_n \quad \forall\, n > m$

$\Rightarrow \quad v_n > \frac{1}{K} u_n \quad \forall\, n > m \qquad (\because\ K > 0)$

Since $\sum u_n$ is divergent $\therefore$ $\sum v_n$ is divergent. [See Test IV]

Particular Case of Test V. (When m = 0).

If $\sum u_n$ and $\sum v_n$ are two positive term series and there exist two positive constants H and K (independent of n) such that $H < \frac{u_n}{v_n} < K \ \forall n$,

then the two series $\sum u_n$ and $\sum v_n$ converge or diverge together.

Note. Prove the particular case of Test V independently yourself by using Test I and Test II.

Test VI. Let Σu_n and Σv_n be two positive term series.

(i) If $\lim\limits_{n \to \infty} \frac{u_n}{v_n} = l$ (finite and non-zero), then Σu_n and Σv_n both converge or diverge together.

(ii) If $\mathop{\text{Lt}}\limits_{n \to \infty} \frac{u_n}{v_n} = 0$ and Σv_n converges, then Σu_n also converges.

(iii) If $\mathop{\text{Lt}}\limits_{n \to \infty} \frac{u_n}{v_n} = \infty$ and Σv_n diverges, the Σu_n also diverges.

Proof. (*i*) Since $\quad u_n > 0, v_n > 0 \qquad \therefore \frac{u_n}{v_n} > 0$

$\therefore \quad \mathop{\text{Lt}}\limits_{n \to \infty} \frac{u_n}{v_n} \geq 0$

But $\quad \mathop{\text{Lt}}\limits_{n \to \infty} \frac{u_n}{v_n} = l \neq 0 \qquad \therefore \quad l > 0$

Now $\quad \mathop{\text{Lt}}\limits_{n \to \infty} \frac{u_n}{v_n} = l$

$\Rightarrow$ given $\varepsilon > 0$, $\exists$ a +ve integer m such that

$$\left| \frac{u_n}{v_n} - l \right| < \varepsilon \qquad \forall\, n > m$$

$\Rightarrow \quad l - \varepsilon < \frac{u_n}{v_n} < l + \varepsilon \qquad \forall\, n > m$

$\Rightarrow \quad (l - \varepsilon)\, v_n < u_n < (l + \varepsilon)\, v_n \quad \forall\, n > m \qquad (\because\ v_n > 0)$

Choose $\varepsilon > 0$ such that $l - \varepsilon > 0$.

Let $\quad l - \varepsilon = H$, $l + \varepsilon = K$ where H, K are > 0

$\therefore \quad Hv_n < u_n < Kv_n \ \ \forall\, n > m \qquad ...(1)$

Case I. When Σu_n is convergent

From (1), $\quad Hv_n < u_n \ \forall\, n > m$

$\Rightarrow \quad v_n < \frac{1}{H} u_n \quad \forall\, n > m \qquad (\because\ H > 0)$

Since $\sum u_n$ is convergent, $\sum v_n$ is also convergent.

Case II. When Σu_n is divergent.

From (1), $\quad Kv_n > u_n \quad \forall\, n > m$

$\Rightarrow \quad v_n > \frac{1}{K} u_n \quad \forall\, n > m \qquad (\because\ K > 0)$

Since $\sum u_n$ is divergent, $\sum v_n$ is also divergent.

Case III. When Σv_n is convergent.

From (1), $\quad u_n < Kv_n \quad \forall\, n > m$

Since $\sum v_n$ is convergent, $\sum u_n$ is also convergent.

Case IV. When Σv_n is divergent.

From (1), $\quad u_n > Hv_n \quad \forall\, n > m$

Since $\sum v_n$ is divergent, $\sum u_n$ is also divergent.

Hence $\sum u_n$ and $\sum v_n$ converge or diverge together.

(*ii*) Here $\quad \mathop{\text{Lt}}\limits_{n \to \infty} \frac{u_n}{v_n} = 0$

$\therefore$ Given $\varepsilon > 0$, $\exists$ a +ve integer m such that

$$\left| \frac{u_n}{v_n} - 0 \right| < \varepsilon \qquad \forall\, n > m$$

$\Rightarrow \quad -\varepsilon < \frac{u_n}{v_n} < \varepsilon \qquad \forall\, n > m$

$\Rightarrow \quad u_n < \varepsilon\, v_n \qquad \forall\, n > m \qquad (\because\ v_n > 0)$

Since $\sum v_n$ is convergent, $\sum u_n$ is also convergent.

(*iii*) Here $\quad \mathop{\text{Lt}}\limits_{n \to \infty} \frac{u_n}{v_n} = \infty$

$\therefore$ Given M > 0, however large, $\exists$ a +ve integer m such that

$$\frac{u_n}{v_n} > M \qquad \forall\, n > m$$

$\Rightarrow \quad u_n > Mv_n \qquad \forall\, n > m$

Since $\sum v_n$ is divergent, $\sum u_n$ is also divergent.

Very Important Note. In practice, we generally apply Test VI (*i*) only and that is why Test VI (*i*) is usually called "Comparison Test". It is very useful and will be very frequently applied. Hence, its statement should be carefully mastered.

Test VII. Let Σu_n and Σv_n be two positive term series.

(*i*) If $\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}}$ $\forall$ n > m and Σv_n is convergent, then Σu_n is also convergent.

(*ii*) If $\frac{u_n}{u_{n+1}} < \frac{v_n}{v_{n+1}}$ $\forall$ n > m and Σv_n is divergent, then Σu_n is also divergent.

Proof. (*i*) $\quad \frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}} \qquad \forall\, n > m$

Putting $n = m+1, m+2, \ldots\ldots, n-1$, we have

$$\frac{u_{m+1}}{u_{m+2}} > \frac{v_{m+1}}{v_{m+2}}$$
$$\frac{u_{m+2}}{u_{m+3}} > \frac{v_{m+2}}{v_{m+3}}$$
$$\frac{u_{m+3}}{u_{m+4}} > \frac{v_{m+3}}{v_{m+4}}$$
$$\cdots\cdots\cdots$$
$$\cdots\cdots\cdots$$
$$\frac{u_{n-1}}{u_n} > \frac{v_{n-1}}{v_n}$$

Multiplying the corresponding sides of the above inequalitites, we have

$$\frac{u_{m+1}}{u_n} > \frac{v_{m+1}}{v_n} \quad \forall\, n > m$$

$$\Rightarrow \quad u_n < \left(\frac{u_{m+1}}{v_{m+1}}\right) v_n \quad \forall\, n > m$$

$$\Rightarrow \quad u_n < k v_n \quad \forall\, n > m,$$

where $k = \dfrac{u_{m+1}}{v_{m+1}}$ is a fixed +ve quantity.

Since Σv_n is convergent, so is Σu_n.

(*ii*) Using $\quad \dfrac{u_n}{u_{n+1}} < \dfrac{v_n}{v_{n+1}} \quad \forall\, n > m$

and proceeding as in part (*i*), we have

$$\frac{u_{m+1}}{u_n} < \frac{v_{m+1}}{v_n} \quad \forall\, n > m$$

$$\Rightarrow \quad u_n > \left(\frac{u_{m+1}}{v_{m+1}}\right) v_n \quad \forall\, n < m$$

$$\Rightarrow \quad u_n > k v_n \quad \forall\, n > m,$$

where $k = \dfrac{u_{m+1}}{v_{m+1}}$ is a fixed +ve quantity.

Since Σv_n is divergent, so is Σu_n.

Article 11. An Important Test for Comparison $\sum \frac{1}{n^p}$. [Hyper harmonic series or p-series]

The series $\sum \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots\ldots + \frac{1}{n^p} + \ldots\ldots$ converges if p > 1 and diverges if p ≤ 1.

Proof. Let S_n be the nth partial sum of $\sum \frac{1}{n^p}$.

Case I. When p > 1

$$\frac{1}{1^p} = 1 \qquad \ldots\text{(one term)}$$

$$\frac{1}{2^p} + \frac{1}{3^p} < \frac{1}{2^p} + \frac{1}{2^p} = \frac{2}{2^p} = \frac{1}{2^{p-1}} \qquad \ldots\text{(2 terms)}$$

$$\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} < \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} = \frac{4}{4^p} = \frac{1}{(2^{p-1})^2} \qquad \ldots(2^2 \text{ terms})$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{1}{(2^n)^p} + \frac{1}{(2^n+1)^p} + \ldots\ldots + \frac{1}{(2^{n+1}-1)^p} < \frac{1}{(2^{p-1})^n} \qquad \ldots(2^n \text{ terms})$$

Adding the above inequalities, we have

$$S_{2^{n+1}-1} < 1 + \frac{1}{2^{p-1}} + \frac{1}{(2^{p-1})^2} + \ldots\ldots + \frac{1}{(2^{p-1})^n}$$

$$\left(\text{a G.P. of } n+1 \text{ terms with common ratio } \frac{1}{2^{p-1}}\right)$$

$$= \frac{1 - \left(\frac{1}{2^{p-1}}\right)^{n+1}}{1 - \frac{1}{2^{p-1}}} = \frac{1}{1 - \frac{1}{2^{p-1}}} - \frac{\left(\frac{1}{2^{p-1}}\right)^{n+1}}{1 - \frac{1}{2^{p-1}}} < \frac{1}{1 - \frac{1}{2^{p-1}}} \qquad \left[\because \frac{\left(\frac{1}{2^{p-1}}\right)^{n+1}}{1 - \frac{1}{2^{p-1}}} > 0\right]$$

$$\Rightarrow \quad S_{2^{n+1}-1} < \frac{1}{1 - \frac{1}{2^{p-1}}} = K \text{ (say)}, \ \forall\, n \in N.$$

Since the series $\sum \frac{1}{n^p}$ is a series of positive terms, the sequence $< S_n >$ of its partial sums is an increasing sequence.

Now $\quad 2^{n+1} - 1 > 2^n > n \quad \forall\, n \in N$

$\Rightarrow \quad S_n < S_{2^n} < S_{2^{n+1}-1} < K \quad \forall\, n$

$\Rightarrow \quad S_n < K \quad \forall\, n$

$\Rightarrow$ $< S_n >$ is bounded above.

$\therefore$ $< S_n >$ converges and hence $\sum \frac{1}{n^p}$ converges.

Case II. When $p = 1$.

$$\sum \frac{1}{n^p} = \sum \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\ldots$$

$$1 + \frac{1}{2} > \frac{1}{2}$$

$$\frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$$

$$\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}>\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{4}{8}=\frac{1}{2}$$

....

....

$$\frac{1}{2^{m-1}+1}+\frac{1}{2^{m-1}+2}+\ldots\ldots+\frac{1}{2^m}>\frac{2^m-2^{m-1}}{2^m}=\frac{2^{m-1}(2-1)}{2^m}=\frac{1}{2}$$

Adding the above inequalities, we have $S_{2^m} > \frac{m}{2} \quad \forall\, m \in N$.

We shall now show that $< S_n >$ is not bounded above.

If K is any number, however large, we can always choose a positive integer m such that $\frac{m}{2} > K$.

With such a choice of m, $S_{2^m} > K$

Whenever $n \geq 2^m$, $\quad S_n > S_{2^m} > K \quad$ [$\because$ $< S_n >$ is an increasing sequence]

Thus the sequence $< S_n >$ of partial sums of the given series is not bounded above.

$\Rightarrow$ $< S_n >$ diverges to ∞ and hence $\sum \frac{1}{n^p}$ diverges to ∞.

Case III. When p < 1

$$p<1 \Rightarrow n^p<n \Rightarrow \frac{1}{n^p}>\frac{1}{n} \quad \forall\, n$$

But the series $\sum \frac{1}{n}$ is divergent (Case II).

Hence $\sum \frac{1}{n^p}$ is also divergent.

☞**Note.** The above test is very useful and will be frequently applied for comparison. Hence, it must be mastered.

ILLUSTRATIVE EXAMPLES

Example 1. *Examine the convergence of the series :*

(*i*) $\frac{3}{5}+\frac{4}{5^2}+\frac{3}{5^3}+\frac{4}{5^4}+\ldots\ldots$ *to* ∞ $\quad$ (*ii*) $\frac{1}{2}+\frac{1}{3^2}+\frac{1}{2^3}+\frac{1}{3^4}+\ldots\ldots$ *to* ∞

(*iii*) $a+b+a^2+b^2+a^3+b^3+\ldots\ldots$ *to* ∞ $\quad$ (*iv*) $1+\frac{1}{4^{2/3}}+\frac{1}{9^{2/3}}+\frac{1}{16^{2/3}}+\ldots\ldots$ *to* ∞.

Sol. (*i*) $\frac{3}{5}+\frac{4}{5^2}+\frac{3}{5^3}+\frac{4}{5^4}+\ldots\ldots$ to ∞

$$=\left(\frac{3}{5}+\frac{3}{5^3}+\ldots\ldots\text{ to }\infty\right)+\left(\frac{4}{5^2}+\frac{4}{5^4}+\ldots\ldots\text{ to }\infty\right)=\Sigma u_n+\Sigma v_n \text{ (say)}$$

Now Σu_n is a G.P. with common ratio $=\frac{1}{5^2}$ which is numerically less than 1,

$\therefore$ Σu_n is convergent.

Σv_n is also a G.P. with common ratio $=\frac{1}{5^2}$ which is numerically less than 1,

$\therefore$ Σv_n is convergent.

$\therefore$ The given series *viz.* $\Sigma(u_n+v_n)$ is also convergent.

(*ii*) *Please try yourself.* **[Ans. Convergent]**

(*iii*) $a+b+a^2+b^2+a^3+b^3+\ldots\ldots$ to ∞

$$=(a+a^2+a^3+\ldots\ldots\text{ to }\infty)+(b+b^2+b^3+\ldots\ldots\text{ to }\infty)=\Sigma u_n+\Sigma v_n\text{ (say)}$$

Now Σu_n is a G.P. with common ratio $= a$ and $\therefore$ converges only when $|a|<1$. Similarly, Σv_n converges only when $|b|<1$.

$\therefore$ The given series *viz.* $\Sigma(u_n+v_n)$ converges only when both $|a|$ and $|b|$ are <1.

(*iv*) $1+\frac{1}{4^{2/3}}+\frac{1}{9^{2/3}}+\frac{1}{16^{2/3}}+\ldots\ldots$ to ∞

$$=1+\frac{1}{(2^2)^{2/3}}+\frac{1}{(3^2)^{2/3}}+\frac{1}{(4^2)^{2/3}}+\ldots\ldots\text{ to }\infty$$

$$=\frac{1}{1^{4/3}}+\frac{1}{2^{4/3}}+\frac{1}{3^{4/3}}+\frac{1}{4^{4/3}}+\ldots\ldots\text{ to }\infty=\sum\frac{1}{n^{4/3}}=\sum\frac{1}{n^p}\text{ with }p=\frac{4}{3}>1$$

$\therefore$ By p-series test, the given series is convergent.

Example 2. (*a*) *Discuss the convergence or divergence of the following series :*

(*i*) $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots\ldots$ $\quad$ (*ii*) $\frac{1}{1.4}+\frac{1}{2.5}+\frac{1}{3.6}+\ldots\ldots$

(*iii*) $\frac{1}{1.2.3}+\frac{3}{2.3.4}+\frac{5}{3.4.5}+\ldots\ldots$ $\quad$ (*iv*) $\frac{1}{1.2}+\frac{2}{3.4}+\frac{3}{5.6}+\ldots\ldots$

(*v*) $\frac{1}{1.3}+\frac{2}{3.5}+\frac{3}{5.7}+\ldots\ldots$

Sol. (*i*) Here $\quad u_n=\frac{1}{n(n+1)}=\frac{1}{n^2\left(1+\frac{1}{n}\right)}$

Let us compare Σu_n with Σv_n where $v_n=\frac{1}{n^2}$

$$\frac{u_n}{v_n}=\frac{1}{1+\frac{1}{n}}$$

$\therefore$ $\quad \underset{n\to\infty}{\text{Lt}}\frac{u_n}{v_n}=\underset{n\to\infty}{\text{Lt}}\frac{1}{1+\frac{1}{n}}=1$ which is finite and $\neq 0$.

$\therefore$ Σu_n and Σv_n behave alike.

Since $\Sigma v_n=\Sigma\frac{1}{n^2}$ is of the form $\Sigma\frac{1}{n^p}$ with $p=2>1$

$\therefore$ Σv_n is convergent

$\Rightarrow$ Σu_n is convergent.

(*ii*) Here $\quad u_n=\frac{1}{(T_n\text{ of }1,2,3,\ldots\ldots)(T_n\text{ of }4,5,6,\ldots\ldots)}=\frac{1}{n[4+(n-1).1]}=\frac{1}{n(n+3)}$

Proceed further yourself. **[Ans. Convergent]** (*iii*) Here $u_n = \dfrac{T_n \text{ of } 1, 3, 5, \ldots\ldots}{n(n+1)(n+2)} = \dfrac{2n-1}{n(n+1)(n+2)} = \dfrac{n\left(2-\frac{1}{n}\right)}{n^3\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)} = \dfrac{2-\frac{1}{n}}{n^2\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$

Let us compare Σu_n with Σv_n where $v_n = \dfrac{1}{n^2}$

$$\frac{u_n}{v_n} = \frac{2-\frac{1}{n}}{n^2\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}} \frac{2-\frac{1}{n}}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)} = \frac{2}{(1)(1)} = 2 \text{ which is finite and } \neq 0.$$

$\therefore$ Σu_n and Σv_n behave alike.

Since $\Sigma v_n = \Sigma \dfrac{1}{n^2}$ is of the form $\Sigma \dfrac{1}{n^p}$ with $p = 2 > 1$.

$\therefore$ Σv_n is convergent $\Rightarrow$ Σu_n is convergent.

(*iv*) *Please try yourself.* **[Ans. Divergent]**

(*v*) *Please try yourself.* **[Ans. Divergent]**

Example 2. (*b*) *Test the following series for convergence :*

(*i*) $\dfrac{1}{2} + \dfrac{2}{3} + \dfrac{3}{4} + \ldots\ldots$ (*ii*) $\dfrac{1}{3.7} + \dfrac{1}{4.9} + \dfrac{1}{5.11} + \dfrac{1}{6.13} + \ldots\ldots$

(*iii*) $\dfrac{1}{3^2} \cdot \dfrac{2}{4^2} + \dfrac{3}{5^2} \cdot \dfrac{4}{6^2} + \dfrac{5}{7^2} \cdot \dfrac{6}{8^2} + \ldots\ldots$

Sol. (*i*) Here $u_n = \dfrac{n}{n+1} = \dfrac{1}{1+\frac{1}{n}}$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1 \neq 0$$

$\Rightarrow$ $\sum u_n$ does not converge.

The given series is a series of +ve terms, it either converges or diverges. Since it does not converge, it must diverge.

Hence the given series is divergent.

(*ii*) Here $u_n = \dfrac{1}{(n\text{th term of } 3, 4, 5, \ldots\ldots)(n\text{th term of } 7, 9, 11, \ldots\ldots)}$

$$= \frac{1}{(n+2)(2n+5)} = \frac{1}{n\left(1+\frac{2}{n}\right).n\left(2+\frac{5}{n}\right)} = \frac{1}{n^2\left(1+\frac{2}{n}\right)\left(2+\frac{5}{n}\right)}$$

Let us compare $\sum u_n$ with $\sum v_n$ where $v_n = \dfrac{1}{n^2}$.

$$\frac{u_n}{v_n} = \frac{1}{\left(1+\frac{2}{n}\right)\left(2+\frac{5}{n}\right)}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{2}{n}\right)\left(2+\frac{5}{n}\right)} = \frac{1}{1\times 2} = \frac{1}{2} \text{ which is non-zero and finite.}$$

$\therefore$ $\sum u_n$ and $\sum v_n$ behave alike.

Since $\sum v_n = \sum \dfrac{1}{n^2}$ is of the form $\sum \dfrac{1}{n^p}$ with $p = 2 > 1$

$\therefore$ $\sum v_n$ is convergent.

Hence $\sum u_n$ is convergent.

(*iii*) Here $u_n = \dfrac{2n-1}{(2n+1)^2} \cdot \dfrac{2n}{(2n+2)^2}$

$$= \frac{2n\left(1-\frac{1}{2n}\right)}{4n^2\left(1+\frac{1}{2n}\right)^2} \cdot \frac{2n}{4n^2\left(1+\frac{1}{n}\right)^2} = \frac{1-\frac{1}{2n}}{4n^2\left(1+\frac{1}{2n}\right)^2\left(1+\frac{1}{n}\right)^2}$$

Let us compare $\sum u_n$ with $\sum v_n$ where $v_n = \dfrac{1}{n^2}$.

$$\frac{u_n}{v_n} = \frac{1-\frac{1}{2n}}{4\left(1+\frac{1}{2n}\right)^2\left(1+\frac{1}{n}\right)^2}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \frac{1}{4\times 1\times 1} = \frac{1}{4}, \text{ which is non-zero and finite.}$$

$\therefore$ $\sum u_n$ and $\sum v_n$ converge or diverge together.

Since $\sum v_n = \sum \dfrac{1}{n^2}$ is of the form $\sum \dfrac{1}{n^p}$ with $p = 2 > 1$.

$\therefore$ $\sum v_n$ is convergent.

Hence $\sum u_n$ is convergent.

Example 3. *Discuss the convergence or divergence of the following series :*

(*i*) $\dfrac{1}{\sqrt{1}+\sqrt{2}} + \dfrac{1}{\sqrt{2}+\sqrt{3}} + \dfrac{1}{\sqrt{3}+\sqrt{4}} + \ldots\ldots$

(*ii*) $\sqrt{\dfrac{1}{4}} + \sqrt{\dfrac{2}{6}} + \sqrt{\dfrac{3}{8}} + \ldots\ldots + \sqrt{\dfrac{n}{2(n+1)}} + \ldots\ldots$

(*iii*) $\dfrac{1}{\sqrt{1.2}} + \dfrac{1}{\sqrt{2.3}} + \dfrac{1}{\sqrt{3.4}} + \ldots\ldots$

Sol. (*i*) Here $$u_n = \frac{1}{\sqrt{n}+\sqrt{n+1}} = \frac{1}{\sqrt{n}\left[1+\sqrt{1+\frac{1}{n}}\right]}$$

Let us compare Σu_n with Σv_n where $v_n = \frac{1}{\sqrt{n}}$

$$\frac{u_n}{v_n} = \frac{1}{1+\sqrt{1+\frac{1}{n}}}$$

$$\operatorname*{Lt}_{n\to\infty} \frac{u_n}{v_n} = \operatorname*{Lt}_{n\to\infty} \frac{1}{1+\sqrt{1+\frac{1}{n}}} = \frac{1}{1+1} = \frac{1}{2} \text{ which is finite and } \neq 0.$$

$\therefore$ Σu_n and Σv_n behave alike.

Since $\Sigma v_n = \Sigma \frac{1}{n^{1/2}}$ is of the form $\Sigma \frac{1}{n^p}$ with $p = \frac{1}{2} < 1$

$\therefore$ Σv_n is divergent $\Rightarrow$ Σu_n is divergent.

(*ii*) Here $$u_n = \sqrt{\frac{n}{2(n+1)}} = \sqrt{\frac{1}{2\left(1+\frac{1}{n}\right)}}$$

$$\operatorname*{Lt}_{n\to\infty} u_n = \operatorname*{Lt}_{n\to\infty} \frac{1}{\sqrt{2\left(1+\frac{1}{n}\right)}} = \frac{1}{\sqrt{2}} \neq 0$$

$\Rightarrow$ Σu_n does not converge.

The given series is a series of +ve terms, it either converges or diverges. Since it does not converge, it must diverge.

Hence the given series is divergent.

(*iii*) Here $$u_n = \frac{1}{\sqrt{n(n+1)}} = \frac{1}{\sqrt{n^2\left(1+\frac{1}{n}\right)}} = \frac{1}{n\sqrt{1+\frac{1}{n}}}.$$

$\left[\textbf{Hint.} \text{ Take } v_n = \frac{1}{n} \text{ then } p = 1\right]$ **[Ans. Divergent]**

Example 4. *Discuss the convergence or divergence of the following series :*

(*i*) $\frac{1}{a.1^2+b} + \frac{1}{a.2^2+b} + \frac{1}{a.3^2+b} + \ldots\ldots$ (*ii*) $\frac{1}{1^p} + \frac{1}{3^p} + \frac{1}{5^p} + \frac{1}{7^p} + \ldots\ldots$

(*iii*) $\frac{2}{1^p} + \frac{3}{2^p} + \frac{4}{3^p} + \frac{5}{4^p} + \ldots\ldots$

Sol. (*i*) Here $$u_n = \frac{1}{an^2+b} = \frac{1}{n^2\left(a+\frac{b}{n^2}\right)}$$

Take $$v_n = \frac{1}{n^2}$$

$$\operatorname*{Lt}_{n\to\infty} \frac{u_n}{v_n} = \operatorname*{Lt}_{n\to\infty} \frac{1}{a+\frac{b}{n^2}} = \frac{1}{a} \text{ which is finite and } \neq 0$$

$\therefore$ Σu_n and Σv_n behave alike.

Since $\Sigma v_n = \Sigma \frac{1}{n^2}$ is of the form $\Sigma \frac{1}{n^p}$ with $p = 2 > 1$

$\therefore$ Σv_n is convergent $\Rightarrow$ Σu_n is convergent.

(*ii*) Here $$u_n = \frac{1}{(2n-1)^p} = \frac{1}{n^p\left(2-\frac{1}{n}\right)^p}$$

Take $$v_n = \frac{1}{n^p}.$$

$$\operatorname*{Lt}_{n\to\infty} \frac{u_n}{v_n} = \operatorname*{Lt}_{n\to\infty} \frac{1}{\left(2-\frac{1}{n}\right)^p} = \frac{1}{2^p} \text{ which is finite and } \neq 0$$

$\therefore$ Σu_n and Σv_n behave alike.

Now $\Sigma v_n = \Sigma \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

$\Rightarrow$ Σu_n converges if $p > 1$ and diverges if $p \leq 1$.

(*iii*) Here $$u_n = \frac{n+1}{n^p} = \frac{n\left(1+\frac{1}{n}\right)}{n^p} = \frac{1+\frac{1}{n}}{n^{p-1}}$$

Take $$v_n = \frac{1}{n^{p-1}}.$$

$$\operatorname*{Lt}_{n\to\infty} \frac{u_n}{v_n} = \operatorname*{Lt}_{n\to\infty} \left(1+\frac{1}{n}\right) = 1 \text{ which is finite and } \neq 0.$$

$\therefore$ Σu_n and Σv_n behave alike.

Now $\Sigma v_n = \Sigma \frac{1}{n^{p-1}}$ converges if $p-1 > 1$ and diverges if $p-1 \leq 1$

i.e., converges if $p > 2$ and diverges if $p \leq 2$.

$\Rightarrow$ Σu_n converges if $p > 2$ and diverges if $p \leq 2$.

Example 5. *Test the convergence of the following series :*

(*i*) $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ (*ii*) $\Sigma\left(n+\frac{1}{2}\right)^{-2}$ (*iii*) $\sum_{k=1}^{\infty} \frac{k+1}{k(2k-1)}$

(*iv*) $\sum_{k=1}^{\infty} \frac{k+3}{k^3-k+1}$ (*v*) $\sum \frac{n^2+n+1}{n^4+1}$.

Sol. Hints.

(*i*) $$u_n = \frac{1}{2n-1} = \frac{1}{n\left(2-\frac{1}{n}\right)}.$$

Take $v_n = \dfrac{1}{n}$, $p = 1$. [Ans. Divergent]

(ii) $u_n = \dfrac{1}{\left(n+\frac{1}{2}\right)^2} = \dfrac{1}{n^2\left(1+\frac{1}{2n}\right)^2}$

Take $v_n = \dfrac{1}{n^2}$, $p = 2$. [Ans. Convergent]

(iii) Here $u_k = \dfrac{k+1}{k(2k-1)}$ changing k to n

$$u_n = \frac{n+1}{n(2n-1)} = \frac{n\left(1+\frac{1}{n}\right)}{n^2\left(2-\frac{1}{n}\right)} = \frac{1+\frac{1}{n}}{n\left(2-\frac{1}{n}\right)}$$

Take $v_n = \dfrac{1}{n}$, $p = 1$. [Ans. Divergent]

(iv) *Please try yourself.* [Ans. Convergent]

(v) $u_n = \dfrac{n^2\left(1+\frac{1}{n}+\frac{1}{n^2}\right)}{n^4\left(1+\frac{1}{n^4}\right)}$

Take $v_n = \dfrac{1}{n^2}$. [Ans. Convergent]

Example 6. *Examine the convergence of the series :*

(i) $\sum \dfrac{n}{n+1}$ (ii) $\sum\limits_{1}^{\infty} \dfrac{1}{n(n+1)}$ (iii) $\sum \dfrac{1}{(an+b)^p}$

(iv) $\sum \dfrac{1}{n^p(n+1)^p}$ (v) $\dfrac{2^p}{1^q}+\dfrac{3^p}{2^q}+\dfrac{4^p}{3^q}+\ldots\ldots$, *p and q being +ve numbers.*

(vi) $1+\dfrac{1}{2^2}+\dfrac{2^2}{3^3}+\dfrac{3^3}{4^4}+\dfrac{4^4}{5^5}+\ldots\ldots$

Sol. (i) $u_n = \dfrac{n}{n+1} = \dfrac{1}{1+\frac{1}{n}}$

$$\operatorname*{Lt}_{n\to\infty} u_n = \operatorname*{Lt}_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1 \neq 0.$$

$\Rightarrow$ Σu_n cannot converge. Being a series of +ve terms, it either converges or diverges. Since it cannot converge, it must diverge.

Hence Σu_n is divergent.

(ii) *Please try yourself.* [Ans. Convergent]

(iii) $u_n = \dfrac{1}{(an+b)^p} = \dfrac{1}{n^p\left(a+\frac{b}{n}\right)^p}$

Take $v_n = \dfrac{1}{n^p}$ etc. [Ans. Convergent if $p > 1$ and divergent if $p \le 1$]

(iv) $u_n = \dfrac{1}{n^p(n+1)^p} = \dfrac{1}{n^p.n^p\left(1+\frac{1}{n}\right)^p} = \dfrac{1}{n^{2p}\left(1+\frac{1}{n}\right)^p}$

Take $v_n = \dfrac{1}{n^{2p}}$ etc. [Ans. Convergent if $p > \frac{1}{2}$ and divergent if $p \le \frac{1}{2}$.]

(v) $u_n = \dfrac{(n+1)^p}{n^q} = \dfrac{n^p\left(1+\frac{1}{n}\right)^p}{n^q} = \dfrac{\left(1+\frac{1}{n}\right)^p}{n^{q-p}}$

Taking $v_n = \dfrac{1}{n^{q-p}}$ etc. [Ans. Convergent if $q-p > 1$ and divergent if $q - p \le 1$

$\Rightarrow$ Convergent if $q > p+1$ and divergent if $q \le p+1$.]

(vi) *Leaving aside the first term* ($\because$ *addition or deletion of a finite number of terms does not alter the nature of the series*), we have

$$u_n = \frac{n^n}{(n+1)^{n+1}} = \frac{n^n}{n^{n+1}\left(1+\frac{1}{n}\right)^{n+1}} = \frac{1}{n\left(1+\frac{1}{n}\right)^{n+1}}$$

Take $v_n = \dfrac{1}{n}$.

$$\operatorname*{Lt}_{n\to\infty}\frac{u_n}{v_n} = \operatorname*{Lt}_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^{n+1}} = \operatorname*{Lt}_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n}\cdot\frac{1}{\left(1+\frac{1}{n}\right)}$$

$$= \frac{1}{e}\cdot\frac{1}{1} \qquad \left[\because \operatorname*{Lt}_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e\right]$$

$= \dfrac{1}{e}$ which is finite and $\neq 0$

$\therefore$ Σu_n and Σv_n behave alike.

Since $\Sigma v_n = \Sigma \dfrac{1}{n}$ is of the form $\Sigma \dfrac{1}{n^p}$ with $p = 1$

$\therefore$ Σv_n is divergent. $\Rightarrow$ Σu_n is divergent.

Example 7. *Examine the convergence of the series :*

(i) $\sum\limits_{n=1}^{\infty}\dfrac{\sqrt{n^2-1}}{n^3+1}$ (ii) $\sum\limits_{n=1}^{\infty}\dfrac{\sqrt{n^2-1}}{2n+1}$ (iii) $\Sigma\dfrac{\sqrt{n}}{n^2+1}$

(iv) $\sum\sqrt{\dfrac{n}{2+3n^3}}$ (v) $\dfrac{\sqrt{2}-1}{3^3-1}+\dfrac{\sqrt{3}-1}{4^3-1}+\dfrac{\sqrt{4}-1}{5^3-1}+\ldots\ldots$

Sol. (*i*) Here $$u_n = \frac{\sqrt{n^2-1}}{n^3-1} = \frac{n\sqrt{1-\frac{1}{n^2}}}{n^3\left(1+\frac{1}{n^3}\right)} = \frac{\sqrt{1-\frac{1}{n^2}}}{n^2\left(1+\frac{1}{n^3}\right)}$$

Take $$v_n = \frac{1}{n^2}$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}} \frac{\sqrt{1-\frac{1}{n^2}}}{1+\frac{1}{n^3}} = \frac{\sqrt{1-0}}{1+0} = 1 \text{ which is finite and} \neq 0$$

$\therefore$ Σu_n and Σv_n behave alike.

Since $\Sigma v_n = \Sigma \frac{1}{n^2}$ is of the form $\Sigma \frac{1}{n^p}$ with $p = 2 > 1$, it converges. $\Rightarrow$ Σu_n is convergent.

(*ii*) **(Hint)** $$u_n = \frac{\sqrt{n^2-1}}{2n+1} = \frac{n\sqrt{1-\frac{1}{n^2}}}{n\left(2+\frac{1}{n}\right)} = \frac{\sqrt{1-\frac{1}{n^2}}}{2+\frac{1}{n}}$$

$$\underset{n\to\infty}{\text{Lt}}\, u_n = \frac{1}{2} \neq 0.$$ **[Ans. Divergent]**

(*iii*) **(Hint)** $$u_n = \frac{\sqrt{n}}{n^2+1} = \frac{n^{1/2}}{n^2\left(1+\frac{1}{n^2}\right)} = \frac{1}{n^{3/2}\left(1+\frac{1}{n^2}\right)}$$

Take $v_n = \frac{1}{n^{3/2}}$, $p = \frac{3}{2} > 1$ etc. **[Ans. Convergent]**

(*iv*) $$u_n = \sqrt{\frac{n}{2+3n^3}} = \frac{n}{\sqrt{n^3\left(3+\frac{2}{n^3}\right)}} = \frac{1}{n\sqrt{3+\frac{2}{n^3}}}$$

Take $v_n = \frac{1}{n}$ etc. $p = 1$. **[Ans. Divergent]**

(*v*) Here $$u_n = \frac{\sqrt{n+1}-1}{(n+2)^3-1} = \frac{\sqrt{n}\left(\sqrt{1+\frac{1}{n}}-\frac{1}{\sqrt{n}}\right)}{n^3\left[\left(1+\frac{2}{n}\right)^3-\frac{1}{n^3}\right]} = \frac{\sqrt{1+\frac{1}{n}}-\frac{1}{\sqrt{n}}}{n^{5/2}\left[\left(1+\frac{2}{n}\right)^3-\frac{1}{n^3}\right]}$$

Take $$v_n = \frac{1}{n^{5/2}}.$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}} \frac{\sqrt{1+\frac{1}{n}}-\frac{1}{\sqrt{n}}}{\left(1+\frac{2}{n}\right)^3-\frac{1}{n^3}} = \frac{\sqrt{1+0}-0}{(1+0)^3-0} = 1 \text{ which is finite and} \neq 0.$$

$\therefore$ Σu_n and Σv_n behave alike.

Since $\Sigma v_n = \Sigma \frac{1}{n^{5/2}}$ is of the form $\Sigma \frac{1}{n^p}$ with $p = \frac{5}{2} > 1$.

Σv_n is convergent. $\Rightarrow$ Σu_n is convergent.

Example 8. *Test the convergence of the series :*

(*i*) $\frac{1}{5}+\frac{\sqrt{2}}{7}+\frac{\sqrt{3}}{9}+\frac{\sqrt{4}}{11}+\ldots\ldots$ (*ii*) $\frac{2^2}{3^2}+\frac{2^2-4^2}{3^2-5^5}+\frac{2^2-4^2-6^2}{3^2-5^2-7^2}+\ldots\ldots$

Sol. (*i*) Here, numerator of each term is the square root of the corresponding natural number.

$\therefore$ Numerator of $u_n = \sqrt{n}$

Denominator of u_n = nth term of 5, 7, 9, 11 ..... (*an A.P.*)

$= 5 + (n-1).2 = 2n+3$

$$\therefore \quad u_n = \frac{\sqrt{n}}{2n+3} = \frac{n^{1/2}}{n\left(2+\frac{3}{n}\right)} = \frac{1}{n^{1/2}\left(2+\frac{3}{n}\right)}$$

Take $v_n = \frac{1}{n^{1/2}}$, $p = \frac{1}{2} < 1$ etc. **[Ans. Divergent]**

(*ii*) Here $$u_n = \frac{2^2-4^2-6^2\ldots\ldots-(2n)^2}{3^2-5^2-7^2\ldots\ldots-(2n+1)^2} = \frac{2.2^2-[2^2+4^2+6^2+\ldots\ldots+(2n)^2]}{2.3^2-[3^2+5^2+7^2+\ldots\ldots+(2n+1)^2]}$$

$$= \frac{8-\sum_1^n(2n)^2}{18-\sum_1^n(2n+1)^2} = \frac{8-4\sum_1^n n^2}{18-4\sum_1^n n^2-4\sum_1^n n - n}$$

$$= \frac{8-4.\frac{n(n+1)(2n+1)}{6}}{18-4.\frac{n(n+1)(2n+1)}{6}-4.\frac{n(n+1)}{2}-n}$$

$$= \frac{24-2n(2n^2+3n+1)}{54-2n(2n^2+3n+1)-6n(n+1)-3n}$$

$$= \frac{-4n^2-6n^2-2n+24}{-4n^3-12n^2-11n+54} = \frac{n^3\left(4+\frac{6}{n}+\frac{2}{n^2}-\frac{24}{n^3}\right)}{n^3\left(4+\frac{12}{n}+\frac{11}{n^2}-\frac{54}{n^3}\right)} = \frac{4+\frac{6}{n}+\frac{2}{n^2}-\frac{24}{n^3}}{4+\frac{12}{n}+\frac{11}{n^2}-\frac{54}{n^3}}$$

$\underset{n\to\infty}{\text{Lt}}\, u_n = 1 \neq 0$ $\Rightarrow$ Σu_n is not convergent. $\Rightarrow$ Σu_n is divergent.

Example 9. *Examine the convergence of the following series :*

(*i*) $\sum_{n=1}^{\infty} (\sqrt{n^3+1} - \sqrt{n^3})$ (*ii*) $\sum (\sqrt{n^2+1} - n)$

(*iii*) $\sum (\sqrt{n^4+1} - n^2)$ (*iv*) $\sum (\sqrt{n+1} - \sqrt{n})$

(*v*) $\sum (\sqrt{n^2+1} - \sqrt{n^2-1})$.

Sol. (*i*) Here $u_n = \sqrt{n^3+1} - \sqrt{n^3}$

$$= (\sqrt{n^3+1} - \sqrt{n^3}) \times \frac{\sqrt{n^3+1}+\sqrt{n^3}}{\sqrt{n^3+1}+\sqrt{n^3}} \quad \textit{(Rationalising)}$$

$$= \frac{(n^3+1)-n^3}{\sqrt{n^3+1}+\sqrt{n^3}} = \frac{1}{\sqrt{n^3}\left(\sqrt{1+\frac{1}{n^3}}+1\right)}$$

Take $v_n = \frac{1}{\sqrt{n^3}}$.

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}} \frac{1}{\sqrt{1+\frac{1}{n^3}}+1} = \frac{1}{\sqrt{1+0}+1} = \frac{1}{2} \text{ which is finite and} \neq 0.$$

$\therefore$ Σu_n and Σv_n behave alike.

Since $\sum v_n = \sum \frac{1}{\sqrt{n^3}} = \sum \frac{1}{n^{3/2}}$ is of the form $\sum \frac{1}{n^p}$ with $p = \frac{3}{2} > 1$.

$\therefore$ Σv_n is convergent $\Rightarrow$ Σu_n is convergent.

(*ii*) Here $u_n = \sqrt{n^2+1} - n = \frac{1}{\sqrt{n^2+1}+n}$ **(Rationalising)**

$$= \frac{1}{n\left(\sqrt{1+\frac{1}{n^2}}+1\right)}$$

Take $v_n = \frac{1}{n}$ etc. **[Ans. Divergent]**

(*iii*) Here $u_n = \sqrt{n^4+1} - n^2 = \frac{1}{\sqrt{n^4+1}+n^2}$ **(Rationalising)**

$$= \frac{1}{n^2\left(\sqrt{1+\frac{1}{n^4}}+1\right)}$$

Take $v_n = \frac{1}{n^2}$ etc. **[Ans. Convergent]**

(*iv*) Here $u_n = \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1}+\sqrt{n}}$ (Rationalising)

$$= \frac{1}{\sqrt{n}\left(\sqrt{1+\frac{1}{n}}+1\right)}$$

Take $v_n = \frac{1}{\sqrt{n}}$ etc. **[Ans. Divergent]**

(*v*) *Please try yourself.* **[Ans. Divergent]**

Example 10. *Examine the convergence of the series :*

(*i*) $(\sqrt{2}-1)+(\sqrt{5}-2)+(\sqrt{10}-3)+\ldots\ldots$ (*ii*) $\sum (\sqrt{n^4+1} - \sqrt{n^4-1})$

(*iii*) $\sum \frac{\sqrt{n+1}-\sqrt{n}}{n^p}$ (*iv*) $\sum \frac{\sqrt{n+1}-\sqrt{n}}{n}$

(*v*) $\sum \frac{\sqrt{n+1}-\sqrt{n-1}}{n}$ (*vi*) $\frac{\sqrt{2}-\sqrt{1}}{1}+\frac{\sqrt{3}-\sqrt{2}}{2}+\frac{\sqrt{4}-\sqrt{3}}{3}+\ldots\ldots$

(*vii*) $\sum \left(\sqrt[3]{n^3+1}-n\right)$ (*viii*) $\sum (\sqrt[3]{n+1}-\sqrt[3]{n})$.

Sol. (*i*) Here $u_n = \sqrt{n^2+1} - n$.

Now proceed as in Ex. 9 (*ii*).

(*ii*) Here $u_n = \sqrt{n^4+1} - \sqrt{n^4-1}$

$$= \left(\sqrt{n^4+1} - \sqrt{n^4-1}\right) \times \frac{\sqrt{n^4+1}+\sqrt{n^4-1}}{\sqrt{n^4+1}+\sqrt{n^4-1}} \quad \text{(Rationalising)}$$

$$= \frac{(n^4+1)-(n^4-1)}{\sqrt{n^4+1}+\sqrt{n^4-1}} = \frac{2}{\sqrt{n^4+1}+\sqrt{n^4-1}}$$

$$= \frac{2}{n^2\left(\sqrt{1+\frac{1}{n^4}}+\sqrt{1-\frac{1}{n^4}}\right)}$$

Take $v_n = \frac{1}{n^2}$.

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}} \frac{2}{\sqrt{1+\frac{1}{n^4}}+\sqrt{1-\frac{1}{n^4}}} = \frac{2}{1+1} = 1 \text{ which is finite and} \neq 0.$$

$\therefore$ Σu_n and Σv_n behave alike.

Since $\sum v_n = \sum \frac{1}{n^2}$ is of the form $\sum \frac{1}{n^p}$ with $p = 2 > 1$.

$\therefore$ Σv_n is convergent $\Rightarrow$ Σu_n is convergent.

(*iii*) Here $u_n = \dfrac{\sqrt{n+1}-\sqrt{n}}{n^p}$

$= \dfrac{\sqrt{n+1}-\sqrt{n}}{n^p} \times \dfrac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$ (Rationalising)

$= \dfrac{(n+1)-n}{n^p \cdot \sqrt{n}\left(\sqrt{1+\frac{1}{n}}+1\right)} = \dfrac{1}{n^{p+1/2}\left(\sqrt{1+\frac{1}{n}}+1\right)}$

Take $v_n = \dfrac{1}{n^{p+1/2}}$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}} \frac{1}{\sqrt{1+\frac{1}{n}}+1} = \frac{1}{1+1} = \frac{1}{2} \text{ which is finite and } \neq 0.$$

$\therefore$ Σu_n and Σv_n behave alike.

Since $\sum v_n = \sum \dfrac{1}{n^{p+1/2}}$ is convergent if $p + \dfrac{1}{2} > 1$ and divergent if $p + \frac{1}{2} \leq 1$.

i.e., convergent if $p > \frac{1}{2}$ and divergent if $p \leq \frac{1}{2}$.

$\therefore$ Σu_n is convergent if $p > \frac{1}{2}$ and divergent if $p \leq \frac{1}{2}$.

(*iv*), (*v*), (*vi*) *Please try yourself.* [**Ans.** Convergent]

(*vii*) Here $u_n = (n^3+1)^{1/3} - n = \left[n^3\left(1+\dfrac{1}{n^3}\right)\right]^{1/3} - n$

$= n\left(1+\dfrac{1}{n^3}\right)^{1/3} - n = n\left[\left(1+\dfrac{1}{n^3}\right)^{1/3} - 1\right]$

$= n\left[1+\dfrac{1}{3}\cdot\dfrac{1}{n^3}+\dfrac{\frac{1}{3}(\frac{1}{3}-1)}{2!}\cdot\left(\dfrac{1}{n^3}\right)^2+\ldots-1\right]$

$= \dfrac{n}{n^3}\left[\dfrac{1}{3}-\dfrac{1}{9}\cdot\dfrac{1}{n^3}\ldots\ldots\right] = \dfrac{1}{n^2}\left[\dfrac{1}{3}-\dfrac{1}{9}\cdot\dfrac{1}{n^3}\ldots\ldots\right]$

Take $v_n = \dfrac{1}{n^2}$.

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}} \left(\frac{1}{3}-\frac{1}{9}\cdot\frac{1}{n^3}\ldots\ldots\right) = \frac{1}{3} \text{ which is finite and } \neq 0.$$

$\therefore$ Σu_n and Σv_n behave alike.

Since $\sum v_n = \sum \dfrac{1}{n^2}$ is of the form $\sum \dfrac{1}{n^p}$ with $p = 2 > 1$

$\therefore$ Σv_n is convergent $\Rightarrow$ Σu_n is convergent.

Note. Rationalisation is effective only when square roots are involved whereas this method of Binomial Expansion is general.

(*viii*) Here $u_n = \sqrt[3]{n+1} - \sqrt[3]{n} = (n+1)^{1/3} - n^{1/3}$

$= n^{1/3}\left(1+\dfrac{1}{n}\right)^{1/3} - n^{1/3} = n^{1/3}\left[1+\dfrac{1}{3}\cdot\dfrac{1}{n}+\dfrac{\frac{1}{3}(\frac{1}{3}-1)}{2!}\cdot\dfrac{1}{n^2}\ldots\ldots-1\right]$

$= \dfrac{n^{1/3}}{n}\left[\dfrac{1}{3}-\dfrac{1}{9}\cdot\dfrac{1}{n}\ldots\ldots\right] = \dfrac{1}{n^{2/3}}\left[\dfrac{1}{3}-\dfrac{1}{9}\cdot\dfrac{1}{n}\ldots\ldots\right]$

Take $v_n = \dfrac{1}{n^{2/3}}$.

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}} \left(\frac{1}{3}-\frac{1}{9}\cdot\frac{1}{n}\ldots\ldots\right) = \frac{1}{3} \text{ which is finite and } \neq 0.$$

$\therefore$ Σu_n and Σv_n behave alike.

Since $\sum v_n = \sum \dfrac{1}{n^{2/3}}$ is of the form $\sum \dfrac{1}{n^p}$ with $p = \dfrac{2}{3} < 1$

$\therefore$ Σv_n is divergent $\Rightarrow$ Σu_n is divergent.

Example 11. *Test the convergence of the series :*

(*i*) $\sum \sin \dfrac{1}{n}$ (*ii*) $\sum \dfrac{1}{n} \sin \dfrac{1}{n}$ (*iii*) $\sum \dfrac{1}{\sqrt{n}} \sin \dfrac{1}{n}$

(*iv*) $\sum \sin \dfrac{1}{n^2}$ (*v*) $\sum \cos \dfrac{1}{n}$ (*vi*) $\sum \tan^{-1} \dfrac{1}{n}$

(*vii*) $\sum \cot^{-1} n^2$ (*viii*) $\sum \dfrac{1}{\sqrt{n}} \tan \dfrac{1}{n}$ (*ix*) $\sum \left(1 - \cos \dfrac{\pi}{n}\right)$

Sol. (*i*) Here $u_n = \sin \dfrac{1}{n}$

$= \dfrac{1}{n} - \dfrac{1}{3!}\cdot\dfrac{1}{n^3} + \dfrac{1}{5!}\cdot\dfrac{1}{n^5}\ldots\ldots$ $\left[\because \sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!}\ldots\ldots\right]$

$= \dfrac{1}{n}\left[1 - \dfrac{1}{3!}\cdot\dfrac{1}{n^2} + \dfrac{1}{5!}\cdot\dfrac{1}{n^4}\ldots\ldots\right]$

Take $v_n = \dfrac{1}{n}$.

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}} \left[1 - \frac{1}{3!}\cdot\frac{1}{n^2} + \frac{1}{5!}\cdot\frac{1}{n^4}\ldots\ldots\right] = 1 \text{ which is finite and } \neq 0.$$

$\therefore$ Σu_n and Σv_n behave alike.

Since $\sum v_n = \sum \dfrac{1}{n}$ is of the form $\sum \dfrac{1}{n^p}$ with $p = 1$.

Σv_n is divergent $\Rightarrow$ Σu_n is divergent.

(*ii*) Here $u_n = \frac{1}{n}\sin\frac{1}{n}$

$$= \frac{1}{n}\left[\frac{1}{n} - \frac{1}{3!}\cdot\frac{1}{n^3} + \frac{1}{5!}\cdot\frac{1}{n^5} \ldots\ldots\right] = \frac{1}{n^2}\left[1 - \frac{1}{3!}\cdot\frac{1}{n^2} + \frac{1}{5!}\cdot\frac{1}{n^4} \ldots\ldots\right]$$

Take $v_n = \frac{1}{n^2}$ etc. **[Ans. Convergent]**

(*iii*) *Please try yourself.* **[Ans. Convergent]**

(*iv*) *Please try yourself.* **[Ans. Convergent]**

(*v*) Here $u_n = \cos\frac{1}{n}$

$$\underset{n\to\infty}{\text{Lt}}\ u_n = \underset{n\to\infty}{\text{Lt}}\ \cos\frac{1}{n} = \cos 0 = 1 \neq 0.$$

$\Rightarrow$ Σu_n does not converge. Being a series of +ve terms, it must, therefore, diverge.

(*vi*) Here $u_n = \tan^{-1}\frac{1}{n} = \frac{1}{n} - \frac{1}{3n^3} + \ldots\ldots \quad \left[\because\ \tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} \ldots\ldots \text{(Gregory's Series)}\right]$

$$= \frac{1}{n}\left[1 - \frac{1}{3n^2} + \ldots\ldots\right]$$

Take $v_n = \frac{1}{n}$ etc. **[Ans. Divergent]**

(*vii*) Here $u_n = \cot^{-1}n^2 = \tan^{-1}\frac{1}{n^2} \quad \left[\because\ \cot^{-1}x = \tan^{-1}\frac{1}{x}\right]$

$$= \frac{1}{n^2} - \frac{1}{3}\left(\frac{1}{n^2}\right)^3 + \ldots\ldots = \frac{1}{n^2}\left[1 - \frac{1}{3n^4} + \ldots\ldots\right]$$

Take $v_n = \frac{1}{n^2}$ etc. **[Ans. Convergent]**

(*viii*) Here $u_n = \frac{1}{\sqrt{n}}\tan\frac{1}{n}$

Take $v_n = \frac{1}{n^{3/2}}$

$$\underset{n\to\infty}{\text{Lt}}\ \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}}\ \frac{\tan\frac{1}{n}}{\frac{1}{n}} = \underset{x\to 0}{\text{Lt}}\ \frac{\tan x}{x} = 1 \text{ where } x = \frac{1}{n}$$

$$\Rightarrow \quad \underset{n\to\infty}{\text{Lt}}\ \frac{u_n}{v_n} = 1 \text{ which is finite and} \neq 0\ .$$

$\therefore$ Σu_n and Σv_n behave alike.

Since $\sum v_n = \sum \frac{1}{n^{3/2}}$ which is of the form $\sum \frac{1}{n^p}$ with $p = \frac{3}{2} > 1$.

$\therefore$ Σv_n is convergent $\Rightarrow$ Σu_n is convergent.

Note. Try part (*i*) and (*ii*) by this method using $\underset{x\to 0}{\text{Lt}}\ \frac{\sin x}{x} = 1$.

(*ix*) Here $u_n = 1 - \cos\frac{\pi}{n}$

$$= 1 - \left[1 - \frac{1}{2!}\left(\frac{\pi}{n}\right)^2 + \frac{1}{4!}\left(\frac{\pi}{n}\right)^4 - \ldots\ldots\right] \quad \left[\because\ \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\ldots\right]$$

$$= \frac{1}{n^2}\left[\frac{\pi^2}{2!} - \frac{\pi^4}{4!\,n^2} + \ldots\ldots\right]$$

Take $v_n = \frac{1}{n^2}$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\left[\frac{\pi^2}{2!} - \frac{\pi^4}{4!\,n^2} + \ldots\ldots\right] = \frac{\pi^2}{2} \text{ which is finite and} \neq 0.$$

$\therefore$ $\sum u_n$ and $\sum v_n$ behave alike.

Since $\sum v_n = \sum \frac{1}{n^2}$ is of the form $\sum \frac{1}{n^p}$ with $p = 2 > 1$

$\therefore$ $\sum v_n$ is convergent $\Rightarrow$ $\sum u_n$ is convergent

Example 12. *Test for convergence the series :*

(*i*) $\frac{1}{\log 2} + \frac{1}{\log 3} + \frac{1}{\log 4} + \ldots\ldots$ (*ii*) $\sum \frac{1}{n^{1+\frac{1}{n}}}$

Sol. (*i*) The given series is $\sum_{n=2}^{\infty}\frac{1}{\log n}$

Since $\log n < n \quad \forall n > 1$

$\therefore$ $\frac{1}{\log n} > \frac{1}{n} \quad \forall\, n \geq 2.$

Since each term of the given series $\sum_{n=2}^{\infty}\frac{1}{\log n}$ is greater than the corresponding term of the series $\sum_{n=2}^{\infty}\frac{1}{n}$ which is divergent, therefore, the given series is divergent.

(*ii*) Here $u_n = \frac{1}{n^{1+\frac{1}{n}}} = \frac{1}{n.n^{1/n}}$

Take $v_n = \dfrac{1}{n}$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}} = \frac{1}{n^{1/n}} = \frac{1}{1} = 1 \text{ which is finite and} \neq 0.$$

$\therefore$ By comparison test, Σu_n and Σv_n behave alike. Since $\Sigma v_n = \Sigma \dfrac{1}{n}$ is divergent

$(\because \ p = 1)$.

$\therefore$ Σu_n is also divergent.

Example 13. *Test the convergence of the series* $\sum \dfrac{x^{n-1}}{1+x^n}$, $x > 0$.

Or

Test the convergence of the series $\dfrac{1}{1+x} + \dfrac{x}{1+x^2} + \dfrac{x^2}{1+x^3} + \ldots\ldots$

Sol. Here $u_n = \dfrac{x^{n-1}}{1+x^n}$.

Case I. When $0 < x < 1$, $x^n \to 0$ as $n \to \infty$.

Take $v_n = x^{n-1}$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}} = \frac{1}{1+x^n} = \frac{1}{1+0} = 1 \text{ which is finite and} \neq 0.$$

$\therefore$ Σu_n and Σv_n either both converge or both diverge.

But $\Sigma v_n = \Sigma x^{n-1}$ is a G.P. with common ratio x.

Also $0 < x < 1$.

$\therefore$ Σv_n is convergent and hence Σu_n is convergent.

Case II. When $x = 1$

$$u_n = \frac{1}{1+1} = \frac{1}{2}$$

$$\underset{n\to\infty}{\text{Lt}} u_n = \frac{1}{2} \neq 0.$$

$\Rightarrow$ Σu_n cannot converge and hence must diverge ($\because$ a series of +ve terms either converges or diverges).

Case III. When $x > 1$, $0 < \dfrac{1}{x} < 1$ $\therefore$ $\dfrac{1}{x^n} \to 0$ as $n \to \infty$

$$u_n = \frac{x^{n-1}}{1+x^n} = \frac{1}{x} \cdot \frac{x^n}{1+x^n} = \frac{1}{x} \cdot \frac{1}{\frac{1}{x^n}+1}$$

$$\underset{n\to\infty}{\text{Lt}} u_n = \frac{1}{x} \cdot \frac{1}{0+1} = \frac{1}{x} \neq 0.$$

$\Rightarrow$ Σu_n cannot converge. Being a series of +ve terms it must diverge.

Example 14. *Show that the series* $\sum \left(\dfrac{1}{n} - \log \dfrac{n+1}{n}\right)$ *is convergent.*

Sol. Here $u_n = \dfrac{1}{n} - \log\left(1+\dfrac{1}{n}\right)$

$$= \frac{1}{n} - \left(\frac{1}{n} - \frac{1}{2} \cdot \frac{1}{n^2} + \frac{1}{3} \cdot \frac{1}{n^3} \ldots\ldots\right) \qquad \left[\because \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} \ldots\ldots\right]$$

$$= \frac{1}{2} \cdot \frac{1}{n^2} - \frac{1}{3} \cdot \frac{1}{n^3} + \ldots = \frac{1}{n^2}\left[\frac{1}{2} - \frac{1}{3n} + \ldots\ldots\right]$$

Take $v_n = \dfrac{1}{n^2}$ and complete the solution yourself.

Example 15. *If* $u_n = \dfrac{1}{n}$ *and* $v_n = \log(n+1) - \log n$, *prove that* Σv_n *diverges.*

Sol. $u_n = \dfrac{1}{n}$

$$v_n = \log(n+1) - \log n = \log \frac{n+1}{n} = \log\left(1+\frac{1}{n}\right)$$

$$\underset{n\to\infty}{\text{Lt}} \frac{v_n}{u_n} = \underset{n\to\infty}{\text{Lt}}\ n \log\left(1+\frac{1}{n}\right) = \underset{n\to\infty}{\text{Lt}} \log\left(1+\frac{1}{n}\right)^n = \log e = 1$$

which is finite and $\neq 0$.

$\therefore$ Σu_n and Σv_n behave alike.

Since $\sum u_n = \sum \dfrac{1}{n}$ is of the form $\sum \dfrac{1}{n^p}$ with $p = 1$.

$\therefore$ Σu_n is divergent $\Rightarrow$ Σv_n is divergent.

Example 16. *If* $\sum u_n$ *converges, prove that* $\sum \dfrac{u_n}{1-u_n}$ $(u_n \neq 1)$ *also converges.*

Sol. Σu_n is convergent (given) $\Rightarrow$ $\underset{n\to\infty}{\text{Lt}} u_n = 0$ ...(1)

Let $v_n = \dfrac{u_n}{1-u_n}$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}} (1-u_n) = 1 - 0 = 1 \qquad \text{[using (1)]}$$

which is finite and $\neq 0$. $\therefore$ Σu_n and Σv_n behave alike.

But Σu_n is given to be convergent.

$\therefore$ $\sum v_n = \sum \dfrac{u_n}{1-u_n}$ also converges.

Example 17. *If* Σu_n^2 *and* Σv_n^2 *are both convergent series, prove that the series* $\Sigma u_n v_n$ *is also convergent.*

Sol. Since Σu_n^2 and Σv_n^2 are both convergent

$\therefore \quad \frac{1}{2}\Sigma(u_n^2 + v_n^2)$ is also convergent. $\Rightarrow \Sigma \frac{u_n^2+v_n^2}{2}$ is also convergent.

Now G.M. between u_n^2 and v_n^2 < A.M. between u_n^2 and v_n^2.

$$\Rightarrow \quad \sqrt{u_n^2 v_n^2} < \frac{u_n^2+v_n^2}{2} \quad \Rightarrow \quad u_n v_n < \frac{u_n^2+v_n^2}{2}$$

$\Rightarrow$ Each term of the series $\Sigma u_n v_n$ is less than the corresponding term of the series $\Sigma \frac{u_n^2+v_n^2}{2}$ which is convergent.

Hence the series $\Sigma u_n v_n$ is also convergent.

Example 18. *If Σu_n is a convergent series of positive terms, prove that Σu_n^2 is also convergent. Is the converse true ? Illustrate your answer with an example.*

Sol. Since Σu_n is convergent. $\therefore \lim_{n\to\infty} u_n = 0$

$\Rightarrow$ given $\varepsilon > 0$, $\exists$ a +ve integer m such that $-\varepsilon < u_n < \varepsilon \quad \forall\, n \ge m$.

Now Σu_n is a series of +ve terms $\Rightarrow \quad u_n > 0 \quad \forall\, n$

Also choose $\varepsilon < 1$

Then $\quad 0 < u_n < 1 \quad \forall\, n \ge m$

$\Rightarrow \quad u_n^2 < u_n \quad \forall\, n \ge m$

$[\because$ if $0 < x < 1$ then $\quad x^n < x \quad \forall\, n > 1]$

$\Rightarrow$ for $n \ge m$, each term of the series Σu_n^2 is less than the corresponding term of the series Σu_n which is given to be convergent.

Hence Σu_n^2 is also convergent.

The converse is not always true. For example, take $u_n = \frac{1}{n}$ then $u_n^2 = \frac{1}{n^2}$.

Now $\Sigma u_n^2 = \Sigma \frac{1}{n^2}$ converges whereas $\Sigma u_n = \Sigma \frac{1}{n}$ diverges.

Example 19. *Discuss the convergence of the series :*

(i) $\Sigma \frac{1}{x^n + x^{-n}}$, $x > 0$ $\qquad$ *(ii)* $\Sigma \frac{x^n}{a^n + x^n}$.

Sol. *(i)* Here $\quad u_n = \frac{1}{x^n + x^{-n}} = \frac{x^n}{x^{2n}+1}$.

Three cases arise.

Case I. *When $0 < x < 1$*

Take $\quad v_n = x^n$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{x^{2n}+1} = \frac{1}{0+1} = 1 \text{ which is finite and} \ne 0.$$

$\therefore \quad \Sigma u_n$ and Σv_n behave alike.

But $\Sigma v_n = \Sigma x^n$ being a geometric series with common ratio $x < 1$ converges.

$\therefore \quad \Sigma u_n$ also converges.

Case II. *When $x > 1$* so that $0 < \frac{1}{x} < 1$

Take $\quad v_n = \left(\frac{1}{x}\right)^n$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{x^n}{x^{2n}+1} \cdot x^n = \lim_{n\to\infty} \frac{x^{2n}}{x^{2n}+1} = \lim_{n\to\infty} \frac{1}{1+\left(\frac{1}{x}\right)^{2n}} = \frac{1}{1+0} = 1$$

which is finite and $\ne 0$.

$\therefore \quad \Sigma u_n$ and Σv_n behave alike.

But $\Sigma v_n = \Sigma \left(\frac{1}{x}\right)^n$ being a geometric series with common ratio $\frac{1}{x} < 1$ converges. $\therefore \Sigma u_n$ also converges.

Case III. *When $x = 1$.*

$$u_n = \frac{1}{1+1} = \frac{1}{2} \quad \therefore \quad \lim_{n\to\infty} u_n = \frac{1}{2} \ne 0.$$

$\Rightarrow \quad \Sigma u_n$ cannot converge. Being a series of +ve terms, it must diverge.

Hence the given series Σu_n converges if $x > 1$ or $x < 1$ and diverges if $x = 1$.

(ii) Here $\quad u_n = \frac{a^n}{a^n + x^n}$.

Three cases arise.

Case I. When x < a *i.e.,* $\frac{x}{a} < 1$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{a^n}{a^n + x^n} = \lim_{n\to\infty} \frac{1}{1+\left(\frac{x}{a}\right)^n} = \frac{1}{1+0} = 1 \ne 0.$$

$\therefore \quad \Sigma u_n$ does not converge. Being a series of +ve terms, it diverges.

Case II. When x > a.

Take $\quad v_n = \left(\frac{a}{x}\right)^n$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{a^n}{a^n+x^n} \cdot \left(\frac{x}{a}\right)^n = \lim_{n\to\infty} \frac{a^n}{a^n+x^n}$$

$$= \lim_{n\to\infty} \frac{1}{\left(\frac{a}{x}\right)^n + 1} = \frac{1}{0+1} = 1 \qquad \left[\because \frac{a}{x} < 1\right]$$

which is finite and $\ne 0$.

$\Rightarrow \quad \Sigma u_n$ and Σv_n behave alike.

But $\Sigma v_n = \Sigma \left(\frac{a}{x}\right)^n$ being a geometric series with common ratio $\frac{a}{x} < 1$ converges. $\therefore$ Σu_n also converges.

Case III. When x = a

$$u_n = \frac{a^n}{a^n + a^n} = \frac{1}{2}$$

$\therefore \underset{n\to\infty}{\text{Lt}}\ u_n = \frac{1}{2} \neq 0. \Rightarrow \Sigma u_n$ does not converge. Being a series of +ve terms, it must diverge.

Hence the given series Σu_n converges if $x > a$ and diverges if $x \le a$.

Example 20. *Examine the following series for convergence*

(i) $\sum \frac{1}{(\log n)^{\log n}}$ (ii) $\sum \frac{1}{(\log \log n)^{\log n}}$ (iii) $\sum r^{\log n}$.

Sol. (i) Since $\underset{n\to\infty}{\text{Lt}} \log(\log n) = \infty$

$\therefore$ We can find n so large that $\log(\log n) > 2$

$\Rightarrow \quad (\log n)[\log(\log n)] > 2 \log n$

$\Rightarrow \quad \log[(\log n)^{\log n}] > \log n^2$

$\Rightarrow \quad (\log n)^{\log n} > n^2 \Rightarrow \frac{1}{(\log n)^{\log n}} < \frac{1}{n^2}$

Since the series $\sum \frac{1}{n^2}$, $(p = 2 > 1)$, converges.

$\therefore$ By comparison test, the series $\sum \frac{1}{(\log n)^{\log n}}$ also converges.

(ii) Since $\underset{n\to\infty}{\text{Lt}} \sum \log(\log \log n) = \infty$

$\therefore$ We can find n so large that $\log(\log \log n) > 2$

$\Rightarrow \quad (\log n)[\log(\log \log n)] > 2 \log n$

$\Rightarrow \quad \log(\log \log n)^{\log n} > \log n^2$

$\Rightarrow \quad (\log \log n)^{\log n} > n^2 \Rightarrow \frac{1}{(\log \log n)^{\log n}} < \frac{1}{n^2}$

Since the series $\sum \frac{1}{n^2}$, $(p = 2 > 1)$, converges

$\therefore$ By comparison test, the series $\sum \frac{1}{(\log \log n)^{\log n}}$ also converges.

(iii) Since multiplication of numbers is commutative

$\therefore \quad \log n \log r = \log r \log n$

$\Rightarrow \quad \log(r^{\log n}) = \log(n^{\log r}) \Rightarrow r^{\log n} = n^{\log r}$

$\therefore \quad \sum r^{\log n} = \sum n^{\log r} = \sum \frac{1}{n^{(-\log r)}}$

converges iff $\quad -\log r > 1$ [p-series test]

i.e., iff $\quad \log r < -1$ or iff $\log r < -\log e$

or iff $\quad \log r < \log e^{-1}$ or iff $r < \frac{1}{e}$

Hence $\sum r^{\log n}$ converges iff $r < \frac{1}{e}$.

Example 21. *Test the following series for convergence :*

(i) $1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \ldots\ldots + \frac{1}{n^n} + \ldots\ldots$

(ii) $\sum_{n=1}^{\infty} \frac{2^n + 3^n}{6^n}$ (iii) $\sum_{n=1}^{\infty} \frac{1}{2^n + 3^n}$ (iv) $\sum_{n=1}^{\infty} \frac{1}{n^3}\left(\frac{n+2}{n+3}\right)^n$

(v) $\sum_{n=1}^{\infty} \frac{1}{n^{\alpha + \frac{\beta}{n}}}$ (vi) $\sum_{n=1}^{\infty} \frac{\sqrt{n+1} - 1}{(n+2)^3 - 1}$ (vii) $\sum_{n=1}^{\infty} (q^{1/n} - 1)$, $q \neq 1$ and $q > 0$

(viii) $\sum_{n=1}^{\infty} \frac{|\cos n|}{n^{3/2}}$ (ix) $\sum_{n=1}^{\infty} \frac{2^{n+1} - 3}{2^{n+1} + 2}$.

Sol. (i) Here $\quad u_n = \frac{1}{n^n}$

Since $\quad n^n > 2^n \quad \forall\, n \in N$

$\therefore \quad \frac{1}{n^n} < \frac{1}{2^n} \quad \forall\, n$

$\Rightarrow \quad u_n < \frac{1}{2^n} \quad \forall\, n$

$\Rightarrow \quad u_n < v_n$ where $v_n = \frac{1}{2^n}$

But Σv_n is a geometric series with common ratio $r = \frac{1}{2}$.

Since $|r| < 1$, the geometric series is convergent.

$\therefore$ By comparison test, Σu_n is also convergent.

(ii) Here $\quad u_n = \frac{2^n + 3^n}{6^n} = \left(\frac{2}{6}\right)^n + \left(\frac{3}{6}\right)^n$

$$= \left(\frac{1}{3}\right)^n + \left(\frac{1}{2}\right)^n = a_n + b_n \text{ where } a_n = \left(\frac{1}{3}\right)^n \text{ and } b_n = \left(\frac{1}{2}\right)^n$$

Since Σa_n and Σb_n are both geometric series with common ratio numerically less than 1, they are convergent.

$\Rightarrow \Sigma(a_n + b_n)$ is convergent $\Rightarrow \Sigma u_n$ is convergent.

(*iii*) Here $$u_n = \frac{1}{2^n + 3^n} = \frac{1}{3^n\left(1+\frac{2^n}{3^n}\right)} = \frac{1}{3^n\left[1+\left(\frac{2}{3}\right)^n\right]}$$

Take $$v_n = \frac{1}{3^n}.$$

Σv_n is a geometric series with common ratio numerically less than 1.

$\therefore$ Σv_n is convergent.

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{1}{1+\left(\frac{2}{3}\right)^n} = \frac{1}{1+0}$$

$$= 1 \neq 0 \qquad \left[\because \lim_{n\to\infty} r^n = 0 \text{ if } 0<r<1\right]$$

$\therefore$ By comparison test, Σu_n is also convergent.

(*iv*) Here $$u_n = \frac{1}{n^3}\left(\frac{n+2}{n+3}\right)^n$$

Take $v_n = \frac{1}{n^3}$, then Σv_n is convergent. $\qquad (\because \ p = 3 > 1)$

$$\frac{u_n}{v_n} = \left(\frac{n+2}{n+3}\right)^n = \frac{\left(1+\frac{2}{n}\right)^n}{\left(1+\frac{3}{n}\right)^n} = \frac{\left[\left(1+\frac{2}{n}\right)^{n/2}\right]^2}{\left[\left(1+\frac{3}{n}\right)^{n/3}\right]^3}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \frac{e^2}{e^3} = \frac{1}{e} \neq 0 \qquad (\because \ 2 < e < 3)$$

$\therefore$ By comparison test, Σu_n is convergent.

(*v*) Here $$u_n = \frac{1}{n^{\alpha+\frac{\beta}{n}}} = \frac{1}{n^\alpha\cdot(n^{1/n})^\beta}$$

Take $v_n = \frac{1}{n^\alpha}$, then Σv_n is convergent if $\alpha > 1$ and divergent if $\alpha \le 1$.

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{1}{(n^{1/n})^\beta} = 1 \neq 0 \qquad \left[\because \lim_{n\to\infty} n^{1/n} = 1\right]$$

$\therefore$ By comparison test, Σu_n converges if $\alpha > 1$ and diverges if $\alpha \le 1$.

(*vi*) Here $$u_n = \frac{\sqrt{n+1}-1}{(n+2)^3-1} = \frac{\sqrt{n}\left(\sqrt{1+\frac{1}{n}}-\frac{1}{\sqrt{n}}\right)}{n^3\left[\left(1+\frac{2}{n}\right)^3-\frac{1}{n^3}\right]} = \frac{1}{n^{5/2}}\cdot\frac{\sqrt{1+\frac{1}{n}}-\frac{1}{\sqrt{n}}}{\left(1+\frac{2}{n}\right)^3-\frac{1}{n^3}}$$

Take $v_n = \frac{1}{n^{5/2}}$ then Σv_n is convergent $\qquad \left(\because \ p = \frac{5}{2} > 1\right)$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{\sqrt{1+\frac{1}{n}}-\frac{1}{\sqrt{n}}}{\left(1+\frac{2}{n}\right)^3-\frac{1}{n^3}} = \frac{1-0}{1-0} = 1 \neq 0.$$

$\therefore$ By comparison test, Σu_n is convergent.

(*vii*) **Case I.** Let $\quad q > 1$

Here $\quad u_n = q^{1/n} - 1$

Take $\quad v_n = \frac{1}{n}$, then Σv_n is divergent $\qquad (\because \ p = 1)$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{q^{1/n}-1}{\frac{1}{n}} = \lim_{m\to 0}\frac{q^m-1}{m} \text{ where } m = \frac{1}{n}$$

$$= \log q \neq 0 \qquad (\because \ q > 1)$$

$\therefore$ By comparison test, Σu_n is divergent.

Case II. Let $0 < q < 1$

Here $$u_n = q^{1/n} - 1 = q^{1/n}\left[1-\left(\frac{1}{q}\right)^{1/n}\right] = -q^{1/n}\left[\left(\frac{1}{q}\right)^{1/n}-1\right]$$

Take $\quad v_n = \frac{1}{n}$ then Σv_n is divergent.

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty} -q^{1/n}\left[\frac{\left(\frac{1}{q}\right)^{1/n}-1}{1/n}\right] = -1\times\log\frac{1}{q} = \log q \neq 0$$

$$(\because \ 0 < q < 1)$$

$\therefore$ By comparison test, Σu_n is divergent.

Hence Σu_n is always divergent.

(*viii*) Here $$u_n = \frac{|\cos n|}{n^{3/2}} \le \frac{1}{n^{3/2}} \qquad [\because \ |\cos n| \le 1 \ \forall\, n]$$

and $\sum \frac{1}{n^{3/2}}$ is convergent $\qquad \left(\because \ p = \frac{3}{2} > 1\right)$

$\therefore$ By comparison test, Σu_n is convergent.

(*ix*) Here $$u_n = \frac{2^{n+1}-3}{2^{n+1}+2} = \frac{2^{n+1}\left(1-\frac{3}{2^{n+1}}\right)}{2^{n+1}\left(1+\frac{2}{2^{n+1}}\right)} = \frac{1-\frac{3}{2^{n+1}}}{1+\frac{2}{2^{n+1}}}$$

$$\lim_{n \to \infty} u_n = \frac{1-0}{1+0} = 1 \neq 0$$

$\Rightarrow$ Σu_n does not converge.

Now Σu_n being a series of positive terms either converges or diverges. Since it does not converge, it must diverge.

Article 12. D' Alembert's Ratio Test

Statement. *If Σu_n is a series of positive terms such that*

(a) $\lim_{n \to \infty} \frac{u_n}{u_{n+1}} = l$, *then*

(i) Σu_n *is convergent if* $l > 1$ (ii) Σu_n *is divergent if* $l < 1$

(iii) Σu_n *may converge or diverge if* $l = 1$ *(i.e., the test fails if* $l = 1$*)*

(b) $\lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \infty$, *then* Σu_n *is convergent.*

Proof. (a) Σu_n is a series of positive terms

$$\Rightarrow \quad u_n > 0 \quad \forall\, n$$

$$\Rightarrow \quad \frac{u_n}{u_{n+1}} > 0 \Rightarrow \lim_{n \to \infty} \frac{u_n}{u_{n+1}} = l \geq 0.$$

Since $\lim_{n \to \infty} \frac{u_n}{u_{n+1}} = l$, therefore, for each $\varepsilon > 0, \exists$ a positive integer m such that

$$\left| \frac{u_n}{u_{n+1}} - l \right| < \varepsilon \qquad \forall\, n \geq m$$

$$\Rightarrow \quad l - \varepsilon < \frac{u_n}{u_{n+1}} < l + \varepsilon \qquad \forall\, n \geq m$$

Replacing n by $m, m+1, m+2, \ldots\ldots, n-1$ in the above inequality, we have

$$l - \varepsilon < \frac{u_m}{u_{m+1}} < l + \varepsilon$$

$$l - \varepsilon < \frac{u_{m+1}}{u_{m+2}} < l + \varepsilon$$

$$l - \varepsilon < \frac{u_{m+2}}{u_{m+3}} < l + \varepsilon$$

....

....

$$l - \varepsilon < \frac{u_{n-1}}{u_n} < l + \varepsilon$$

Multiplying the above $(n-m)$ inequalities, we have

$$(l-\varepsilon)^{n-m} < \frac{u_m}{u_n} < (l+\varepsilon)^{n-m} \qquad \ldots(1)$$

(i) Let $l > 1$

Choose $\varepsilon > 0$ such that $l - \varepsilon > 1$. (This is always possible since we have only to choose an ε such that $0 < \varepsilon < l - 1$)

From (1), we have $\quad (l-\varepsilon)^{n-m} < \frac{u_m}{u_n} \Rightarrow u_n < \frac{u_m}{(l-\varepsilon)^{n-m}}$

$$\Rightarrow \quad u_n < u_m (l-\varepsilon)^m \cdot \frac{1}{(l-\varepsilon)^n}$$

$$\Rightarrow \quad u_n < K \cdot \frac{1}{(l-\varepsilon)^n} \ \forall\, n \geq m \text{ where } K = u_m (l-\varepsilon)^m$$

Now, $\sum_{n=1}^{\infty} \frac{1}{(l-\varepsilon)^n}$ being a geometric series with common ratio $\frac{1}{l-\varepsilon} < 1$ is convergent.

Therefore, by comparison test, the series $\sum_{n=1}^{\infty} u_n$ is convergent.

(ii) Let $l < 1$

Choose $\varepsilon < 0$ such that $l + \varepsilon < 1$ (This is always possible since we have only to choose an ε such that $0 < \varepsilon < 1 - l$)

Since $\quad l \geq 0, 0 < l + \varepsilon < 1$

From (1), we have $\quad \frac{u_m}{u_n} < (l+\varepsilon)^{n-m} \Rightarrow u_n > \frac{u_m}{(l+\varepsilon)^{n-m}}$

$$\Rightarrow \quad u_n > u_m (l+\varepsilon)^m \cdot \frac{1}{(l+\varepsilon)^n}$$

$$\Rightarrow \quad u_n > k \cdot \frac{1}{(l+\varepsilon)^n} \quad \forall\, n \geq m \text{ where } k = u_m (l+\varepsilon)^m$$

Now, $\sum_{n=1}^{\infty} \frac{1}{(l+\varepsilon)^n}$ being a geometric series with common ratio $\frac{1}{l+\varepsilon} > 1$ is divergent.

Therefore, by comparison test, the series $\sum_{n=1}^{\infty} u_n$ is divergent.

(iii) Let $\quad l = 1$

First consider the series $\quad \sum \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots + \frac{1}{n} + \ldots\ldots$

$$u_n = \frac{1}{n}, u_{n+1} = \frac{1}{n+1}$$

$$\frac{u_n}{u_{n+1}} = \frac{n+1}{n} = 1 + \frac{1}{n} \text{ so that } \lim_{n \to \infty} \frac{u_n}{u_{n+1}} = 1.$$

Since the series $\sum \frac{1}{n}$ is divergent, we find that if $l = 1$, a series may diverge.

Next consider the series

$$\sum \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \ldots\ldots + \frac{1}{n^2} + \ldots\ldots$$

$$u_n = \frac{1}{n^2},\ u_{n+1} = \frac{1}{(n+1)^2}$$

$$\frac{u_n}{u_{n+1}} = \frac{(n+1)^2}{n^2} = \left(\frac{n+1}{n}\right)^2 = \left(1+\frac{1}{n}\right)^2 \text{ so that } \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = 1$$

Since the series $\sum \frac{1}{n^2}$ is convergent, we find that if $l = 1$, a series may converge.

The above two examples show that if $l = 1$, a series may converge or diverge. Hence the test fails when $l = 1$.

(b) $$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \infty$$

Then there exists a positive integer m such that $\frac{u_n}{u_{n+1}} > 2 \quad \forall \quad n \geq m$

Replacing n by $m, m+1, m+2, \ldots\ldots, n-1$ in the above inequality, we have

$$\frac{u_m}{u_{m+1}} > 2$$

$$\frac{u_{m+1}}{u_{m+2}} > 2$$

$$\frac{u_{m+2}}{u_{m+3}} > 2$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{u_{n-1}}{u_n} > 2$$

Multiplying the above $(n-m)$ inequalities, we have

$$\frac{u_m}{u_n} > 2^{n-m} \Rightarrow \frac{u_n}{u_m} < \frac{1}{2^{n-m}}$$

$$\Rightarrow \quad u_n < (u_m \cdot 2^m) \cdot \frac{1}{2^n}$$

$$\Rightarrow \quad u_n < K \cdot \frac{1}{2^n} \quad \forall\, n \geq m, \text{ where } K = u_m \cdot 2^m$$

Now $\sum_{n=1}^{\infty} \frac{1}{2^n}$ being a geometric series with common ratio $\frac{1}{2} < 1$ is convergent. Therefore, by comparison test, the series $\sum_{n=1}^{\infty} u_n$ is convergent.

ILLUSTRATIVE EXAMPLES

Example 1. *Discuss the convergence of the following series:*

(i) $1 + \frac{2!}{2^2} + \frac{3!}{3^3} + \frac{4!}{4^4} + \ldots\ldots$

(ii) $1 + \frac{2^p}{2!} + \frac{3^p}{3!} + \frac{4^p}{4!} + \ldots\ldots, (p > 0)$

(iii) $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \ldots\ldots + \frac{1}{2^{n-1}+1} + \ldots\ldots$

(iv) $\frac{1^2 \cdot 2^2}{1!} + \frac{2^2 \cdot 3^2}{2!} + \frac{3^2 \cdot 4^2}{3!} + \frac{4^2 \cdot 5^2}{4!} + \ldots\ldots$

Sol. (i) Here $\quad u_n = \frac{n!}{n^n} \qquad \left[\because 1 = \frac{1!}{1^1}\right.$

$$\therefore \quad u_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$$

$$\frac{u_n}{u_{n+1}} = \frac{n!}{n^n} \cdot \frac{(n+1)^{n+1}}{(n+1)!} = \frac{n!}{n^n} \cdot \frac{(n+1)^{n+1}}{(n+1)\,n!} = \frac{(n+1)^n}{n^n} = \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n$$

$$\operatorname*{Lt}_{n\to\infty} \frac{u_n}{u_{n+1}} = \operatorname*{Lt}_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e > 1 \qquad (\because\ 2 < e < 3)$$

$\therefore$ By D' Alembert's Ratio Test, Σu_n is convergent.

(ii) Here $\quad u_n = \frac{n^p}{n!} \qquad \left[\because 1 = \frac{1^p}{1!}\right]$

$$\therefore \quad u_{n+1} = \frac{(n+1)^p}{(n+1)!}$$

$$\frac{u_n}{u_{n+1}} = \frac{n^p}{n!} \cdot \frac{(n+1)!}{(n+1)^p} = \frac{n^p \cdot (n+1)\,n!}{n!\,(n+1)^p}$$

$$= \frac{n^p}{(n+1)^{p-1}} = \frac{n^p}{n^{p-1}\left(1+\frac{1}{n}\right)^{p-1}} = \frac{n}{\left(1+\frac{1}{n}\right)^{p-1}}$$

$$\operatorname*{Lt}_{n\to\infty} \frac{u_n}{u_{n+1}} = \operatorname*{Lt}_{n\to\infty} \frac{n}{\left(1+\frac{1}{n}\right)^{p-1}} = \infty$$

$\therefore$ By D' Alembert's Ratio Test, Σu_n is convergent.

(iii) Here $\quad u_n = \frac{1}{2^{n-1}+1} \quad \therefore \quad u_{n+1} = \frac{1}{2^n+1}$

$$\frac{u_n}{u_{n+1}} = \frac{2^n+1}{2^{n-1}+1} = \frac{2^n\left(1+\frac{1}{2^n}\right)}{2^{n-1}\left(1+\frac{1}{2^{n-1}}\right)} = 2 \cdot \frac{1+\frac{1}{2^n}}{1+\frac{1}{2^{n-1}}}$$

$$\underset{n\to\infty}{\text{Lt}}\frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}}\ 2.\frac{1+\frac{1}{2^n}}{1+\frac{1}{2^{n-1}}} = 2 > 1$$

$\therefore$ By D' Alembert's Ratio Test, Σu_n is convergent.

(*iv*) Here $\quad u_n = \dfrac{n^2(n+1)^2}{n!} \quad \therefore \quad u_{n+1} = \dfrac{(n+1)^2(n+2)^2}{(n+1)!}$

$$\frac{u_n}{u_{n+1}} = \frac{n^2(n+1)^2}{n!}\cdot\frac{(n+1)!}{(n+1)^2(n+2)^2} = \frac{n^2.(n+1)n!}{n!(n+2)^2}$$

$$= \frac{n^3\left(1+\frac{1}{n}\right)}{n^2\left(1+\frac{2}{n}\right)^2} = n.\frac{1+\frac{1}{n}}{\left(1+\frac{2}{n}\right)^2}$$

$$\underset{n\to\infty}{\text{Lt}}\frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}}\ n.\frac{1+\frac{1}{n}}{\left(1+\frac{1}{n}\right)^2} = \infty$$

$\therefore$ By D' Alembert's Ratio Test, Σu_n is convergent.

Example 2. *Test the convergence of the following series :*

(*i*) $\dfrac{2}{1^2+1}+\dfrac{2^2}{2^2+1}+\dfrac{2^3}{3^2+1}+\ldots\ldots$ $\qquad$ (*ii*) $\dfrac{1}{3}+\dfrac{8}{9}+\dfrac{27}{27}+\dfrac{64}{81}+\dfrac{125}{243}+\ldots\ldots$

(*iii*) $\dfrac{2!}{3}+\dfrac{3!}{3^2}+\dfrac{4!}{3^3}+\ldots\ldots$ $\qquad$ (*iv*) $\dfrac{1}{5}+\dfrac{2!}{5^2}+\dfrac{3!}{5^3}+\ldots\ldots$

(*v*) $\dfrac{1}{2}+\dfrac{2!}{8}+\dfrac{3!}{32}+\dfrac{4!}{128}+\ldots\ldots$

Sol. (*i*) Here $\quad u_n = \dfrac{2^n}{n^2+1} \quad \therefore \quad u_{n+1} = \dfrac{2^{n+1}}{(n+1)^2+1}$

$$\frac{u_n}{u_{n+1}} = \frac{2^n}{2^{n+1}}\cdot\frac{(n+1)^2+1}{n^2+1} = \frac{1}{2}\cdot\frac{n^2+2n+2}{n^2+1} = \frac{1}{2}\cdot\frac{1+\frac{2}{n}+\frac{2}{n^2}}{1+\frac{1}{n^2}}$$

$$\underset{n\to\infty}{\text{Lt}}\frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}}\frac{1}{2}\cdot\frac{1+\frac{2}{n}+\frac{2}{n^2}}{1+\frac{1}{n^2}} = \frac{1}{2} < 1$$

$\therefore$ By D' Alembert's Ratio Test, Σu_n is divergent.

(*ii*) The given series is $\dfrac{1^3}{3^1}+\dfrac{2^3}{3^2}+\dfrac{3^3}{3^3}+\dfrac{4^3}{3^4}+\dfrac{5^3}{3^5}+\ldots\ldots$

Here $\quad u_n = \dfrac{n^3}{3^n} \quad \therefore \quad u_{n+1} = \dfrac{(n+1)^3}{3^{n+1}}$

$$\frac{u_n}{u_{n+1}} = \frac{n^3}{(n+1)^3}\cdot\frac{3^{n+1}}{3^n} = 3.\frac{n^3}{n^3\left(1+\frac{1}{n}\right)^3} = \frac{3}{\left(1+\frac{1}{n}\right)^3}$$

$$\underset{n\to\infty}{\text{Lt}}\frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}}\frac{3}{\left(1+\frac{1}{n}\right)^3} = 3 > 1$$

$\therefore$ By D' Alembert's Ratio Test, Σu_n is convergent.

(*iii*) Here $\quad u_n = \dfrac{(n+1)!}{3^n} \quad \therefore \quad u_{n+1} = \dfrac{(n+2)!}{3^{n+1}}$

$$\frac{u_n}{u_{n+1}} = \frac{(n+1)!}{(n+2)!}\cdot\frac{3^{n+1}}{3^n} = 3.\frac{(n+1)!}{(n+2)(n+1)!} = \frac{3}{n+2}$$

$$\underset{n\to\infty}{\text{Lt}}\frac{u_n}{u_{n+1}} = \frac{3}{\infty} = 0 < 1$$

$\therefore$ By D' Alembert's Ratio Test, Σu_n is divergent.

(*iv*) *Please try yourself.* **[Ans.** Divergent]

(*v*) The given series is $\dfrac{1!}{2^1}+\dfrac{2!}{2^3}+\dfrac{3!}{2^5}+\dfrac{4!}{2^7}+\ldots\ldots$

Here $\quad u_n = \dfrac{n!}{2^{2n-1}} \quad \therefore \quad u_{n+1} = \dfrac{(n+1)!}{2^{2n+1}}$

$$\frac{u_n}{u_{n+1}} = \frac{n!}{(n+1)!}\cdot\frac{2^{2n+1}}{2^{2n-1}} = \frac{4}{n+1}$$

$$\underset{n\to\infty}{\text{Lt}}\frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}}\frac{4}{n+1} = \frac{4}{\infty} = 0 < 1$$

$\therefore$ By D' Alembert's Ratio Test, Σu_n is divergent.

Example 3. *Examine the convergence or divergence of the following series :*

(*i*) $\dfrac{1}{3}+\dfrac{1.2}{3.5}+\dfrac{1.2.3}{3.5.7}+\dfrac{1.2.3.4}{3.5.7.9}+\ldots\ldots$ (*ii*) $\dfrac{5}{2+5}+\dfrac{5^2}{2^2+5}+\dfrac{5^3}{2^3+5}+\ldots\ldots$

(*iii*) $\sum\dfrac{2^{n-1}}{3^n+1}$.

Sol. (*i*) Here $\quad u_n = \dfrac{1.2.3\ldots\ldots n}{3.5.7\ldots\ldots[3+(n-1).2]} = \dfrac{n}{3.5.7\ldots\ldots(2n+1)}$

$\therefore \quad u_{n+1} = \dfrac{(n+1)!}{3.5.7\ldots\ldots(2n+1)(2n+3)}$

$$\frac{u_n}{u_{n+1}} = \frac{n!.(2n+3)}{(n+1)!} = \frac{2n+3}{n+1} = \frac{2+\frac{3}{n}}{1+\frac{1}{n}}$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}} \frac{2+\frac{3}{n}}{1+\frac{1}{n}} = 2 > 1$$

$\therefore$ By D' Alembert's Ratio Test, Σu_n is convergent.

(*ii*) Here $\quad u_n = \dfrac{5^n}{2^n+5} \quad \therefore \quad u_{n+1} = \dfrac{5^{n+1}}{2^{n+1}+5}$

$$\frac{u_n}{u_{n+1}} = \frac{5^n}{5^{n+1}} \cdot \frac{2^{n+1}+5}{2^n+5} = \frac{1}{5} \cdot \frac{2^{n+1}\left(1+\frac{5}{2^{n+1}}\right)}{2^n\left(1+\frac{5}{2^n}\right)} = \frac{2}{5} \cdot \frac{1+\frac{5}{2^{n+1}}}{1+\frac{5}{2^n}}$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}} \frac{2}{5} \cdot \frac{1+\frac{5}{2^{n+1}}}{1+\frac{5}{2^n}} = \frac{2}{3} < 1$$

$\therefore$ By D' Alembert's Ratio Test, Σu_n is divergent.

(*iii*) *Please try yourself.* **[Ans.** Convergent]

Example 4. *Test the convergence of the following series :*

(*i*) $\sum \dfrac{1}{n!}$ $\qquad$ (*ii*) $\sum \dfrac{n^3+a}{2^n+a}$

(*iii*) $\sum \dfrac{n^2(n+1)^2}{n!}$ $\qquad$ (*iv*) $\sum \dfrac{2^n . n!}{n^n}$.

Sol. (*i*) *Please try yourself.* **[Ans.** Convergent]

(*ii*) Here $\quad u_n = \dfrac{n^3+a}{2^n+a} \quad \therefore \quad u_{n+1} = \dfrac{(n+1)^3+a}{2^{n+1}+a}$

$$\frac{u_n}{u_{n+1}} = \frac{n^3+a}{(n+1)^3+a} \cdot \frac{2^{n+1}+a}{2^n+a}$$

$$= \frac{n^3\left(1+\frac{a}{n^3}\right)}{n^3\left[\left(1+\frac{1}{n}\right)^3+\frac{a}{n^3}\right]} \cdot \frac{2^{n+1}\left(1+\frac{1}{2^{n+1}}\right)}{2^n\left(1+\frac{a}{2^n}\right)} = \frac{1+\frac{a}{n^3}}{\left(1+\frac{1}{n}\right)^3+\frac{a}{n^3}} \cdot \frac{2\left(1+\frac{a}{2^{n+1}}\right)}{1+\frac{a}{2^n}}$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \frac{1+0}{1+0} . 2 . \frac{1+0}{1+0} = 2 > 1$$

$\therefore$ By D' Alembert's Ratio Test, Σu_n is convergent.

(*iii*) Here $\quad u_n = \dfrac{n^2(n+1)^2}{n!}$ $\qquad$ | See Ex. 1 (*iv*)

(*iv*) Here $\quad u_n = \dfrac{2^n . n!}{n^n} \quad \therefore \quad u_{n+1} = \dfrac{2^{n+1}(n+1)!}{(n+1)^{n+1}}$

$$\frac{u_n}{u_{n+1}} = \frac{2^n . n!}{2^{n+1}(n+1)!} \cdot \frac{(n+1)^{n+1}}{n^n} = \frac{1}{2(n+1)} \cdot \frac{(n+1)^{n+1}}{n^n}$$

$$= \frac{1}{2} \cdot \frac{(n+1)^n}{n^n} = \frac{1}{2} \cdot \left(\frac{n+1}{n}\right)^n = \frac{1}{2}\left(1+\frac{1}{n}\right)^n$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}} \frac{1}{2}\left(1+\frac{1}{n}\right)^n = \frac{e}{2}$$

Now $\quad 2 < e < 3 \quad \Rightarrow \quad 1 < \dfrac{e}{2} < \dfrac{3}{2}$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \frac{e}{2} > 1 \quad \Rightarrow \quad \Sigma u_n \text{ is convergent.}$$

Example 5. *Discuss the convergence or divergence of the following series :*

(*i*) $\sum \dfrac{n^2}{n!}$ $\qquad$ (*ii*) $\sum \dfrac{n!}{n^n}$ $\qquad$ (*iii*) $\sum \dfrac{3^n . n!}{n^n}$.

Sol. (*i*) Here $\quad u_n = \dfrac{n^2}{n!}$

$$\therefore \quad u_{n+1} = \frac{(n+1)^2}{(n+1)!} = \frac{(n+1)^2}{(n+1)\,n!} = \frac{n+1}{n!}$$

$$\frac{u_n}{u_{n+1}} = \frac{n^2}{n+1} = \frac{1}{\frac{1}{n}+\frac{1}{n^2}}$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}} \frac{1}{\frac{1}{n}+\frac{1}{n^2}} = \frac{1}{0} = \infty$$

$\therefore$ By D' Alembert's Ratio Test, Σu_n is convergent.

(*ii*) Here $\quad u_n = \dfrac{n!}{n^n}$

$$\therefore \quad u_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}} = \frac{(n+1)n!}{(n+1)^{n+1}} = \frac{n!}{(n+1)^n}$$

$$\frac{u_n}{u_{n+1}} = \frac{(n+1)^n}{n^n} = \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}} \left(1+\frac{1}{n}\right)^n = e > 1 \qquad (\because\ 2 < e < 3)$$

$\therefore$ By D' Alembert's Ratio Test, Σu_n is convergent.

(*iii*) *Please try yourself.* **[Ans. Divergent]**

$$\left(\because\ 2<e<3\ \therefore\ \frac{e}{3}<1\right)$$

Example 6. *Discuss the convergence or divergence of the following series :*

(*i*) $\sum \frac{x^n}{3^n . n^2}, x > 0$ (*ii*) $\sum \frac{x^n}{a+\sqrt{n}}$

(*iii*) $\sum \sqrt{\frac{n+1}{n^3+1}} . x^n$ (*iv*) $\sum \frac{\sqrt{n}}{\sqrt{n^2+1}} . x^n\ (x>0)$

(*v*) $\sum \frac{x^n}{n}, x > 0$ (*vi*) $\sum \frac{n}{n^2+1} . x^n, x > 0.$

Sol. (*i*) Here $u_n = \frac{x^n}{3^n . n^2}$ $\therefore$ $u_{n+1} = \frac{x^{n+1}}{3^{n+1}(n+1)^2}$

$$\frac{u_n}{u_{n+1}} = \frac{x^n}{x^{n+1}} . \frac{3^{n+1}}{3^n} . \frac{(n+1)^2}{n^2} = \frac{3}{x} . \left(1+\frac{1}{n}\right)^2$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}} \frac{3}{x}\left(1+\frac{1}{n}\right)^2 = \frac{3}{x}$$

$\therefore$ By D' Alembert's Ratio Test, Σu_n converges if $\frac{3}{x} > 1$ *i.e.*, $x < 3$ and diverges if $\frac{3}{x} < 1$, *i.e.*, $x > 3$

When $x = 3$, $\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = 1$

$\therefore$ Ratio Test fails.

Now, when $x = 3$, $u_n = \frac{3^n}{3^n . n^2} = \frac{1}{n^2}$

$\therefore$ $\sum u_n = \sum \frac{1}{n^2}$ is of the form $\sum \frac{1}{n^p}$ with $p = 2 > 1$

$\therefore$ Σu_n is convergent.

Hence the given series Σu_n converges if $x \le 3$ and diverges if $x > 3$.

(*ii*) Here $u_n = \frac{x^n}{a+\sqrt{n}}$ $\therefore$ $u_{n+1} = \frac{x^{n+1}}{a+\sqrt{n+1}}$

$$\frac{u_n}{u_{n+1}} = \frac{x^n}{x^{n+1}} . \frac{a+\sqrt{n+1}}{a+\sqrt{n}} = \frac{1}{x} \frac{\sqrt{n}\left(\frac{a}{\sqrt{n}}+\sqrt{1+\frac{1}{n}}\right)}{\sqrt{n}\left(\frac{a}{\sqrt{n}}+1\right)} = \frac{1}{x} . \frac{\frac{a}{\sqrt{n}}+\sqrt{1+\frac{1}{n}}}{\frac{a}{\sqrt{n}}+1}$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}} \frac{1}{x} . \frac{\frac{a}{\sqrt{n}}+\sqrt{1+\frac{1}{n}}}{\frac{a}{\sqrt{n}}+1} = \frac{1}{x} . \frac{0+\sqrt{1+0}}{0+1} = \frac{1}{x}$$

$\therefore$ By D' Alembert's Ratio Test, Σu_n converges if $\frac{1}{x} > 1$ *i.e.*, $x < 1$ and diverges if $\frac{1}{x} < 1$ *i.e.*, $x > 1$.

When $x = 1$, the Ratio Test fails.

When $x = 1$, $u_n = \frac{1}{a+\sqrt{n}} = \frac{1}{\sqrt{n}\left(\frac{a}{\sqrt{n}}+1\right)}$

Take $v_n = \frac{1}{\sqrt{n}}$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}} \frac{1}{\frac{a}{\sqrt{n}}+1} = 1 \text{ which is finite and} \ne 0.$$

Since $\sum v_n = \sum \frac{1}{n^{1/2}}$ is of the form $\sum \frac{1}{n^p}$ with $p = \frac{1}{2} < 1$

$\therefore$ Σv_n is divergent $\Rightarrow$ Σu_n is divergent.

Hence the given series Σu_n converges if $x < 1$ and diverges if $x \ge 1$.

(*iii*) Here $u_n = \sqrt{\frac{n+1}{n^3+1}} . x^n$ $\therefore$ $u_{n+1} = \sqrt{\frac{n+2}{(n+1)^3+1}} . x^{n+1}$

$$\frac{u_n}{u_{n+1}} = \sqrt{\frac{n+1}{n^3+1}} . \sqrt{\frac{(n+1)^3+1}{n+2}} . \frac{1}{x}$$

$$= \sqrt{\frac{n\left(1+\frac{1}{n}\right)n^3\left\{\left(1+\frac{1}{n}\right)^3+\frac{1}{n^3}\right\}}{n\left(1+\frac{2}{n}\right) . n^3\left(1+\frac{1}{n^3}\right)}} . \frac{1}{x}$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}} \sqrt{\frac{1+\frac{1}{n}}{1+\frac{2}{n}} . \frac{\left(1+\frac{1}{n}\right)^3+\frac{1}{n^3}}{1+\frac{1}{n^3}}} . \frac{1}{x} = \frac{1}{x}$$

$\therefore$ By D' Alembert's Ratio Test, $\sum u_n$ converges if $\frac{1}{x} > 1$.

i.e., $x < 1$ and diverges if $\frac{1}{x} < 1$ *i.e.*, $x > 1$.

When $x = 1$, the Ratio Test fails.

When $x = 1$, $$u_n = \sqrt{\frac{n+1}{n^3+1}} = \sqrt{\frac{n\left(1+\frac{1}{n}\right)}{n^3\left(1+\frac{1}{n^3}\right)}} = \frac{1}{n}\cdot\sqrt{\frac{1+\frac{1}{n}}{1+\frac{1}{n^3}}}$$

Take $v_n = \frac{1}{n}$, $\lim\limits_{n\to\infty}\frac{u_n}{v_n} = \lim\limits_{n\to\infty}\sqrt{\frac{1+\frac{1}{n}}{1+\frac{1}{n^3}}} = 1$ which is finite and $\neq 0$.

$\therefore$ Σu_n and Σv_n behave alike.

Since $\sum v_n = \sum \frac{1}{n}$ is of the form $\sum \frac{1}{n^p}$ with $p = 1$.

$\therefore$ Σv_n diverges $\Rightarrow$ Σu_n diverges.

Hence the given series Σu_n converges if $x < 1$ and diverges if $x \geq 1$.

(*iv*) Here $$u_n = \sqrt{\frac{n}{n^2+1}}\, x^n$$

$\therefore$ $$u_{n+1} = \sqrt{\frac{n+1}{(n+1)^2+1}}\cdot x^{n+1}$$

$$\frac{u_n}{u_{n+1}} = \sqrt{\frac{n}{n+1}\cdot\frac{n^2+2n+2}{n^2+1}}\cdot\frac{1}{x} = \sqrt{\frac{1}{1+\frac{1}{n}}\cdot\frac{1+\frac{2}{n}+\frac{2}{n^2}}{1+\frac{1}{n^2}}}\cdot\frac{1}{x}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\sqrt{\frac{1}{1+\frac{1}{n}}\cdot\frac{1+\frac{2}{n}+\frac{2}{n^2}}{1+\frac{1}{n^2}}}\cdot\frac{1}{x} = \frac{1}{x}$$

$\therefore$ By D' Alembert's Ratio Test,

Σu_n converges if $\frac{1}{x} > 1$ *i.e.*, $x < 1$ and diverges if $\frac{1}{x} < 1$ *i.e.*, $x > 1$

When $x = 1$, the Ratio Test fails.

When $x = 1$, $$u_n = \sqrt{\frac{n}{n^2+1}} = \sqrt{\frac{n}{n^2\left(1+\frac{1}{n^2}\right)}} = \frac{1}{\sqrt{n}}\cdot\frac{1}{\sqrt{1+\frac{1}{n^2}}}$$

Take $v_n = \frac{1}{\sqrt{n}}$, $\lim\limits_{n\to\infty}\frac{u_n}{v_n} = \lim\limits_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n^2}}} = 1$ which is finite and $\neq 0$.

$\therefore$ By Comparison Test, Σu_n and Σv_n behave alike.

Since $\sum v_n = \sum \frac{1}{\sqrt{n}}$ is of the form $\sum \frac{1}{n^p}$ with $p = \frac{1}{2} < 1$

Σv_n diverges $\Rightarrow$ Σu_n diverges.

Hence the given series Σu_n converges if $x < 1$ and diverges if $x \geq 1$.

(*v*) *Please try yourself.* [**Ans.** Convergent if $x < 1$ and divergent if $x \geq 1$]

(*vi*) Here $$u_n = \frac{n}{n^2+1}\cdot x^n, x > 0$$

$\therefore$ $$u_{n+1} = \frac{n+1}{(n+1)^2+n}\cdot x^{n+1}$$

$$\frac{u_n}{u_{n+1}} = \frac{n}{n+1}\cdot\frac{n^2+2n+2}{n^2+1}\cdot\frac{1}{x} = \frac{1}{1+\frac{1}{n}}\cdot\frac{1+\frac{2}{n}+\frac{2}{n^2}}{1+\frac{1}{n^2}}\cdot\frac{1}{x}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{1}{1+\frac{1}{n}}\cdot\frac{1+\frac{2}{n}+\frac{2}{n^2}}{1+\frac{1}{n^2}}\cdot\frac{1}{x} = \frac{1}{x}$$

$\therefore$ By D' Alembert's Ratio Test,

$\sum u_n$ converges if $\frac{1}{x} > 1$ *i.e.*, $x < 1$ and diverges if $\frac{1}{x} < 1$ *i.e.*, $x > 1$.

When $x = 1$, the Ratio Test fails.

However, when $x = 1$, $u_n = \frac{n}{n^2+1} = \frac{n}{n^2\left(1+\frac{1}{n^2}\right)} = \frac{1}{n\left(1+\frac{1}{n^2}\right)}$

Take $v_n = \frac{1}{n}$ so that $\frac{u_n}{v_n} = \frac{1}{1+\frac{1}{n^2}}$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \frac{1}{1+0} = 1 \text{ which is finite and } \neq 0.$$

By Comparison Test, Σu_n and Σv_n behave alike.

Since $\sum v_n = \sum \frac{1}{n}$ $(p = 1)$ is divergent.

$\therefore$ Σu_n is divergent.

Hence Σu_n is convergent if $x < 1$ and divergent if $x \geq 1$.

Example 7. *Test the convergence or divergence of the following series :*

(*i*) $1 + 2x + 3x^2 + 4x^3 + \ldots\ldots$ (*ii*) $x + \frac{x^2}{4} + \frac{x^3}{9} + \frac{x^4}{16} + \ldots\ldots$

(*iii*) $\frac{1}{3} + \frac{x}{36} + \frac{x^2}{243} + \ldots\ldots + \frac{x^{n-1}}{x^n . n^2} + \ldots\ldots (x > 0)$

(*iv*) $1 + \frac{x}{2} + \frac{x^2}{5} + \frac{x^3}{10} + \ldots\ldots + \frac{x^n}{n^2+1} + \ldots\ldots$

(*v*) $1 + 3x + 5x^2 + 7x^3 + 9x^4 + \ldots\ldots$

(*vi*) $a + (a+d)x + (a+2d)x^2 + (a+3d)x^3 + \ldots\ldots$

Sol. (*i*) Here $u_n = nx^{n-1}$ $\therefore$ $u_{n+1} = (n+1)x^n$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \frac{1}{x}$$

$\therefore$ $\sum u_n$ converges if $\frac{1}{x} > 1$ *i.e.*, $x < 1$ and diverges if $\frac{1}{x} < 1$ *i.e.*, $x > 1$.

When $x = 1$, $u_n = \frac{1}{n^{-1}}$, $p = -1 < 1$ $\therefore$ Σu_n diverges

Hence Σu_n converges if $x < 1$ and diverges if $x \geq 1$.

(*ii*) Here $u_n = \frac{x^n}{n^2}$

Proceed further yourself. **[Ans.** Convergent if $x < 1$ and divergent if $x \geq 1$]

(*iii*) Here $u_n = \frac{x^{n-1}}{3^n . n^2}$. Proceed further yourself. [See Ex. 6 (*i*)]

[Ans. Convergent if $x \leq 3$ and divergent if $x > 3$]

(*iv*) Leaving the first term, $u_n = \frac{x}{n^2+1}$. **[Ans.** Convergent if $x \leq 1$ and divergent if $x > 1$]

(*v*) Here $u_n = [1 + (n-1).2]x^{n-1} = (2n-1)x^{n-1}$

$u_{n+1} = [2(n+1) - 1]x^n = (2n+1)x^n$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}} \frac{2n-1}{2n+1} . \frac{1}{x} = \underset{n\to\infty}{\text{Lt}} \frac{2 - \frac{1}{n}}{2 + \frac{1}{n}} . \frac{1}{x} = \frac{1}{x}$$

$\therefore$ $\sum u_n$ converges if $\frac{1}{x} > 1$ *i.e.*, $x < 1$ and diverges if $\frac{1}{x} < 1$ *i.e.* $x > 1$.

When $x = 1$, $u_n = 2n - 1$ $\underset{n\to\infty}{\text{Lt}} u_n = \infty \neq 0$.

$\Rightarrow$ Σu_n does not converge. Being a series of +ve terms, it diverges.

Hence Σu_n is convergent if $x < 1$ and divergent if $x \geq 1$.

(*vi*) Here $u_n = [a + (n-1)d]x^{n-1}$. $u_{n+1} = (a + nd)x^n$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}} \frac{a+(n-1)d}{a+nd} . \frac{1}{x} = \underset{n\to\infty}{\text{Lt}} \frac{\frac{a}{n} + \left(1 - \frac{1}{n}\right)d}{\frac{a}{n} + d} . \frac{1}{x} = \frac{d}{d} . \frac{1}{x} = \frac{1}{x}$$

$\therefore$ $\sum u_n$ converges if $\frac{1}{x} > 1$ *i.e.*, $x < 1$ and diverges if $\frac{1}{x} < 1$ *i.e.*, $x > 1$.

When $x = 1$, $u_n = a + (n-1)d$ $\underset{n\to\infty}{\text{Lt}} u_n = \infty \neq 0$

$\Rightarrow$ Σu_n does not converge. Being a series of +ve terms, it diverges.

Hence Σu_n is convergent if $x < 1$ and divergent if $x \geq 1$.

Example 8. *Examine the convergence or divergence of the following series :*

(*i*) $\frac{x}{1.3} + \frac{x^2}{3.5} + \frac{x^3}{5.7} + \ldots\ldots$ (*ii*) $\frac{x}{1.2} + \frac{x^2}{2.3} + \frac{x^3}{3.4} + \ldots\ldots$

(*iii*) $\frac{1}{1.2.3} + \frac{x}{4.5.6} + \frac{x^2}{7.8.9} + \ldots\ldots$ (*iv*) $x + \frac{3}{5}x^2 + \frac{8}{10}x^3 + \frac{15}{17}x^4 + \ldots\ldots + \frac{n^2-1}{n^2+1}x^n + \ldots\ldots$

(*v*) $\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \frac{x^6}{5\sqrt{4}} + \ldots\ldots$

Sol. (*i*) Here $u_n = \frac{x^n}{[1+(n-1).2][3+(n-1).2]} = \frac{x^n}{(2n-1)(2n+1)}$

$\therefore$ $u_{n+1} = \frac{x^{n+1}}{(2n+1)(2n+3)}$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}} \frac{2n+3}{2n-1} . \frac{1}{x} = \underset{n\to\infty}{\text{Lt}} \frac{2 + \frac{3}{n}}{2 - \frac{1}{n}} . \frac{1}{x} = \frac{1}{x}$$

$\therefore$ By D' Alembert's Ratio Test, $\sum u_n$ converges if $\frac{1}{x} > 1$ *i.e.*, $x < 1$ and diverges if $\frac{1}{x} < 1$, *i.e.*, $x > 1$.

If $x = 1$, $u_n = \frac{1}{(2n-1)(2n+1)} = \frac{1}{4n^2 - 1} = \frac{1}{n^2\left(4 - \frac{1}{n^2}\right)}$

Take $v_n = \frac{1}{n^2}$, $\underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}} \frac{1}{4 - \frac{1}{n^2}} = \frac{1}{4}$ which is finite and $\neq 0$.

$\therefore$ $\sum u_n$ and $\sum v_n$ behave alike.

Since $\sum v_n = \sum \frac{1}{n^2}$ is of the form $\sum \frac{1}{n^p}$ with $p = 2 > 1$, $\sum v_n$ converges.

$\therefore \quad \sum u_n$ converges.

Hence $\sum u_n$ converges if $x \le 1$ and diverges if $x > 1$.

(*ii*) *Please try yourself.* **[Ans.** Convergent if $x \le 1$ and divergent if $x > 1$]

(*iii*) Here $\quad u_n = \dfrac{x^{n-1}}{(3n-2)(3n-1)3n}$ $\quad$ [$\because$ T_n of 1, 4, 7, ... $= 1 + (n-1).3 = 3n - 2$]

$$u_{n+1} = \frac{x^n}{(3n+1)(3n+2)(3n+3)}$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}} \frac{(3n+1)(3n+2)(3n+3)}{(3n-2)(3n-1)3n} \cdot \frac{1}{x}$$

$$= \underset{n\to\infty}{\text{Lt}} \frac{\left(3+\frac{1}{n}\right)\left(3+\frac{2}{n}\right)\left(3+\frac{3}{n}\right)}{\left(3-\frac{2}{n}\right)\left(3-\frac{1}{n}\right)(3)} \cdot \frac{1}{x} = \frac{1}{x} \text{ etc.}$$

When $x = 1$, $\quad u_n = \dfrac{1}{(3n-2)(3n-1)3n} = \dfrac{1}{n^3} \cdot \dfrac{1}{\left(3-\frac{2}{n}\right)\left(3-\frac{1}{n}\right).3}$

Take $\quad v_n = \dfrac{1}{n^3}$ etc.

$\therefore \quad \Sigma u_n$ converges if $x \le 1$ and diverges if $x > 1$.

(*iv*) *Please try yourself.* **[Ans.** Convergent if $x < 1$ and divergent if $x \ge 1$]

(*v*) Here $\quad u_n = \dfrac{x^{2n-2}}{(n+1)\sqrt{n}} \quad \therefore \quad u_{n+1} = \dfrac{x^{2n}}{(n+2)\sqrt{n+1}}$

$$\frac{u_n}{u_{n+1}} = \frac{(n+2)\sqrt{n+1}}{(n+1)\sqrt{n}} \cdot \frac{1}{x^2} = \frac{1+\frac{2}{n}}{1+\frac{1}{n}} \cdot \sqrt{1+\frac{1}{n}} \cdot \frac{1}{x^2}$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \frac{1}{x^2}$$

By D' Alembert's Ratio Test, $\sum u_n$ converges if $\dfrac{1}{x^2} > 1$ *i.e.*, $x^2 < 1$ and diverges if $\dfrac{1}{x^2} < 1$ *i.e.*, $x^2 > 1$.

When $x^2 = 1$, $\quad u_n = \dfrac{1}{(n+1)\sqrt{n}} = \dfrac{1}{n^{3/2}\left(1+\frac{1}{n}\right)}$

Take $\quad v_n = \dfrac{1}{n^{3/2}}$ etc. Σu_n is convergent $\quad (\because \; p > 1)$

Hence Σu_n is convergent if $x^2 \le 1$ and divergent if $x^2 > 1$.

Example 9. *Examine the covergence or divergence of the following series :*

(*i*) $1 + \dfrac{2}{5}x + \dfrac{6}{9}x^2 + \dfrac{14}{17}x^3 + \ldots\ldots + \dfrac{2^{n+1}-2}{2^{n+1}+1}x^n + \ldots\ldots$

(*ii*) $\dfrac{x^2}{2\sqrt{1}} + \dfrac{x^3}{3\sqrt{2}} + \dfrac{x^4}{4\sqrt{3}} + \ldots\ldots (x > 0)$

(*iii*) $\dfrac{x}{2\sqrt{3}} + \dfrac{x^2}{3\sqrt{4}} + \dfrac{x^3}{4\sqrt{5}} + \ldots\ldots$ $\qquad$ (*iv*) $\dfrac{1}{4} + \dfrac{7}{10}x + \dfrac{25}{28}x^2 + \ldots\ldots + \dfrac{3^n-2}{3^n+1}x^{n-1} + \ldots\ldots$

Sol. (*i*) Here, leaving the first term,

$$u_n = \frac{2^{n+1}-2}{2^{n+1}+2}x^n \quad \therefore \quad u_{n+1} = \frac{2^{n+2}-2}{2^{n+2}+2}x^{n+1}$$

$$\frac{u_n}{u_{n+1}} = \frac{2^{n+1}-2}{2^{n+1}+2} \cdot \frac{2^{n+2}+2}{2^{n+2}-2} \cdot \frac{1}{x} = \frac{2^{n+1}\left(1-\frac{2}{2^{n+1}}\right)}{2^{n+1}\left(1+\frac{2}{2^{n+1}}\right)} \cdot \frac{2^{n+2}\left(1+\frac{2}{2^{n+2}}\right)}{2^{n+2}\left(1-\frac{2}{2^{n+2}}\right)} \cdot \frac{1}{x}$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}} \frac{1-\frac{1}{2^n}}{1+\frac{1}{2^n}} \cdot \frac{1+\frac{1}{2^{n+1}}}{1-\frac{1}{2^{n+1}}} \cdot \frac{1}{x} = \frac{1}{x}$$

$\therefore \quad$ By D' Alembert's Ratio Test,

$\sum u_n$ converges if $\dfrac{1}{x} > 1$ *i.e.*, $x < 1$ and diverges if $\dfrac{1}{x} < 1$ *i.e.*, $x > 1$.

When $x = 1$, $\quad u_n = \dfrac{2^{n+1}-2}{2^{n+1}+2} = \dfrac{2^{n+1}\left(1-\frac{2}{2^{n+1}}\right)}{2^{n+1}\left(1+\frac{2}{2^{n+2}}\right)} = \dfrac{1-\frac{1}{2^n}}{1+\frac{1}{2^n}}$

$\underset{n\to\infty}{\text{Lt}} u_n = 1 \ne 0 \quad \Rightarrow \quad \Sigma u_n$ does not converge. Being a series of +ve terms, it must diverge.

Hence Σu_n is convergent if $x < 1$ and divergent if $x \ge 1$.

(*ii*) *Please try yourself.* **[Ans.** Convergent if $x \le 1$ and divergent if $x > 1$]

(*iii*) *Please try yourself.* **[Ans.** Convergent if $x \le 1$ and divergent if $x > 1$]

(*iv*) *Please try yourself.* **[Ans.** Convergent if $x < 1$ and divergent if $x \ge 1$]

Example 10. *Test for convergence the positive term series :*

$$1 + \frac{\alpha+1}{\beta+1} + \frac{(\alpha+1)(2\alpha+1)}{(\beta+1)(2\beta+1)} + \frac{(\alpha+1)(2\alpha+1)(3\alpha+1)}{(\beta+1)(2\beta+1)(3\beta+1)} + \ldots\ldots$$

Sol. Leaving the first term

$$u_n = \frac{(\alpha+1)(2\alpha+1)\ldots\ldots(n\alpha+1)}{(\beta+1)(2\beta+1)\ldots\ldots(n\beta+1)}$$

$$\therefore \quad u_{n+1} = \frac{(\alpha+1)(2\alpha+1)\ldots\ldots(n\alpha+1)[(n+1)\alpha+1]}{(\beta+1)(2\beta+1)\ldots\ldots(n\beta+1)[(n+1)\beta+1]}$$

$$\frac{u_n}{u_{n+1}} = \frac{(n+1)\beta+1}{(n+1)\alpha+1} = \frac{\left(1+\frac{1}{n}\right)\beta+\frac{1}{n}}{\left(1+\frac{1}{n}\right)\alpha+\frac{1}{n}}$$

$$\operatorname{Lt}_{n\to\infty} \frac{u_n}{u_{n+1}} = \operatorname{Lt}_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)\beta+\frac{1}{n}}{\left(1+\frac{1}{n}\right)\alpha+\frac{1}{n}} = \frac{\beta}{\alpha}$$

$\therefore$ By D'Alembert's Ratio Test

Σu_n converges if $\frac{\beta}{\alpha} > 1$, *i.e.*, $\beta > \alpha > 0$ and diverges if $\frac{\beta}{\alpha} < 1$, *i.e.* $\beta < \alpha$ or $\alpha > \beta > 0$

When $\alpha = \beta$, the Ratio Test fails.

When $\alpha = \beta$, $u_n = 1$, $\operatorname{Lt}_{n\to\infty} u_n = 1 \neq 0$

$\Rightarrow$ Σu_n does not converge. Being a series of +ve terms, it must diverge.

Hence the given series is convergent if $\beta > \alpha > 0$ and divergent if $\alpha \geq \beta > 0$.

Example 11. *Test the following series for convergence :*

(*i*) $1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots, x > 0$

(*ii*) $1 + \frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + \ldots\ldots, x > 0.$

Sol. (*i*) Leaving the first term

$$u_n = \frac{x^n}{n!} \quad \therefore \quad u_{n+1} = \frac{x^{n+1}}{(n+1)!}$$

$$\frac{u_n}{u_{n+1}} = \frac{(n+1)!}{n!} \cdot \frac{1}{x} = \frac{n+1}{x}$$

$$\operatorname{Lt}_{n\to\infty} \frac{u_n}{u_{n+1}} = \operatorname{Lt}_{n\to\infty} \frac{n+1}{x} = \infty > 1$$

$\therefore$ By Ratio Test, the given series is convergent.

(*ii*) Leaving the first term $u_n = \frac{x^{2n}}{2n}$ $\therefore$ $u_{n+1} = \frac{x^{2n+2}}{2n+2}$

$$\frac{u_n}{u_{n+1}} = \frac{2n+2}{2n} \cdot \frac{1}{x^2} = \left(1+\frac{1}{n}\right) \cdot \frac{1}{x^2}$$

$$\operatorname{Lt}_{n\to\infty} \frac{u_n}{u_{n+1}} = \operatorname{Lt}_{n\to\infty} \left(1+\frac{1}{n}\right) \cdot \frac{1}{x^2} = \frac{1}{x^2}$$

$\therefore$ By Ratio Test, the series is convergent if $\frac{1}{x^2} > 1$ *i.e.*, if $x^2 < 1$ *i.e.*, if $x < 1$

The series is divergent if $\frac{1}{x^2} < 1$ *i.e.*, if $x^2 > 1$ *i.e.*, if $x > 1$

When $x^2 = 1$ *i.e.*, $x = 1$, the Ratio Test fails.

However, when $x = 1$, the series becomes $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \ldots..$

Leaving the first term, $u_n = \frac{1}{2n}$

Taking $\quad v_n = \frac{1}{n}, \frac{u_n}{v_n} = \frac{1}{2}$

$$\operatorname{Lt}_{n\to\infty} \frac{u_n}{v_n} = \frac{1}{2} \text{ which is finite and non-zero.}$$

$\therefore$ By Comparison Test, the series Σu_n and Σv_n converge or diverge together.

But the series $\Sigma v_n = \Sigma \frac{1}{n}$ diverges (by *p*-series test)

$\therefore$ Σu_n diverges.

Hence Σu_n converges when $x < 1$ and diverges when $x \geq 1$.

Example 12. *Test for convergence the following series :*

(*i*) $\frac{1}{2} + \frac{1.3}{2.5} + \frac{1.3.5}{2.5.8} + \ldots\ldots$ (*ii*) $\frac{1}{2} + \frac{2!}{2^3} + \frac{3!}{2^5} + \frac{4!}{2^7} + \ldots\ldots$

(*iii*) $\sum_{n=1}^{\infty} \frac{5^n}{n^2+7}$ (*iv*) $\sum_{n=1}^{\infty} \frac{1}{5^n+k}, k > 0$

(*v*) $\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^{1/n}$ (*vi*) $\frac{a}{3} + \frac{a^2}{6} + \frac{a^3}{11} + \ldots\ldots + \frac{a^n}{n^2+2} + \ldots..(a > 0)$

(*vii*) $\frac{a}{6} + \frac{5a^2}{10} + \frac{13a^3}{18} + \ldots\ldots + \frac{2^{n+1}-3}{2^{n+1}+2}a^n + \ldots..(a > 0)$

(*viii*) $\frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\ldots (x > 0)$ (*ix*) $\sum_{n=1}^{\infty} x^n \cos\frac{1}{n}$

Sol. (*i*) Here $\quad u_n = \frac{1.3.5\ldots\ldots(2n-1)}{2.5.8\ldots\ldots(3n-1)}$, $u_{n+1} = \frac{1.3.5\ldots\ldots(2n-1)(2n+1)}{2.5.8\ldots\ldots(3n-1)(3n+2)}$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{3n+2}{2n+1} = \frac{3+\frac{2}{n}}{2+\frac{1}{n}}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{3}{2} > 1$$

$\therefore$ By ratio test, the series is convergent.

(*ii*) Here $\quad u_n = \dfrac{n!}{2^{2n-1}}, \quad u_{n+1} = \dfrac{(n+1)!}{2^{2n+1}}$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{n!}{(n+1)!} \cdot \frac{2^{2n+1}}{2^{2n-1}} = \frac{4}{n+1}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = 0 < 1$$

$\therefore$ By ratio test, the series is divergent.

(*iii*) Here $\quad u_n = \dfrac{5^n}{n^2+7}, \quad u_{n+1} = \dfrac{5^{n+1}}{(n+1)^2+7}$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{5^n}{5^{n+1}} \cdot \frac{(n+1)^2+7}{n^2+7} = \frac{n^2+2n+8}{5(n^2+7)} = \frac{1+\frac{2}{n}+\frac{8}{n^2}}{5\left(1+\frac{7}{n^2}\right)}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{5} < 1$$

$\therefore$ By ratio test, the series is divergent.

(*iv*) Here $\quad u_n = \dfrac{1}{5^n+k}, \quad u_{n+1} = \dfrac{1}{5^{n+1}+k}$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{5^{n+1}+k}{5^n+k} = \frac{5+\frac{k}{5^n}}{1+\frac{k}{5^n}}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = 5 > 1$$

$\therefore$ By ratio test, the series is convergent.

(*v*) Here $\quad u_n = \left(\dfrac{1}{n}\right)^{1/n} = \dfrac{1}{n^{1/n}}$

$$\lim_{n\to\infty} u_n = 1, \text{ since } \lim_{n\to\infty} n^{1/n} = 1$$

Since $\lim\limits_{n\to\infty} u_n \neq 0$, the series of positive terms is divergent.

(*vi*) Here $\quad u_n = \dfrac{a^n}{n^2+2}, \quad u_{n+1} = \dfrac{a^{n+1}}{(n+1)^2+2}$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{a^n}{a^{n+1}} \cdot \frac{(n+1)^2+2}{n^2+2} = \frac{n^2+2n+3}{a(n^2+2)} = \frac{1+\frac{2}{n}+\frac{3}{n^2}}{a\left(1+\frac{2}{n^2}\right)}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{a}$$

$\therefore$ By ratio test, Σu_n is convergent if $\dfrac{1}{a} > 1$ *i.e.*, if $a < 1$ and divergent if $\dfrac{1}{a} < 1$, *i.e.*, if $a > 1$.

When $a = 1$, ratio test fails.

Now, when $a = 1$, $\quad u_n = \dfrac{1}{n^2+2}$

Take $v_n = \dfrac{1}{n^2}$ so that Σv_n is convergent $\qquad (\because \; p = 2 > 1)$

$$\frac{u_n}{v_n} = \frac{n^2}{n^2+2} = \frac{1}{1+\frac{2}{n^2}}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = 1 \text{ which is non-zero, finite.}$$

$\therefore$ By comparison test, Σu_n and Σv_n converge or diverge together. Since Σv_n is convergent, therefore, Σu_n is convergent.

Hence Σu_n is convergent if $0 < a \le 1$ and divergent if $a > 1$.

(*vii*) Here $\quad u_n = \dfrac{2^{n+1}-3}{2^{n+1}+2} a^n, \quad u_{n+1} = \dfrac{2^{n+2}-3}{2^{n+2}+2} a^{n+1}$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{2^{n+1}-3}{2^{n+2}-3} \times \frac{2^{n+2}+2}{2^{n+1}+2} \cdot \frac{1}{a} = \frac{1-\frac{3}{2^{n+1}}}{2-\frac{3}{2^{n+1}}} \times \frac{2+\frac{2}{2^{n+1}}}{1+\frac{2}{2^{n+1}}} \cdot \frac{1}{a}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{2} \times \frac{2}{1} \times \frac{1}{a} = \frac{1}{a}$$

$\therefore$ By ratio test, Σu_n is convergent if $\dfrac{1}{a} > 1$, *i.e.*, if $a < 1$ and divergent if $\dfrac{1}{a} < 1$, *i.e.*, if $a > 1$.

When $a = 1$, ratio test fails.

Now, when $a = 1$, $\quad u_n = \dfrac{2^{n+1}-3}{2^{n+1}+2} = \dfrac{1-\frac{3}{2^{n+1}}}{1+\frac{2}{2^{n+1}}}$

$$\lim_{n\to\infty} u_n = 1 \neq 0$$

$\therefore$ Σu_n is divergent.

Hence Σu_n is convergent if $0 < a < 1$ and divergent if $a \geq 1$.

(*viii*) Here $$u_n = \frac{x^{2n-1}}{(2n-1)!}$$

$$u_{n+1} = \frac{x^{2n+1}}{(2n+1)!}$$

$\therefore$ $$\frac{u_n}{u_{n+1}} = \frac{x^{2n-1}}{x^{2n+1}} \cdot \frac{(2n+1)!}{(2n-1)!} = \frac{(2n+1)(2n)}{x^2}$$

$$\lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \infty$$

$\therefore$ By ratio test, the series is convergent.

(*ix*) Here $$u_n = x^n \cos \frac{1}{n}$$

$$u_{n+1} = x^{n+1} \cos \frac{1}{n+1}$$

$\therefore$ $$\frac{u_n}{u_{n+1}} = \frac{1}{x} \cdot \frac{\cos \frac{1}{n}}{\cos \frac{1}{n+1}}$$

$$\lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \frac{1}{x}$$

$\therefore$ By ratio test, Σu_n is convergent if $\frac{1}{x} > 1$, *i.e.* if $x < 1$ and divergent if $\frac{1}{x} < 1$, *i.e.*, if $x > 1$.

When $x = 1$, ratio test fails.

Now, when $x = 1$, $\quad u_n = \cos \frac{1}{n}$

$$\lim_{n \to \infty} u_n = 1 \neq 0$$

$\therefore$ Σu_n is divergent.

Hence Σu_n is convergent if $x < 1$ and divergent if $x \geq 1$.

Example 13. *Test for convergence the series :*

$$\sum_{n=1}^{\infty} \frac{1}{3^n + x}, \; x > 0.$$

Sol. Here $\quad u_n = \dfrac{1}{3^n + x} \quad \therefore \quad u_{n+1} = \dfrac{1}{3^{n+1} + x}$

$$\frac{u_n}{u_{n+1}} = \frac{3^{n+1} + x}{3^n + x} = \frac{3^n\left(3 + \frac{x}{3^n}\right)}{3^n\left(1 + \frac{x}{3^n}\right)} = \frac{3 + \frac{x}{3^n}}{1 + \frac{x}{3^n}}$$

$$\lim_{n \to \infty} \frac{u_n}{u_{n+1}} = 3 > 1$$

$\therefore$ By ratio test, Σu_n is convergent.

Article 13. Cauchy's Root Test

Statement. *If Σu_n is a series of positive terms such that*

(*a*) $\lim_{n \to \infty} (u_n)^{1/n} = l$, *then*

(*i*) *Σu_n is convergent if $l < 1$* $\qquad$ (*ii*) *Σu_n is divergent if $l > 1$*

(*iii*) *Σu_n may converge or diverge if $l = 1$ (i.e., the test fails if $l = 1$)*

(*b*) $\lim_{n \to \infty} (u_n)^{1/n} = \infty$, *then Σu_n is divergent.*

Proof. (*a*) Σu_n is a series of positive terms

$\Rightarrow$ $\quad u_n > 0 \;\; \forall \, n$

$\Rightarrow$ $(u_n)^{1/n}$ is the positive nth root of u_n

$\Rightarrow$ $\quad \lim_{n \to \infty} (u_n)^{1/n} = l \geq 0$

Since $\lim_{n \to \infty} (u_n)^{1/n} = l$, therefore, for each $\varepsilon > 0$, $\exists$ a positive integer m such that

$$|(u_n)^{1/n} - l| < \varepsilon \qquad \forall \, n \geq m$$

$\Rightarrow$ $$l - \varepsilon < (u_n)^{1/n} < l + \varepsilon \qquad \forall \, n \geq m$$

$\Rightarrow$ $$(l - \varepsilon)^n < u_n < (l + \varepsilon)^n \qquad \forall \, n \geq m \qquad \ldots(1)$$

(*i*) Let $l < 1$

Choose $\varepsilon > 0$ such that $\quad r = l + \varepsilon < 1$

From (1), we have $\quad u_n < r^n \quad \forall \, n \geq m$

Now, $\sum_{n=1}^{\infty} r^n$ being a geometric series with common ratio $r < 1$ is convergent. Therefore, by comparison test, the series $\sum_{n=1}^{\infty} u_n$ is convergent.

(*ii*) Let $l > 1$

Choose $\varepsilon > 0$ such that $\quad R = l - \varepsilon > 1$

From (1), we have $\quad u_n > R^n \quad \forall \, n \geq m$

Now, $\sum_{n=1}^{\infty} R^n$ being a geometric series with common ratio $R > 1$ is divergent. Therefore, by comparison test, the series $\sum_{n=1}^{\infty} u_n$ is divergent.

(*iii*) Let $l = 1$

First consider the series $\sum \frac{1}{n}$

$$u_n = \frac{1}{n},\ \lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty}\left(\frac{1}{n}\right)^{1/n} = 1$$

Since the series $\sum \frac{1}{n}$ is divergent, we find that if $l = 1$, a series may diverge.

Next consider the series $\sum \frac{1}{n^2}$

$$u_n = \frac{1}{n^2},\ \lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty}\left(\frac{1}{n^2}\right)^{1/n} = \lim_{n\to\infty}\left(\frac{1}{n^{1/n}}\right)^2 = 1$$

Since the series $\sum \frac{1}{n^2}$ is convergent, we find that if $l = 1$, a series may converge.

The above two examples show that if $l = 1$, a series may converge or diverge. Hence the test fails when $l = 1$.

(*b*) $\lim_{n\to\infty} (u_n)^{1/n} = \infty$

Then there exists a positive integer m such that $(u_n)^{1/n} > 2 \quad \forall\, n \geq m$

$\Rightarrow \quad u_n > 2^n \quad \forall\, n \geq m$

Now, $\sum_{n=1}^{\infty} 2^n$ being a geometric series with common ratio $2 > 1$ is divergent. Therefore, by comparison test, the series $\sum_{n=1}^{\infty} u_n$ is divergent.

ILLUSTRATIVE EXAMPLES

Example 1. *Test the convergence of the following series :*

(*i*) $\sum \left(\frac{n}{n+1}\right)^{n^2}$ or $\sum \left(1+\frac{1}{n}\right)^{-n^2}$

(*ii*) $\sum \frac{x^n}{n^n}$ (*iii*) $\sum_{n=2}^{\infty} \frac{1}{(\log n)^n}$

(*iv*) $\sum n^n x^n\ (x > 0)$ (*v*) $\sum \left(\frac{n+1}{3n}\right)^n$

Sol. (*i*) Here $u_n = \left(\frac{n}{n+1}\right)^{n^2}$

$$\therefore \quad (u_n)^{1/n} = \left[\left(\frac{n}{n+1}\right)^{n^2}\right]^{1/n} = \left(\frac{n}{n+1}\right)^n = \left(\frac{n+1}{n}\right)^{-n} = \left[\left(1+\frac{1}{n}\right)^n\right]^{-1}$$

$$\operatorname*{Lt}_{n\to\infty} (u_n)^{1/n} = \operatorname*{Lt}_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^n\right]^{-1} = e^{-1} = \frac{1}{e} < 1 \qquad (\because\ 2 < e < 3)$$

$\therefore$ By Cauchy's Root Test, the given series Σu_n is convergent.

(*ii*) **Hint.** $\operatorname*{Lt}_{n\to\infty} (u_n)^{1/n} = \operatorname*{Lt}_{n\to\infty} \frac{x}{n} = 0 < 1.$ **[Ans. Convergent]**

(*iii*) **Hint.** $\operatorname*{Lt}_{n\to\infty} (u_n)^{1/n} = \operatorname*{Lt}_{n\to\infty} \frac{1}{\log n} = 0 < 1.$ **[Ans. Convergent]**

(*iv*) **Hint.** $\operatorname*{Lt}_{n\to\infty} (u_n)^{1/n} = \operatorname*{Lt}_{n\to\infty} nx = \infty\ (\because\ x > 0)$ **[Ans. Divergent]**

(*v*) **Hint.** $\operatorname*{Lt}_{n\to\infty} (u_n)^{1/n} = \operatorname*{Lt}_{n\to\infty} \frac{n+1}{3n} = \operatorname*{Lt}_{n\to\infty} \frac{1}{3}\left(1+\frac{1}{n}\right) = \frac{1}{3} < 1.$ **[Ans. Convergent]**

Example 2. *Examine the convergence of the following series :*

(*i*) $\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4}+\frac{4}{3}\right)^{-3} + \ldots\ldots$

(*ii*) $\sum \frac{(n-\log n)^n}{2^2 . n^n}$ (*iii*) $\frac{1}{2}+\frac{2}{3}x+\left(\frac{3}{4}\right)^2 x^2 + \left(\frac{4}{5}\right)^3 x^3 + \ldots\ldots$

(*iv*) $\sum \left(\frac{nx}{n+1}\right)^n$ (*v*) $\sum \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$

Sol. (*i*) Here $u_n = \left[\left(\frac{n+1}{n}\right)^{n+1} - \frac{n+1}{n}\right]^{-n}$

$$(u_n)^{1/n} = \left[\left(\frac{n+1}{n}\right)^{n+1} - \frac{n+1}{n}\right]^{-1} = \left[\left(1+\frac{1}{n}\right)^{n+1} - \left(1+\frac{1}{n}\right)\right]^{-1}$$

$$= \left[\left(1+\frac{1}{n}\right)^n . \left(1+\frac{1}{n}\right) - \left(1+\frac{1}{n}\right)\right]^{-1}$$

$$\operatorname*{Lt}_{n\to\infty} (u_n)^{1/n} = \operatorname*{Lt}_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^n . \left(1+\frac{1}{n}\right) - \left(1+\frac{1}{n}\right)\right]^{-1} = (e.1-1)^{-1} = \frac{1}{e-1} < 1$$

$$\left(\because\ e > 2\ \therefore\ e-1 > 1 \Rightarrow \frac{1}{e-1} < 1\right)$$

$\therefore$ By Cauchy's Root Test, Σu_n is convergent.

(*ii*) Here $u_n = \frac{(n-\log n)^n}{2^n . n^n}$

$$\therefore \quad (u_n)^{1/n} = \frac{n-\log n}{2n} = \frac{1}{2}\left(1-\frac{\log n}{n}\right)$$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}} (u_n)^{1/n} = \underset{n\to\infty}{\text{Lt}} \frac{1}{2}\left(1-\frac{\log n}{n}\right) = \frac{1}{2}(1-0) = \frac{1}{2} < 1 \quad \left[\underset{n\to\infty}{\text{Lt}} \frac{\log n}{n} = \underset{n\to\infty}{\text{Lt}} \frac{\frac{1}{n}}{1} = 0\right]$$

$\therefore$ By Cauchy's Root Test, Σu_n is convergent.

(*iii*) Leaving the first term, $u_n = \left(\frac{n+1}{n+2}\right)^n . x^n$

$$\therefore \quad (u_n)^{1/n} = \frac{n+1}{n+2} . x = \frac{1+\frac{1}{n}}{1+\frac{2}{n}} . x$$

$$\underset{n\to\infty}{\text{Lt}} (u_n)^{1/n} = \underset{n\to\infty}{\text{Lt}} \frac{1+\frac{1}{n}}{1+\frac{2}{n}} . x = x$$

$\therefore$ By Cauchy's Root Test, Σu_n converges if $x < 1$ and diverges if $x > 1$.
When $x = 1$, the Root Test fails.

When $x = 1$,
$$u_n = \left(\frac{n+1}{n+2}\right)^n = \frac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{2}{n}\right)^n}$$

$$\underset{n\to\infty}{\text{Lt}} u_n = \underset{n\to\infty}{\text{Lt}} \frac{\left(1+\frac{1}{n}\right)^n}{\left[\left(1+\frac{2}{n}\right)^{n/2}\right]^2} = \frac{e}{e^2} = \frac{1}{e} \neq 0.$$

$\Rightarrow$ Σu_n does not converge. Being a series of +ve terms, it must diverge. Hence the given series is convergent if $x < 1$ and divergent if $x \geq 1$.

(*iv*) **Hint.**
$$u_n = \left(\frac{nx}{n+1}\right)^n \quad \therefore \quad (u_n)^{1/n} = \frac{nx}{n+1} = \frac{x}{1+\frac{1}{n}}$$

$$\underset{n\to\infty}{\text{Lt}} (u_n)^{1/n} = \underset{n\to\infty}{\text{Lt}} \frac{x}{1+\frac{1}{n}} = x \text{ etc.}$$

When $x = 1$,
$$\underset{n\to\infty}{\text{Lt}} u_n = \underset{n\to\infty}{\text{Lt}} \left(\frac{n}{n+1}\right)^n = \underset{n\to\infty}{\text{Lt}} \left(\frac{n+1}{n}\right)^{-n}$$

$$= \underset{n\to\infty}{\text{Lt}} \left[\left(1+\frac{1}{n}\right)^n\right]^{-1} = e^{-1} = \frac{1}{e} \neq 0, \text{ etc.}$$

[**Ans.** Convergent if $x < 1$, Divergent if $x \geq 1$]

(*v*) **Hint.**
$$\underset{n\to\infty}{\text{Lt}} (u_n)^{1/n} = \underset{n\to\infty}{\text{Lt}} \left[\left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}\right]^{1/n} = \underset{n\to\infty}{\text{Lt}} \left[\left(1+\frac{1}{\sqrt{n}}\right)^{-n\sqrt{n}}\right]^{1/n}$$

$$= \underset{n\to\infty}{\text{Lt}} \left(1+\frac{1}{\sqrt{n}}\right)^{-\sqrt{n}} = \underset{n\to\infty}{\text{Lt}} \left[\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}\right]^{-1}$$

$$= e^{-1} = \frac{1}{e} < 1.$$

[**Ans.** Convergent]

Example 3. *Test the convergence of the following series :*

(*i*) $\sum \frac{(1+nx)^n}{n^n}$ (*ii*) $\Sigma 5^{-n-(-1)^n}$.

Sol. (*i*) Here
$$u_n = \sum \frac{(1+nx)^n}{n^n} \quad \therefore \quad (u_n)^{1/n} = \frac{1+nx}{n} = \frac{1}{n} + x.$$

$$\underset{n\to\infty}{\text{Lt}} (u_n)^{1/n} = \underset{n\to\infty}{\text{Lt}} \left(\frac{1}{n}+x\right) = x \text{ etc.}$$

When $x = 1$,
$$u_n = \frac{(1+n)^n}{n^n} = \left(\frac{1+n}{n}\right)^n = \left(1+\frac{1}{n}\right)^n$$

$$\underset{n\to\infty}{\text{Lt}} u_n = \underset{n\to\infty}{\text{Lt}} \left(1+\frac{1}{n}\right)^n = e \neq 0 \text{ etc.}$$

[**Ans.** Convergent if $x < 1$ and divergent if $x \geq 1$].

(*ii*) Here
$$u_n = 5^{-n-(-1)^n}$$

$$\therefore \quad (u_n)^{1/n} = 5^{-1-\frac{(-1)^n}{n}}$$

$$\underset{n\to\infty}{\text{Lt}} (u_n)^{1/n} = 5^{-1} = \frac{1}{5} < 1$$

$\therefore$ By Cauchy's Root Test, Σu_n is convergent.

Example 4. *Discuss the convergence or divergence of the following series :*

(*i*) $\frac{1^3}{3} + \frac{2^3}{3^2} + 1 + \frac{4^3}{3^4} + \ldots\ldots$ (*ii*) $\frac{2}{1^2}x + \frac{3^2}{2^3}x^2 + \frac{4^3}{3^4}x^3 + \ldots\ldots$ *or* $\sum \frac{(n+1)^n x^n}{n^{n+1}}$.

Sol. (*i*) Here
$$u_n = \frac{n^3}{3^n} \quad \therefore \quad (u_n)^{1/n} = \frac{n^{3/n}}{3} = \frac{1}{3}(n^{1/n})^3$$

$$\underset{n\to\infty}{\text{Lt}} (u_n)^{1/n} = \underset{n\to\infty}{\text{Lt}} \frac{1}{3}(n^{1/n})^3 = \frac{1}{3} < 1 \qquad \left[\because \underset{n\to\infty}{\text{Lt}} n^{1/n} = 1\right]$$

$\therefore$ By Cauchy's Root Test, Σu_n is convergent.

Note. Try this question by using D' Alembert's Test.

(*ii*) Here
$$u_n = \frac{(n+1)^n}{n^{n+1}} x^n, \; (u_n)^{1/n} = \frac{n+1}{n^{1+1/n}} . x = \frac{n+1}{n . n^{1/n}} . x = \left(1+\frac{1}{n}\right) . \frac{x}{n^{1/n}} .$$

$$\underset{n\to\infty}{\text{Lt}} (u_n)^{1/n} = \underset{n\to\infty}{\text{Lt}} \left(1+\frac{1}{n}\right)\cdot\frac{x}{n^{1/n}} = x \text{ etc.} \qquad \left[\because \underset{n\to\infty}{\text{Lt}} n^{1/n} = 1\right]$$

When $x = 1$, $\quad u_n = \frac{(n+1)^n}{n^{n+1}} = \frac{(n+1)^n}{n\cdot n^n} = \frac{1}{n}\left(\frac{n+1}{n}\right)^n = \frac{1}{n}\left(1+\frac{1}{n}\right)^n$

Take $\quad v_n = \frac{1}{n}, \quad \underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}} \left(1+\frac{1}{n}\right)^n = e$

which is finite and $\neq 0$ etc. [**Ans.** Convergent if $x < 1$ and divergent if $x \geq 1$]

Example 5. *Discuss the convergence of the following series :*

(a) $\Sigma n^k x^n$ (b) $\Sigma q^{n^2} \cdot r^2 \quad q, r < 0$

(c) $\Sigma e^{\sqrt{n}} \cdot r^n, r > 0$ (d) $\Sigma (n^{1/n} - 1)^n$.

Sol. (a) Here $\quad u_n = n^k x^n \Rightarrow u_n^{1/n} = n^{k/n} \cdot x = (n^{1/n})^k \cdot x$

$$\underset{n\to\infty}{\text{Lt}} u_n^{1/n} = \underset{n\to\infty}{\text{Lt}} (n^{1/n})^k \cdot x = x \qquad \left| \because \underset{n\to\infty}{\text{Lt}} n^{1/n} = 1\right.$$

$\therefore$ By Cauchy's Root Test, Σu_n is convergent if $x < 1$ and divergent if $x > 1$.

If $x = 1$, this test fails

For $x = 1$, $\quad u_n = n^k = \frac{1}{n^{-k}}$.

$\therefore$ Σu_n is convergent if $-k > 1$ *i.e.*, $k < -1$ and divergent if $-k \leq 1$ *i.e.*, $k \geq -1$.

Hence Σu_n is convergent if (*i*) $x < 1$ or (*ii*) $x = 1$ and $k < -1$

Σu_n is divergent if (*i*) $x > 1$ or (*ii*) $x = 1$ and $k \geq -1$.

(b) Here $\quad u_n = q^{n^2} \cdot r^n \Rightarrow u_n^{1/n} = q^n r$

$$\underset{n\to\infty}{\text{Lt}} u_n^{1/n} = \underset{n\to\infty}{\text{Lt}} q^n r = \begin{cases} 0 \text{ if } 0 < q < 1 \\ \infty \text{ if } q > 1 \end{cases}$$

$\therefore$ By Cauchy's Root Test, Σu_n is convergent if $0 < q < 1$ and divergent if $q > 1$.

If $q = 1$, $u_n = r^n$ so that $\Sigma u_n = \Sigma r^n$ is a G.P. with common ratio $r > 0$.

If $0 < r < 1$, Σu_n is convergent.

If $r \geq 1$, Σu_n is divergent.

Hence Σu_n is convergent if (*i*) $0 < q < 1$ or (*ii*) $q = 1$ and $0 < r < 1$

Σu_n is divergent if (*i*) $q > 1$ or (*ii*) $q = 1$ and $r \geq 1$.

(c) Here $\quad u_n = e^{\sqrt{n}} r^n \Rightarrow u_n^{1/n} = e^{1/\sqrt{n}} \cdot r$

$$\underset{n\to\infty}{\text{Lt}} u_n^{1/n} = \underset{n\to\infty}{\text{Lt}} e^{1/\sqrt{n}} \cdot r = r$$

$\therefore$ By Cauchy's Root Test, Σu_n is convergent if $0 < r < 1$ and divergent if $r > 1$.

If $\quad r = 1, \quad u_n = e^{\sqrt{n}}$

$$\underset{n\to\infty}{\text{Lt}} u_n = \underset{n\to\infty}{\text{Lt}} e^{\sqrt{n}} = \infty \neq 0$$

$\therefore$ Σu_n is divergent.

Hence Σu_n is convergent if $0 < r < 1$ and divergent if $r \geq 1$.

(d) Here $\quad u_n = (n^{1/n} - 1)^n \Rightarrow u_n^{1/n} = n^{1/n} - 1$

$$\underset{n\to\infty}{\text{Lt}} u_n^{1/n} = \underset{n\to\infty}{\text{Lt}} (n^{1/n} - 1) = 1 - 1 = 0 < 1.$$

$\therefore$ Σu_n is convergent.

Example 6. *Test for convergence the series :*

(*i*) $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{-n^2}$ (*ii*) $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n x^n, x > 0$

(*iii*) $\sum_{n=2}^{\infty} \frac{1}{[\log(\log n)]^n}$ (*iv*) $\sum_{n=1}^{\infty} 3^{-2n-5(-1)^n}$.

Sol. (*i*) Here $\quad u_n = \left(1+\frac{1}{n}\right)^{-n^2}$

$\therefore \quad u_n^{1/n} = \left(1+\frac{1}{n}\right)^{-n}$

$$\lim_{n\to\infty} u_n^{\frac{1}{n}} = \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^n\right]^{-1} = e^{-1} = \frac{1}{e} < 1$$

$\therefore$ By Cauchy's root test, the given series is convergent.

(*ii*) Here $\quad u_n = \left(1+\frac{1}{n}\right)^n x^n$

$\therefore \quad u_n^{1/n} = \left(1+\frac{1}{n}\right)x$

$$\lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)x = x$$

$\therefore$ By Cauchy's root test, the series is convergent if $x < 1$ and divergent if $x > 1$.

When $x = 1$, Cauchy's root test fails. However, when $x = 1$, $u_n = \left(1+\frac{1}{n}\right)^n$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \neq 0.$$

$\therefore$ The series is divergent.

Hence the series is convergent if $0 < x < 1$ and divergent if $x \geq 1$.

(*iii*) Here $\quad u_n = \frac{1}{[\log(\log n)]^n}$

$$\lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty} \frac{1}{\log(\log n)} = 0 < 1$$

$\therefore$ By Cauchy's root test, the series is convergent.

(*iv*) Here $u_n = 3^{-2n-5(-1)^n}$

$$\lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty} [3^{-2n-5(-1)^n}]^{1/n} = \lim_{n\to\infty} 3^{-2-\frac{5(-1)^n}{n}}$$

$$= \lim_{n\to\infty} 3^{-2} \cdot 3^{-\frac{5(-1)^n}{n}} = 3^{-2} \times 3^0 = \frac{1}{9} < 1.$$

$\therefore$ By Cauchy's root test, the series is convergent.

Article 14. Cauchy's root test is more general than D' Alembert's ratio test.

This is so because

(*i*) $\underset{n\to\infty}{\text{Lt}} \frac{u_{n+1}}{u_n}$ exists $\Rightarrow$ $\underset{n\to\infty}{\text{Lt}} (u_n)^{1/n}$ exists

and $\underset{n\to\infty}{\text{Lt}} \frac{u_{n+1}}{u_n} = \underset{n\to\infty}{\text{Lt}} (u_n)^{1/n}$ [Cauchy's second theorem on limits]

$\therefore$ Whenever Ratio Tests is applicable, so is the Root Test.

(*ii*) $\underset{n\to\infty}{\text{Lt}} (u_n)^{1/n}$ exists *need not imply* $\underset{n\to\infty}{\text{Lt}} \frac{u_{n+1}}{u_n}$ exists *i.e.*, $\underset{n\to\infty}{\text{Lt}} \frac{u_{n+1}}{u_n}$ may not exist.

$\therefore$ When the Ratio Test fails, the Root Test succeeds.

Hence the Root Test is more general than Ratio Test.

Example 1. *Show that Cauchy's root test establishes the convergence of the series* Σu_n *where*

$$u_n = \begin{cases} 2^{-n} & \text{if } n \text{ is odd} \\ 2^{-n+2} & \text{if } n \text{ is even} \end{cases} \quad \text{while D' Alembert's ratio test fails to do so.}$$

Sol. When n is odd, $\underset{n\to\infty}{\text{Lt}} (u_n)^{1/n} = \underset{n\to\infty}{\text{Lt}}\ 2^{-1} = \frac{1}{2} < 1$

When n is even $\underset{n\to\infty}{\text{Lt}} (u_n)^{1/n} = \underset{n\to\infty}{\text{Lt}}\ 2^{-1+2/n} = 2^{-1} = \frac{1}{2} < 1$

$\therefore$ $\underset{n\to\infty}{\text{Lt}} (u_n)^{1/n} = \frac{1}{2} < 1 \quad \forall\, n$, even or odd

$\therefore$ By Cauchy's root test, Σu_n is convergent.

Now when n is odd (so that $n+1$ is even) $u_n = 2^{-n}$, $u_{n+1} = 2^{-(n+1)+2} = 2^{-n+1}$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{2^{-n}}{2^{-n+1}} = \lim_{n\to\infty} \frac{1}{2} = \frac{1}{2} < 1$$

When n is even (so that $n+1$ is odd)

$$u_n = 2^{-n+2}, \quad u_{n+1} = 2^{-(n+1)}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{2^{-n+2}}{2^{-n-1}} = \lim_{n\to\infty} 2^3 = 8 > 1.$$

Thus $\lim_{n\to\infty} \frac{u_n}{u_{n+1}}$ does not exist since we have different results for n odd and even.

$\therefore$ D' Alembert's ratio test fails.

Example 2. *Show that Cauchy's root test establishes the convergence of the series* $\Sigma 3^{-n-(-1)^n}$ *while D' Alembert's ratio test fails to do so.*

Sol. Here $u_n = 3^{-n-(-1)^n}$

$$\lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty} 3^{-1-\frac{(-1)^n}{n}} = 3^{-1} = \frac{1}{3} < 1$$

$\therefore$ By Cauchy's root test, the series is convergent.

Now, when n is odd (so that $n+1$ is even) $u_n = 3^{-n+1}$, $u_{n+1} = 3^{-(n+1)-1} = 3^{-n-2}$

$$\therefore \quad \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{3^{-n+1}}{3^{-n-2}} = \lim_{n\to\infty} 3^3 = 27 > 1$$

When n is even (so that $n+1$ is odd) $u_n = 3^{-n-1}$, $u_{n+1} = 3^{-(n+1)+1} = 3^{-n}$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{3^{-n-1}}{3^{-n}} = \lim_{n\to\infty} 3^{-1} = \frac{1}{3} < 1$$

Thus $\lim_{n\to\infty} \frac{u_n}{u_{n+1}}$ does not exist since we have different results for n odd and even.

$\therefore$ D' Alembert's ratio test fails.

Article 15. Raabe's Test

Statement. *If* Σu_n *is a series of positive terms such that*

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = l, \text{ then}$$

(*i*) Σu_n *is convergent if* $l > 1$, *and* (*ii*) Σu_n *is divergent if* $l < 1$.

Proof. Let us compare the given series Σu_n with an auxiliary series $\sum v_n = \sum \frac{1}{n^p}$ which we know converges if $p > 1$ and diverges if $p \le 1$.

Now
$$\frac{v_n}{v_{n+1}} = \frac{\frac{1}{n^p}}{\frac{1}{(n+1)^p}} = \left(\frac{n+1}{n}\right)^p = \left(1+\frac{1}{n}\right)^p = 1 + \frac{p}{n} + \frac{p(p-1)}{2!} \cdot \frac{1}{n^2} + \ldots\ldots$$

Case (I) Choose a number p such that $l \ge p > 1$.

Then Σv_n is convergent.

By comparison test, the series Σu_n will be convergent if $\exists$ a natural number m such that $\forall\, n \ge m$,

$$\frac{u_n}{u_{n+1}} \ge \frac{v_n}{v_{n+1}}$$

i.e., if
$$\frac{u_n}{u_{n+1}} \ge 1 + \frac{p}{n} + \frac{p(p-1)}{2!} \cdot \frac{1}{n^2} + \ldots\ldots$$

i.e., if
$$\frac{u_n}{u_{n+1}} - 1 \ge \frac{p}{n} + \frac{p(p-1)}{2!} \cdot \frac{1}{n^2} + \ldots\ldots$$

i.e., if $n\left(\frac{u_n}{u_{n+1}}-1\right) \geq p + \frac{p(p-1)}{2!}\cdot\frac{1}{n}+\ldots\ldots$

i.e., if $\lim\limits_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) \geq \lim\limits_{n\to\infty}\left[p+\frac{p(p-1)}{2!}\cdot\frac{1}{n}+\ldots\ldots\right]$

i.e., if $l \geq p$

i.e., if $l \geq p > 1$ or if $l > 1$

$\therefore$ Σu_n is convergent if $l > 1$.

Case (II) Choose a number p such that $l \leq p < 1$.

Then Σv_n is divergent.

By comparison test, the series Σu_n will be divergent if $\exists$ a natural number m such that $\forall\, n \geq m$,

$$\frac{u_n}{u_{n+1}} \leq \frac{v_n}{v_{n+1}}$$

i.e., if $\frac{u_n}{u_{n+1}} \leq 1+\frac{p}{n}+\frac{p(-1)}{2!}\cdot\frac{1}{n^2}+\ldots\ldots$

i.e., if $\frac{u_n}{u_{n+1}}-1 \leq \frac{p}{n}+\frac{p(p-1)}{2!}\cdot\frac{1}{n}+\ldots\ldots$

i.e., if $n\left(\frac{u_n}{u_{n+1}}-1\right) \leq p+\frac{p(p-1)}{2!}\cdot\frac{1}{n}+\ldots\ldots$

i.e., if $\lim\limits_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) \leq \lim\limits_{n\to\infty}\left[p+\frac{p(p-1)}{2!}\cdot\frac{1}{n}+\ldots\ldots\right]$

i.e., if $l \leq p$

i.e., if $l \leq p < 1$ or if $l < 1$

$\therefore$ Σu_n is divergent if $l < 1$.

Note 1. Raabe's test is inconclusive when $l = 1$.

For example, consider the series $\sum\limits_{n=1}^{\infty}\frac{1}{n}$ which we know is divergent.

Then $u_n = \frac{1}{n}$

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty} n\left[\frac{\frac{1}{n}}{\frac{1}{n+1}}-1\right]$$

$$= \lim_{n\to\infty} n\left(\frac{n+1}{n}-1\right) = \lim_{n\to\infty} n\left(\frac{1}{n}\right) = \lim_{n\to\infty} 1 = 1$$

Now consider the series $\sum\limits_{n=2}^{\infty}\frac{1}{n(\log n)^2}$

Since $\sum\limits_{n=2}^{\infty}\frac{1}{n(\log n)^p}$ is convergent if $p > 1$ (see examples with Cauchy's Condensation Test or Cauchy's Integral Test).

$\therefore$ $\sum\limits_{n=2}^{\infty}\frac{1}{n(\log n)^2}$ is convergent $\qquad (\because\ p = 2 > 1)$

Here $u_n = \frac{1}{n(\log n)^2}$

$\therefore$
$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty} n\left[\frac{(n+1)(\log(n+1))^2}{n(\log n)^2}-1\right]$$

$$= \lim_{n\to\infty}\frac{(n+1)(\log(n+1))^2 - n(\log n)^2}{(\log n)^2}$$

$$= \lim_{n\to\infty}\frac{n[(\log(n+1))^2-(\log n)^2]+[\log(n+1)]^2}{(\log n)^2}$$

$$= \lim_{n\to\infty}\frac{n[\log(n+1)+\log n][\log(n+1)-\log n]}{(\log n)^2}+\lim_{n\to\infty}\left(\frac{\log(n+1)}{\log n}\right)^2$$

$$= \lim_{n\to\infty}\frac{n[\log(n+1)+\log n]\log\frac{n+1}{n}}{(\log n)^2}+1$$

$$\left[\because\ \lim_{n\to\infty}\frac{\log(n+1)}{\log n} \qquad \text{Form } \frac{\infty}{\infty}\right.$$

$$= \lim_{n\to\infty}\frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{n}{n+1} \qquad \text{Form } \frac{\infty}{\infty}$$

$$\left.= \lim_{n\to\infty}\frac{1}{1} = 1\right]$$

$$= \lim_{n\to\infty}\frac{n\log(n+1)\log\left(1+\frac{1}{n}\right)+n\log n\log\left(1+\frac{1}{n}\right)}{(\log n)^2}+1$$

$$= \lim_{n\to\infty}\left[\frac{\log(n+1)}{\log n}\cdot\log\left(1+\frac{1}{n}\right)^n\cdot\frac{1}{\log n}+\log\left(1+\frac{1}{n}\right)^n\cdot\frac{1}{\log n}\right]+1$$

$$= 1 \times 1 \times 0 + 1 \times 0 + 1 = 1$$

$$\left[\because \lim_{n\to\infty} \frac{\log(n+1)}{\log n} = 1,\ \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e,\ \lim_{n\to\infty}\frac{1}{\log n} = 0\right]$$

Thus, $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = 1$ in both cases and the series Σu_n is convergent in one case and divergent in the other.

Note 2. Raabe's Test is stronger than D' Alembert's Ratio Test and may succeed where Ratio Test fails.

For example, consider the series $\sum \frac{1}{n^2}$.

Here $$u_n = \frac{1}{n^2}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{(n+1)^2}{n^2} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2 = 1.$$

$\therefore$ Ratio test fails.

But, $$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} n\left[\frac{(n+1)^2}{n^2} - 1\right] = \lim_{n\to\infty} \frac{2n+1}{n} = \lim_{n\to\infty}\left(2+\frac{1}{n}\right) = 2 > 1$$

$\therefore$ By Raabe's Test, the series is convergent.

Note 3. If $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \infty$, then Σu_n is convergent.

If $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = -\infty$, then Σu_n is divergent.

Note 4. In general, Raabe's test is used when D' Alembert's ratio test fails and the ratio $\frac{u_n}{u_{n-1}}$ does not involve the number e. When $\frac{u_n}{u_{n+1}}$ involves e, we apply logarithmic test after the ratio test and not Raabe's test.

Article 16. Logarithmic Test

Statement. *If Σu_n is a series of positive terms such that $\lim_{n\to\infty} n \log \frac{u_n}{u_{n+1}} = l$, then*

(i) Σu_n is convergent if $l > 1$, and *(ii) Σu_n is divergent if $l < 1$.*

Proof. Let us compare the given series Σu_n with an auxiliary series $\Sigma v_n = \sum \frac{1}{n^p}$ which we know is convergent if $p > 1$ and divergent if $p \le 1$.

Now $$\frac{v_n}{v_{n+1}} = \frac{(n+1)^p}{n^p} = \left(\frac{n+1}{n}\right)^p = \left(1+\frac{1}{n}\right)^p$$

$\therefore$ $$\log \frac{v_n}{v_{n+1}} = \log\left(1+\frac{1}{n}\right)^p = p \log\left(1+\frac{1}{n}\right) = p\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \ldots\right)$$

$$\left[\because \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\right]$$

Case (I) Choose a number p such that $l \ge p > 1$.

Then Σv_n is convergent.

By comparison test, the series Σu_n will be convergent if $\exists$ a natural number m such that $\forall\ n \ge m$,

$$\frac{u_n}{u_{n+1}} \ge \frac{v_n}{v_{n+1}}$$

i.e., if $$\log \frac{u_n}{u_{n+1}} \ge \log \frac{v_n}{v_{n+1}}$$ [$\because$ log x is an increasing function]

i.e., if $$\log \frac{u_n}{u_{n+1}} \ge p\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \ldots\right)$$

i.e., if $$n \log \frac{u_n}{u_{n+1}} \ge p\left(1 - \frac{1}{2n} + \frac{1}{3n^2} - \ldots\right)$$

i.e., if $$\lim_{n\to\infty} n \log \frac{u_n}{u_{n+1}} \ge p$$

i.e., if $l \ge p$

i.e., if $l \ge p > 1$ or if $l > 1$

$\therefore$ Σu_n is convergent if $l > 1$.

Case (II) Choose a number p such that $l \le p < 1$.

Then Σv_n is divergent.

By comparison test, the series Σu_n will be divergent if $\exists$ a natural number m such that $\forall\ n \ge m$,

$$\frac{u_n}{u_{n+1}} \le \frac{v_n}{v_{n+1}}$$

i.e., if $$\log \frac{u_n}{u_{n+1}} \le \log \frac{v_n}{v_{n+1}}$$ [$\because$ log x is an increasing function]

i.e., if $$\log \frac{u_n}{u_{n+1}} \le p\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \ldots\right)$$

i.e., if $$n \log \frac{u_n}{u_{n+1}} \le p\left(1 - \frac{1}{2n} + \frac{1}{3n^2} - \ldots\right)$$

i.e., if $$\lim_{n\to\infty} n \log \frac{u_n}{u_{n+1}} \le p$$

i.e., if $\quad l \le p$

i.e., if $\quad l \le p < 1$ or if $l < 1$

$\therefore$ Σu_n is divergent if $l < 1$.

Note 1. The test fails if $\lim\limits_{n\to\infty} n \log \dfrac{u_n}{u_{n+1}} = 1$.

Note 2. The test is applied after the failure of Ratio Test and when $\dfrac{u_n}{u_{n+1}}$ involves e.

Article 17. D'Morgan and Bertrand's Test

Statement. *If Σu_n is a series of positive terms such that*

$$\lim_{n\to\infty}\left[\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\log n\right]=l, \text{ then}$$

(i) Σu_n is convergent if $l > 1$, and *(ii) Σu_n is divergent if $l < 1$.*

Proof. Let us compare the series Σu_n with an auxiliary series

$$\sum_{n=2}^{\infty} v_n = \sum_{n=2}^{\infty}\frac{1}{n(\log n)^p}$$

which we know is convergent if $p > 1$ and divergent if $p \le 1$.

Now $\dfrac{v_n}{v_{n+1}} = \dfrac{(n+1)(\log(n+1))^p}{n(\log n)^p} \quad \forall\, n \ge 2$

$$=\frac{n+1}{n}\cdot\left[\frac{\log(n+1)}{\log n}\right]^p=\left(1+\frac{1}{n}\right)\left[\frac{\log n\left(1+\frac{1}{n}\right)}{\log n}\right]^p$$

$$=\left(1+\frac{1}{n}\right)\left[\frac{\log n+\log\left(1+\frac{1}{n}\right)}{\log n}\right]^p=\left(1+\frac{1}{n}\right)\left[1+\frac{1}{\log n}\cdot\log\left(1+\frac{1}{n}\right)\right]^p$$

$$=\left(1+\frac{1}{n}\right)\left[1+\frac{1}{\log n}\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\ldots\right)\right]^p \quad \left[\because \log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots\right]$$

$$=\left(1+\frac{1}{n}\right)\left[1+\frac{p}{\log n}\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\ldots\right)+\ldots\right]$$

$$=1+\frac{1}{n}+\frac{p}{\log n}\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\ldots\right)+\frac{p}{n\log n}\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\ldots\right)+\ldots$$

$$=1+\frac{1}{n}+\frac{p}{n\log n}\left(1-\frac{1}{2n}+\frac{1}{n}+\frac{1}{3n^2}-\frac{1}{2n^2}\ldots\right)+\ldots$$

$$=1+\frac{1}{n}+\frac{p}{n\log n}\left(1+\frac{1}{2n}-\frac{1}{6n^2}-\ldots\right)+\ldots$$

Case (I) Choose a number p such that $l \ge p > 1$.

Then Σv_n is convergent.

By comparison test, the series Σu_n will be convergent if $\exists$ a natural number m such that $\forall\, n \ge m$,

$$\frac{u_n}{u_{n+1}} \ge \frac{v_n}{v_{n+1}}$$

i.e., if $\quad \dfrac{u_n}{u_{n+1}} \ge 1+\dfrac{1}{n}+\dfrac{p}{n\log n}\left(1+\dfrac{1}{2n}-\dfrac{1}{6n^2}-\ldots\right)+\ldots$

i.e., if $\quad \dfrac{u_n}{u_{n+1}}-1 \ge \dfrac{1}{n}+\dfrac{p}{n\log n}\left(1+\dfrac{1}{2n}-\dfrac{1}{6n^2}-\ldots\right)+\ldots$

i.e., if $\quad n\left(\dfrac{u_n}{u_{n+1}}-1\right) \ge 1+\dfrac{p}{\log n}\left(1+\dfrac{1}{2n}-\dfrac{1}{6n^2}-\ldots\right)+\ldots$

i.e., if $\quad n\left(\dfrac{u_n}{u_{n+1}}-1\right)-1 \ge \dfrac{p}{\log n}\left(1+\dfrac{1}{2n}-\dfrac{1}{6n^2}-\ldots\right)+\ldots$

i.e., if $\quad \left\{n\left(\dfrac{u_n}{u_{n+1}}-1\right)-1\right\}\log n \ge p\left(1+\dfrac{1}{2n}-\dfrac{1}{6n^2}-\ldots\right)+\ldots$

i.e., if $\quad \lim\limits_{n\to\infty}\left\{n\left(\dfrac{u_n}{u_{n+1}}-1\right)-1\right\}\log n \ge p$

i.e., if $\quad l \ge p$

i.e., if $\quad l \ge p > 1$ or if $l > 1$.

$\therefore$ Σu_n is convergent if $l > 1$.

Case (II) Choose a number p such that $l \le p < 1$.

Then Σv_n is divergent.

By comparison test, the series Σu_n will be divergent if $\exists$ a natural number m such that $\forall\, n \ge m$,

$$\frac{u_n}{u_{n+1}} \le \frac{v_n}{v_{n+1}}$$

i.e., if $\quad \dfrac{u_n}{u_{n+1}} \le 1+\dfrac{1}{n}+\dfrac{p}{n\log n}\left(1+\dfrac{1}{2n}-\dfrac{1}{6n^2}-\ldots\right)+\ldots$

i.e., if $\quad n\left(\dfrac{u_n}{u_{n+1}}-1\right) \le 1+\dfrac{p}{\log n}\left(1+\dfrac{1}{2n}-\dfrac{1}{6n^2}-\ldots\right)+\ldots$

i.e., if $\quad n\left(\dfrac{u_n}{u_{n+1}}-1\right)-1 \le \dfrac{p}{\log n}\left(1+\dfrac{1}{2n}-\dfrac{1}{6n^2}\ldots\right)+\ldots$

i.e., if $\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\log n \le p\left(1+\frac{1}{2n}-\frac{1}{6n^2}\ldots\right)+\ldots$

i.e., if $\lim\limits_{n\to\infty}\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\log n \le p$

i.e., if $l \le p$

i.e., if $l \le p < 1$ or if $l < 1$

$\therefore$ Σu_n is divergent if $l < 1$.

Note. The test fails if $l = 1$.

Article 18. Gauss Test

Statement. *If Σu_n is a series of positive terms such that $\frac{u_n}{u_{n+1}} = 1+\frac{\lambda}{n}+\frac{\alpha_n}{n^{1+\delta}}$ where $\delta > 0$ and $< \alpha_n >$ is a bounded sequence, then*

(i) Σu_n is convergent if $\lambda > 1$, and (ii) Σu_n is divergent if $\lambda \le 1$.

Proof. $$\frac{u_n}{u_{n+1}} = 1+\frac{\lambda}{n}+\frac{\alpha_n}{n^{1+\delta}}$$

$$\Rightarrow \frac{u_n}{u_{n+1}}-1 = \frac{\lambda}{n}+\frac{\alpha_n}{n^{1+\delta}} \Rightarrow n\left(\frac{u_n}{u_{n+1}}-1\right) = \lambda+\frac{\alpha_n}{n^{\delta}}$$

$$\Rightarrow \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty}\left(\lambda+\alpha_n\cdot\frac{1}{n^{\delta}}\right) = \lambda$$

$$\left[\because < \alpha_n > \text{ is bounded and } \frac{1}{n^{\delta}} \to 0 \text{ since } \delta > 0\right]$$

$\therefore$ By Raabe's test, Σu_n converges if $\lambda > 1$ and diverges if $\lambda < 1$.

If $\lambda = 1$, Raabe's test fails.

Now, when $\lambda = 1$, we have

$$n\left(\frac{u_n}{u_{n+1}}-1\right) = 1+\frac{\alpha_n}{n^{\delta}} \Rightarrow n\left(\frac{u_n}{u_{n+1}}-1\right)-1 = \frac{\alpha_n}{n^{\delta}}$$

$$\Rightarrow \left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\log n = \frac{\alpha_n \log n}{n^{\delta}}$$

$$\Rightarrow \lim_{n\to\infty}\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\log n = \lim_{n\to\infty}\alpha_n\cdot\frac{\log n}{n^{\delta}} = 0 < 1$$

$$\left[\because < \alpha_n > \text{ is bounded and } \lim_{n\to\infty}\frac{\log n}{n^{\delta}} = 0 \text{ since } \delta > 0\right]$$

$\therefore$ By D' Morgan and Bertrand's Test, Σu_n diverges.

Hence Σu_n is convergent if $\lambda > 1$ and divergent if $\lambda \le 1$.

Note. The test never fails as we know that the series diverges for $\lambda = 1$. Moreover, the test is applied after the failure of Ratio test and when it is possible to expand $\frac{u_n}{u_{n+1}}$ in powers of $\frac{1}{n}$ by Binomial Theorem or by any other method.

Article 19. Working rule in examples and summary of results.

(*a*) Comparison test is applied only when u_n does not involve any power of x involving n and when u_n does not involve factorials.

(*b*) Cauchy's root test is applied when u_n involves the nth power of itself as a whole for example,

$$u_n = \left(\frac{n+1}{n}\right)^{n^2}, \frac{1}{(\log n)^n} \text{ etc.}$$

(*c*) When the series involves increasing powers of x, we start directly with D' Alembert's ratio test, *i.e.,*

If $\text{Lt}\ \frac{u_n}{u_{n+1}} > 1$, Σu_n converges

< 1, Σu_n diverges.

When $\text{Lt}\ \frac{u_n}{u_{n+1}} = 1$, the ratio test fails. Then choose

(*i*) Raabe's test or Gauss test if $\frac{u_n}{u_{n+1}}$ does not involve the number e.

(*ii*) If $\frac{u_n}{u_{n+1}}$ involves the number e, apply logarithmic test *i.e.,* Σu_n converges or diverges according as

$$\text{Lt}\ n\log\frac{u_n}{u_{n+1}} > 1 \quad\text{or}\quad < 1.$$

The application of the above tests is explained by the following solved examples and **all these solved examples are very important from the examination point of view and should be thoroughly practised.**

Note. For application of Gauss Test, expand $\frac{u_n}{u_{n+1}}$ in powers of $\frac{1}{n}$ as $\frac{u_n}{u_{n+1}} = 1+\frac{\lambda}{n}+O\left(\frac{1}{n^2}\right)$ where $O\left(\frac{1}{n^2}\right)$ stands for terms of order $\frac{1}{n^2}$ and higher powers of $\frac{1}{n}$.

ILLUSTRATIVE EXAMPLES

Example 1. *Discuss the convergence of the series :* $\frac{1}{2}+\frac{1.3}{2.4}+\frac{1.3.5}{2.4.6}+\ldots$

Sol. Here $$u_n = \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n}$$

$$\Rightarrow \quad u_{n+1} = \frac{1.3.5.....(2n-1)(2n+1)}{2.4.6......(2n)(2n+2)}$$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{1.3.5.....(2n-1)}{2.4.6......2n} \times \frac{2.4.6......2n(2n+2)}{1.3.5......(2n-1)(2n+1)}$$

$$= \frac{2n+2}{2n+1} = \frac{1+\frac{1}{n}}{1+\frac{1}{2n}} \to 1 \text{ as } n \to \infty.$$

$\therefore$ D' Alembert's Ratio test fails.

$$n\left[\frac{u_n}{u_{n+1}} - 1\right] = n\left[\frac{2n+2}{2n+1} - 1\right] = \frac{n}{2n+1} = \frac{1}{2+\frac{1}{n}}$$

$$\therefore \quad \underset{n\to\infty}{\text{Lt.}}\; n\left[\frac{u_n}{u_{n+1}} - 1\right] = \frac{1}{2} < 1.$$

$\therefore$ By Raabe's test, Σu_n diverges.

The student should have very good practice of writing the general term carefully and correctly. Unless he writes u_n correctly, all his further work will be incorrect.

Example 2. *Discuss the convergence of the series :* $\frac{1^2}{2^2} + \frac{1^2 . 3^2}{2^2 . 4^2} + \frac{1^2 . 3^3 . 5^2}{2^2 . 4^2 . 6^2} + ...$

Sol. Here $\quad u_n = \dfrac{1^2 . 3^2 . 5^2 ...... (2n-1)^2}{2^2 . 4^2 . 6^2 ...... (2n)^2}$

$$\Rightarrow \quad u_{n+1} = \frac{1^2 . 3^2 . 5^2 ...... (2n-1)^2 (2n+1)^2}{2^2 . 4^2 . 6^2 ...... (2n)^2 (2n+2)^2}$$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{(2n+2)^2}{(2n+1)^2} = \frac{4n^2\left(1+\frac{1}{n}\right)^2}{4n^2\left(1+\frac{1}{2n}\right)^2} = \frac{\left(1+\frac{1}{n}\right)^2}{\left(1+\frac{1}{2n}\right)^2}$$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = 1.$$

Hence the ratio test fails.

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) = n\left[\frac{(2n+2)^2}{(2n+1)^2} - 1\right] = n\left[\frac{4n^2+8n+4-(4n^2+4n+1)}{(2n+1)^2}\right]$$

$$= n\,\frac{(4n+3)}{(2n+1)^2} = \frac{4n^2+3n}{(2n+1)^2} = \frac{1+\frac{3}{4n}}{\left(1+\frac{1}{2n}\right)^2} \to 1 \text{ as } n \to \infty$$

$\therefore$ Raabe's test also fails.

When D' Alembert ratio test fails, we can directly apply Gauss test. Now,

$$\frac{u_n}{u_{n+1}} = \frac{(2n+2)^2}{(2n+1)^2} = \frac{\left(1+\frac{1}{n}\right)^2}{\left(1+\frac{1}{2n}\right)^2} = \left(1+\frac{1}{n}\right)^2\left(1+\frac{1}{2n}\right)^{-2}$$

$$= \left(1+\frac{2}{n}+\frac{1}{n^2}\right)\left(1-\frac{2}{2n}+\frac{3}{4n^2}-......\right) = 1+\frac{1}{n}+\frac{1}{n^2}\left(1-2+\frac{3}{4}\right)+......$$

$$= 1+\frac{1}{n}-\frac{1}{4n^2}+...... = 1+\frac{1}{n}+O\left(\frac{1}{n^2}\right)$$

Comparing it with $\frac{u_n}{u_{n+1}} = 1+\frac{\lambda}{n}+O\left(\frac{1}{n^2}\right)$ we have $\lambda = 1$. Thus by Gauss test, the series Σu_n **diverges.**

Example 3. *Discuss the convergence of the series :* $1 + \frac{2^2}{3^3} + \frac{2^2 . 4^2}{3^2 . 5^2} + \frac{2^2 . 4^2 . 6^2}{3^2 . 5^2 . 7^2} + ...$

Sol. Omitting the first term, we have

$$u_n = \frac{2^2 . 4^2 . 6^2 ...... (2n)^2}{3^2 . 5^2 . 7^2 ...... (2n+1)^2}$$

$$\Rightarrow \quad u_{n+1} = \frac{2^2 . 4^2 . 6^2 ...... (2n)^2 (2n+2)^2}{3^2 . 5^2 . 7^2 ...... (2n+1)^2 (2n+3)^2}$$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{(2n+3)^2}{(2n+2)^2} = \frac{\left(1+\frac{3}{2n}\right)^2}{\left(1+\frac{1}{n}\right)^2} \to 1 \text{ as } n \to \infty$$

$\therefore$ Ratio test fails.

Now $\quad \dfrac{u_n}{u_{n+1}} = \left(1+\frac{3}{2n}\right)^2\left(1+\frac{1}{n}\right)^{-2}$

$$= \left(1+\frac{3}{n}+\frac{9}{4n^2}\right)\left(1-\frac{2}{n}+\frac{3}{n^2}-.....\right)$$

[On expanding by Binomial Theorem]

$$= 1+\frac{3}{n}-\frac{2}{n}+\frac{9}{4n^2}-\frac{6}{n^2}+\frac{3}{n^2}+......$$

$$= 1+\frac{1}{n}-\frac{3}{4n^2}+...... = 1+\frac{1}{n}+O\left(\frac{1}{n^2}\right)$$

Comparing it with $\frac{u_n}{u_{n+1}} = 1+\frac{\lambda}{n}+O\left(\frac{1}{n^2}\right)$ we have $\lambda = 1$ $\therefore$ By Gauss test, the series Σu_n **diverges.**

Example 4. *Discuss the convergence of the series :* $\frac{1^2}{4^2} \cdot \frac{1^2 . 5^2}{4^2 . 8^2} + \frac{1^2 . 5^2 . 9^2}{4^2 . 8^2 . 12^2} + \ldots\ldots$

Sol. *Please try yourself.* **(Convergent)**

Example 5. *Discuss the convergence of the series :* $1 + \frac{3}{7}x + \frac{3 . 6}{7 . 10}x^2 + \frac{3 . 6 . 9}{7 . 10 . 13}x^3 + \ldots\ldots$

Sol. Neglecting the first term, we have

$$u_n = \frac{3 . 6 \ldots\ldots (3n)}{7 . 10 . 13 \ldots\ldots (3n+4)} . x^n$$

$$\Rightarrow \quad u_{n+1} = \frac{3 . 6 . 9 \ldots\ldots (3n)(3n+3)}{7 . 10 . 13 \ldots\ldots (3n+4)(3n+7)} x^{n+1}$$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{3n+7}{3n+3} . \frac{1}{x} = \frac{1 + \frac{7}{3n}}{1 + \frac{1}{n}} . \frac{1}{x} \to \frac{1}{x} \text{ as } n \to \infty$$

$\therefore$ By Ratio test, Σu_n *converges* if $\frac{1}{x} > 1$ *i.e.*, $x < 1$ and *diverges* if $\frac{1}{x} < 1$ or $x > 1$.

If $x = 1$, then the ratio test fails.

When $x = 1$, $\quad \frac{u_n}{u_{n+1}} = \frac{3n+7}{3n+3}$

$$\therefore \quad n\left(\frac{u_n}{u_{n+1}} - 1\right) = n\left[\frac{3n+7}{3n+3} - 1\right] = n\left[\frac{4}{3n+3}\right] = \frac{4n}{3n+3}$$

$$\therefore \quad \underset{n \to \infty}{\text{Lt}}\, n\left(\frac{u_n}{u_{n+1}} - 1\right) = \underset{n \to \infty}{\text{Lt}} \frac{4n}{3n+3} = \underset{n \to \infty}{\text{Lt}} \frac{4}{3 + \frac{3}{n}} = \frac{4}{3} > 1.$$

$\therefore$ By Raabe's test, the series *converges*.

Hence the given series **converges if x ≤ 1 and diverges if x > 1.**

Example 6. *Discuss the convergence of the series :* $\frac{x}{1} + \frac{1}{2} . \frac{x^3}{3} + \frac{1 . 3}{2 . 4} . \frac{x^5}{5} + \frac{1 . 3 . 5}{2 . 4 . 6} . \frac{x^7}{7} + \ldots$

Sol. Neglecting the first term, we have

$$u_n = \frac{1 . 3 . 5 \ldots\ldots (2n-1)}{2 . 4 . 6 \ldots\ldots (2n)} . \frac{x^{2n+1}}{2n+1}$$

$$\Rightarrow \quad u_{n+1} = \frac{1 . 3 . 5 \ldots\ldots (2n-1)(2n+1)}{2 . 4 . 6 \ldots\ldots (2n)(2n+2)} . \frac{x^{2n+3}}{2n+3}$$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{2n+2}{2n+1} . \frac{2n+3}{2n+1} . \frac{1}{x^2}$$

$$= \frac{2n\left(1+\frac{1}{n}\right) . 2n\left(1+\frac{3}{2n}\right)}{2n\left(1+\frac{1}{2n}\right) . 2n\left(1+\frac{1}{2n}\right)} . \frac{1}{x^2} = \frac{\left(1+\frac{1}{n}\right)\left(1+\frac{3}{2n}\right)}{\left(1+\frac{1}{2n}\right)^2} . \frac{1}{x^2}$$

$$\underset{n \to \infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n \to \infty}{\text{Lt}} \frac{\left(1+\frac{1}{n}\right)\left(1+\frac{3}{2n}\right)}{\left(1+\frac{1}{2n}\right)^2} . \frac{1}{x^2} = \frac{1}{x^2}$$

$\therefore$ By Ratio Test, Σu_n is convergent if $\frac{1}{x^2} > 1$ *i.e.*, if $x^2 < 1$ and divergent if $\frac{1}{x^2} < 1$ *i.e.*, $x^2 > 1$.

If $x^2 = 1$, then Ratio Test fails.
When $x^2 = 1$, we have

$$\frac{u_n}{u_{n+1}} = \frac{(2n+2)(2n+3)}{(2n+1)^2} = \frac{4n^2 + 10n + 6}{4n^2 + 4n + 1}$$

$$\underset{n \to \infty}{\text{Lt}}\, n\left(\frac{u_n}{u_{n+1}} - 1\right) = \underset{n \to \infty}{\text{Lt}}\, n\left(\frac{4n^2 + 10n + 6}{4n^2 + 4n + 1} - 1\right)$$

$$= \underset{n \to \infty}{\text{Lt}} \frac{6n^2 + 5n}{4n^2 + 4n + 1} = \underset{n \to \infty}{\text{Lt}} \frac{6 + \frac{5n}{n}}{4 + \frac{4}{n} + \frac{1}{n^2}} = \frac{6}{4} = \frac{3}{2} > 1.$$

$\therefore$ By Raabe's Test, the series converges.
Hence Σu_n is convergent if $x^2 \le 1$ and divergent if $x^2 > 1$.
Example 7. *Discuss the convergence of the series :*

$$x^2 + \frac{2^2}{3 . 4}x^4 + \frac{2^2 . 4^2}{3 . 4 . 5 . 6}x^6 + \frac{2^3 . 4^2 . 6^2}{3 . 4 . 5 . 6 . 7 . 8}x^8 + \ldots\ldots$$

Sol. Neglecting the first term, $u_n = \frac{2^2 . 4^2 . 6^2 \ldots\ldots (2n)^2}{3 . 4 . 5 . 6 . 7 . 8 \ldots\ldots (2n+1)(2n+2)} . x^{2n+2}$

(with every new term, two new factors are introduced in the denominator).

$$u_{n+1} = \frac{2^2 . 4^2 . 6^2 \ldots\ldots (2n)^2 (2n+2)^2}{3 . 4 . 5 . 6 . 7 . 8 \ldots\ldots (2n+1)(2n+2) . (2n+3)(2n+4)} x^{2n+4}$$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{(2n+3)(2n+4)}{(2n+2)^2} . \frac{1}{x^2} = \frac{4n^2 + 14n + 12}{4n^2 + 8n + 4} . \frac{1}{x^2}$$

$$= \frac{4 + \frac{14}{n} + \frac{12}{n^2}}{4 + \frac{8}{n} + \frac{4}{n^2}} . \frac{1}{x^2} \to \frac{1}{x^2} \text{ as } n \to \infty$$

$\therefore$ By Ratio Test, the series converges if $\frac{1}{x^2} > 1$ or if $x^2 < 1$ and diverges if $\frac{1}{x^2} < 1$ or if $x^2 > 1$.

If $x^2 = 1$, the ratio test, fails. When $x^2 = 1$, we have

$$\frac{u_n}{u_{n+1}} = \frac{4n^2+14n+12}{4n^2+8n+4}$$

$$\therefore \quad n\left(\frac{u_n}{u_{n+1}} - 1\right) = n\left[\frac{4n^2+14n+12}{4n^2+8n+4} - 1\right] = \frac{6n^2+8n}{4n^2+8n+4}$$

$$= \frac{6+\frac{8}{n}}{4+\frac{8}{n}+\frac{4}{n^2}} \to \frac{6}{4} > 1 \text{ as } n \to \infty$$

$\therefore$ The series converges by Raabe's test.

Hence the given series *converges* if $x^2 \le 1$ *and diverges if* $x^2 > 1$.

Example 8. *Discuss the convergence of the series :*

(a) $1 + \frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + \ldots\ldots$ (b) $x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots\ldots$

Sol. *Please try yourself.*

Ans. (a) Convergent if $x^2 < 1$, Divergent if $x^2 \ge 1$

(b) Convergent.

Example 9. *Discuss the convergence of the series :* $1 + \frac{2}{1}\cdot\frac{1}{2} + \frac{2.4}{1.3}\cdot\frac{1}{3} + \frac{2.4.6}{1.3.5}\cdot\frac{1}{4} + \ldots\ldots$

Sol. Neglecting the first term, we have

$$u_n = \frac{2.4.6\ldots\ldots(2n)}{1.3.5\ldots\ldots(2n-1)}\cdot\frac{1}{n+1}$$

$$\Rightarrow \quad u_{n+1} = \frac{2.4.6\ldots\ldots(2n)(2n+2)}{1.3.5\ldots\ldots(2n-1)(2n+1)}\cdot\frac{1}{n+2}$$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{2n+1}{2n+2}\cdot\frac{n+2}{n+1} = \frac{2n^2+5n+2}{2n^2+4n+2}.$$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}} \frac{2n^2+5n+2}{2n^2+4n+2} = \underset{n\to\infty}{\text{Lt}} \frac{2+\frac{5}{n}+\frac{2}{n^2}}{2+\frac{4}{n}+\frac{2}{n^2}} = \frac{2}{2} = 1$$

$\therefore$ Ratio test fails.

Now $$n\left(\frac{u_n}{u_{n+1}} - 1\right) = n\left(\frac{2n^2+5n+2}{2n^2+4n+2} - 1\right) = n\left[\frac{n}{2n^2+4n+2}\right]$$

$$= \frac{n^2}{2n^2+4n+2} = \frac{1}{2+\frac{4}{n}+\frac{2}{n^2}} \to \frac{1}{2} < 1 \text{ as } n \to \infty$$

$\therefore$ The series *diverges* by Raabe's test.

Example 10. *Discuss the convergence of the series :* $\frac{1^2}{2^2} + \frac{1^2.3^2}{2^2.4^2}x + \frac{1^2.3^2.5^2}{2^2.4^2.6^2}x^2 + \ldots\ldots (x > 0)$.

Sol. *Please try yourself.* [Converges if $x < 1$ and diverges if $x \ge 1$]

Example 11. *Discuss the convergence of the series :* $1 + \frac{x}{2} + \frac{2!}{3^2}x^2 + \frac{3!}{4^3}x^3 + \frac{4!}{5^4}x^4 + \ldots\ldots$

Sol. Neglecting the first term, we have $u_n = \frac{n!}{(n+1)^n}\cdot x^n$ and $u_{n+1} = \frac{(n+1)!}{(n+2)^{n+1}}\cdot x^{n+1}$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}} \frac{n!}{(n+1)^n}x^n \cdot \frac{(n+2)^{n+1}}{(n+1)!}\cdot\frac{1}{x^{n+1}}$$

$$= \underset{n\to\infty}{\text{Lt}} \frac{1}{n^n\left(1+\frac{1}{n}\right)^n}\cdot\frac{n^{n+1}\left(1+\frac{2}{n}\right)^{n+1}}{(n+1)}\cdot\frac{1}{x}$$

$$= \underset{n\to\infty}{\text{Lt}} \frac{\left(1+\frac{2}{n}\right)^n\left(1+\frac{2}{n}\right)}{\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right)}\cdot\frac{1}{x} = \frac{e^2}{e}\cdot\frac{1}{x} = \frac{e}{x}$$

$$\left[\because \underset{n\to\infty}{\text{Lt}}\left(1+\frac{a}{n}\right)^n = \underset{n\to\infty}{\text{Lt}}\left[\left(1+\frac{a}{n}\right)^{n/a}\right]^a = e^a\right]$$

$\therefore$ By D' Alembert's ratio test, the series converges if $\frac{e}{x} > 1$ or if $x < e$ and diverges if $\frac{e}{x} < 1$ or if $x > e$.

If $x = e$, the ratio test fails, $\because \underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = 1$.

Now when $x = e$, $$\frac{u_n}{u_{n+1}} = \frac{\left(1+\frac{2}{n}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^{n+1}}\cdot\frac{1}{e}.$$

☞ **Since the expression $\frac{u_n}{u_{n+1}}$ involves the number e, so we do not apply Raabe's test but apply logarithmic test.**

$$\therefore \quad \log\frac{u_n}{u_{n+1}} = (n+1)\log\left(1+\frac{2}{n}\right) - (n+1)\log\left(1+\frac{1}{n}\right) - \log e$$

$$= (n+1)\left[\log\left(1+\frac{2}{n}\right) - \log\left(1+\frac{1}{n}\right)\right] - 1$$

$$= (n+1)\left[\left(\frac{2}{n}-\frac{1}{2}\cdot\frac{4}{n^2}+\frac{1}{3}\cdot\frac{8}{n^3}-\ldots\right)-\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}+\ldots\right)\right]-1$$

$$= (n+1)\left[\frac{1}{n}-\frac{3}{2n^2}+\ldots\right]-1 = 1-\frac{3}{2n}+\frac{1}{n}-\frac{3}{2n^2}+\ldots-1 = -\frac{1}{2n}-\frac{3}{2n^2}+\ldots$$

$\therefore$ $$\underset{n\to\infty}{\text{Lt}}\; n\log\frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}}\; n\left[-\frac{1}{2n}-\frac{3}{2n^2}+\ldots\right] = \underset{n\to\infty}{\text{Lt}}\left(-\frac{1}{2}-\frac{3}{2n}+\ldots\right) = -\frac{1}{2} < 1$$

$\therefore$ By log test, the series *diverges*.

Hence the given series $\sum u_n$ converges if $x < e$ and diverges if $x \geq e$.

Example 12. *Discuss the convergence of the series :* $x + \frac{2^2 x^2}{2!} + \frac{3^3 x^3}{3!} + \frac{4^4 x^4}{4!} + \frac{5^5 x^5}{5!} + \ldots$

Sol. Here $$u_n = \frac{n^n x^n}{n!} \quad\text{and}\quad u_{n+1} = \frac{(n+1)^{n+1}}{(n+1)!}x^{n+1}$$

$\therefore$ $$\frac{u_n}{u_{n+1}} = \frac{n^n x^n}{n!}\cdot\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{1}{x^{n+1}} = \frac{n^n}{n!}\cdot\frac{(n+1)\,n!}{(n+1)^{n+1}}\cdot\frac{1}{x} = \frac{1}{\left(1+\frac{1}{n}\right)^n}\cdot\frac{1}{x}$$

$\therefore$ $$\underset{n\to\infty}{\text{Lt}}\;\frac{u_n}{u_{n+1}} = \frac{1}{ex}.$$

$\therefore$ By D' Alembert's Ratio test, the series converges if $\frac{1}{ex} > 1$, *i.e.*, if $x < \frac{1}{e}$ and diverges if $\frac{1}{ex} < 1$ or if $x > \frac{1}{e}$.

If $x = \frac{1}{e}$, then Ratio test fails.

Now when $x = \frac{1}{e}$, we have $$\frac{u_n}{u_{n+1}} = \frac{1}{\left(1+\frac{1}{n}\right)^n}\cdot e$$

Since $\frac{u_n}{u_{n+1}}$ involves the number e, we prefer to apply log test rather than Raabe's test.

$\therefore$ $$\log\frac{u_n}{u_{n+1}} = \log e - n\log\left(1+\frac{1}{n}\right)$$

$$= 1-n\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\ldots\right) = \frac{1}{2n}-\frac{1}{3n^2}-\ldots$$

$\therefore$ $$\underset{n\to\infty}{\text{Lt}}\; n\log\frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}}\left(\frac{1}{2}-\frac{1}{3n}+\ldots\right) = \frac{1}{2} < 1.$$

$\therefore$ The series diverges by log test.

Hence the given series $\sum u_n$ *converges if* $x < \frac{1}{e}$ *and diverges if* $x \geq \frac{1}{e}$.

Example 13. *Discuss the convergence of the series :* $1 + \frac{2x}{2!} + \frac{3^2 x^2}{3!} + \frac{4^3 x^3}{4!} + \frac{5^4 x^4}{5!} + \ldots$

Sol. Here $$u_n = \frac{n^{n-1}x^{n-1}}{n!} \quad\text{and}\quad u_{n+1} = \frac{(n+1)^n x^n}{(n+1)!}$$

$\therefore$ $$\frac{u_n}{u_{n+1}} = \frac{n^{n-1}x^{n-1}}{n!}\cdot\frac{(n+1)!}{(n+1)^n x^n} = \frac{(n+1)n^{n-1}}{(n+1)^n}\cdot\frac{1}{x}$$

$$= \frac{n^n\left(1+\frac{1}{n}\right)}{n^n\left(1+\frac{1}{n}\right)^n}\cdot\frac{1}{x} = \frac{\left(1+\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)^n}\cdot\frac{1}{x} \to \frac{1}{ex} \text{ as } n\to\infty$$

$\therefore$ By D' Alembert's ratio test, the series converges if $\frac{1}{ex} > 1$ or if $x < \frac{1}{e}$ and diverges if $\frac{1}{ex} < 1$ *i.e.*, if $x > \frac{1}{e}$. If $x = \frac{1}{e}$, the test fails $\because$ $\underset{n\to\infty}{\text{Lt}}\;\frac{u_n}{u_{n+1}} = 1$.

Now when $x = \frac{1}{e}$, we have $$\frac{u_n}{u_{n+1}} = \frac{1}{\left(1+\frac{1}{n}\right)^{n-1}}\cdot e.$$

Since the expression for $\frac{u_n}{u_{n+1}}$ involves the number e, we use log test after the failure of ratio test.

$$\log\frac{u_n}{u_{n+1}} = \log e - (n-1)\log\left(1+\frac{1}{n}\right)$$

$$= 1-(n-1)\left[\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\ldots\right] = 1-n\left[\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\ldots\right]+\frac{1}{n}-\frac{1}{2n^2}+\ldots$$

$$= 1-\left(1-\frac{1}{2n}+\frac{1}{3n^2}+\ldots\right)+\frac{1}{n}-\frac{1}{2n^2}+\ldots = \frac{3}{2n}-\frac{5}{6n^2}+\ldots$$

$\therefore$ $$\underset{n\to\infty}{\text{Lt}}\; n\log\frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}}\left[\frac{3}{2}-\frac{5}{6n}+\ldots\right] = \frac{3}{2} > 1.$$

$\therefore$ The series converges by log test.

Hence the given series converges if $x \leq \frac{1}{e}$ and diverges if $x > \frac{1}{e}$.

Example 14. *Discuss the convergence of the series :* $\frac{a+x}{1} + \frac{(a+2x)^2}{2!} + \frac{(a+3x)^3}{3!} = \ldots$

Sol. Here $u_n = \dfrac{(a+nx)^n}{n!}$ and $u_{n+1} = \dfrac{[a+(n+1)x]^{n+1}}{(n+1)!}$

$\therefore \quad \dfrac{u_n}{u_{n+1}} = \dfrac{(n+1)!}{n!} \cdot \dfrac{(a+nx)^n}{[a+(n+1)x]^{n+1}}$

$$= \frac{(n+1)\,.\,n^n x^n\left(1+\frac{a}{nx}\right)^n}{(n+1)^{n+1}x^{n+1}\left[1+\frac{a}{(n+1)x}\right]^{n+1}} = \frac{\left(1+\frac{a}{nx}\right)^n}{\left(1+\frac{1}{n}\right)^n\left[1+\frac{a}{(n+1)x}\right]^{n+1}} \cdot \frac{1}{x} \quad ...(1)$$

$\therefore \quad \underset{n\to\infty}{\text{Lt}} \dfrac{u_n}{u_{n+1}} = \dfrac{e^{a/x}}{e\,.\,e^{a/x}} \cdot \dfrac{1}{x} = \dfrac{1}{ex}$ $\left[\because \ \underset{n\to\infty}{\text{Lt}}\left(1+\frac{p}{n}\right)^n = e^p\right]$

$\therefore$ By D' Alembert's Ratio test, the series converges if $\dfrac{1}{ex} > 1$ *i.e.*, if $x < \dfrac{1}{e}$ and diverges if $\dfrac{1}{ex} < 1$ or if $x > \dfrac{1}{e}$.

If $x = \dfrac{1}{e}$, the Ratio test fails, $\therefore \quad \underset{n\to\infty}{\text{Lt}} \dfrac{u_n}{u_{n+1}} = 1$

When $x = \dfrac{1}{e}$, we have $\left[\text{on putting } x = \frac{1}{e} \text{ in (1)}\right]$

$$\frac{u_n}{u_{n+1}} = \frac{e\left(1+\frac{ae}{n}\right)^n}{\left(1+\frac{ae}{n+1}\right)^{n+1}\left(1+\frac{1}{n}\right)^n}$$

$$\therefore \quad \log\frac{u_n}{u_{n+1}} = \left[\log e - n\log\left(1+\frac{1}{n}\right)\right] + \left[n\log\left(1+\frac{ae}{n}\right) - (n+1)\log\left(1+\frac{ae}{n+1}\right)\right]$$

$$= 1 - n\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \ldots\ldots\right) + \left[n\left(\frac{ae}{n} - \frac{a^2e^2}{2n^2} + \frac{a^3e^3}{3n^3} - \ldots\ldots\right)\right.$$
$$\left. - (n+1)\left\{\frac{ae}{n+1} - \frac{a^2e^2}{2(n+1)^2} + \frac{a^3e^3}{3(n+1)^3} - \ldots\ldots\right\}\right]$$

$$= \left(\frac{1}{2n} - \frac{1}{3n^2} + \ldots\ldots\right) + \left[\left(-\frac{a^2e^2}{2n} + \frac{a^3e^3}{3n^2} - \ldots\right) - \left(-\frac{a^2e^2}{2(n+1)} + \frac{a^3e^3}{3(n+1)^2} - \ldots\right)\right]$$

$$\therefore \quad n\log\frac{u_n}{u_{n+1}} = \left(\frac{1}{2} - \frac{1}{3n} + \ldots\ldots\right) + \left(-\frac{a^2e^2}{2} + \frac{a^3e^3}{3n} + \ldots\right) + \frac{n}{n+1}\left(\frac{a^2e^2}{2} - \frac{a^3e^3}{3(n+1)} + \ldots\right)$$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}}\ n\log\frac{u_n}{u_{n+1}} = \frac{1}{2} - \frac{a^2e^2}{2} + \frac{a^2e^2}{2} = \frac{1}{2} < 1$$

$\therefore$ By log test, the series diverges.

Hence the given series *converges* if $x < \dfrac{1}{e}$ *and diverges if* $x \geq \dfrac{1}{e}$.

Example 15. *Discuss the convergence of the series :* $1^p + \left(\dfrac{1}{2}\right)^p + \left(\dfrac{1.3}{2.4}\right)^p + \left(\dfrac{1.3.5}{2.4.6}\right)^p + \ldots\ldots$

Sol. Neglecting the first term, we have

$$u_n = \left[\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots\ldots2n}\right]^p \quad \text{and} \quad u_{n+1} = \left[\frac{1.3.5\ldots(2n-1)(2n+1)}{2.4.6\ldots\ldots2n\,.(2n+2)}\right]^p$$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \left(\frac{2n+2}{2n+1}\right)^p = \frac{\left(1+\frac{1}{n}\right)^p}{\left(1+\frac{1}{2n}\right)^p}$$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = 1 \quad \therefore \quad \text{Ratio test fails.}$$

Now
$$\frac{u_n}{u_{n+1}} = \left(1+\frac{1}{n}\right)^p\left(1+\frac{1}{2n}\right)^{-p}$$

$$= \left[1+\frac{p}{n}+\frac{p(p-1)}{2!}\cdot\frac{1}{n^2}+\ldots\right]\left[1-\frac{p}{2n}+\frac{p(p+1)}{2!}\cdot\frac{1}{4n^2}+\ldots\right]$$

$$= 1+\frac{p}{n}-\frac{p}{2n}+O\left(\frac{1}{n^2}\right)$$

where $O\left(\dfrac{1}{n^2}\right)$ **represents terms of the order** $\dfrac{1}{n^2}$ **and higher powers of** $\dfrac{1}{n^2}$.

$$= 1+\frac{\frac{p}{2}}{n}+O\left(\frac{1}{n^2}\right).$$

Comparing this with $\dfrac{u_n}{u_{n+1}} = 1+\dfrac{\lambda}{n}+O\left(\dfrac{1}{n^2}\right)$. We have $\lambda = \dfrac{p}{2}$.

$\therefore$ By Gauss test, the given series Σu_n converges if $\dfrac{p}{2} > 1$ *i.e.*, $p > 2$ and diverges if $\dfrac{p}{2} \leq 1$ or if $p \leq 2$.

Hence the series *converges if* $p > 2$ *and diverges if* $p \leq 2$.

Example 16. *Discuss the convergence of the series :* $1 + a + \frac{a(a+1)}{1.2} + \frac{a(a+1)(a+2)}{1.2.3} + \ldots (a > 0)$.

Sol. Neglecting the first term, we have

$$u_n = \frac{a(a+1)(a+2)\ldots\ldots(a+n-1)}{1.2.3\ldots\ldots n}$$

and
$$u_{n+1} = \frac{a(a+1)(a+2)\ldots\ldots(a+n-1)(a+n)}{1.2.3\ldots\ldots n.(n+1)}$$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{n+1}{a+n} = \frac{1+\frac{1}{n}}{1+\frac{a}{n}}$$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = 1 \quad \therefore \quad \text{Ratio Test fails.}$$

Now expanding $\frac{u_n}{u_{n+1}}$ in terms of $\frac{1}{n}$, we have

$$\frac{u_n}{u_{n+1}} = \left(1+\frac{1}{n}\right)\left(1+\frac{a}{n}\right)^{-1} = \left(1+\frac{1}{n}\right)\left[1-\frac{a}{n}+\frac{a^2}{n^2}+\ldots\right] = 1+\frac{1-a}{n}+O\left(\frac{1}{n^2}\right).$$

$\therefore$ By Gauss test, the given series Σu_n converges if $1-a > 1$ *i.e.,* if $a < 0$ and diverges if $1 - a \le 1$ *i.e.* if $a \ge 0$.

Since $a > 0$ the given series diverges.

Example 17. *Discuss the convergence of the series :* $x^2 (\log 2)^q + x^3 (\log 3)^q + x^4 (\log 4)^q + \ldots$

Sol. Here $u_n = x^{n+1} [\log (n+1)]^q$ and $u_{n+1} = x^{n+2} [\log (n+2)]^q$.

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{1}{x}\cdot\left[\frac{\log(n+1)}{\log(n+2)}\right]^q = \left[\frac{\log\left\{n\left(1+\frac{1}{n}\right)\right\}}{\log\left\{n\left(1+\frac{2}{n}\right)\right\}}\right]^q \cdot \frac{1}{x}$$

$$= \left[\frac{\log n + \log\left(1+\frac{1}{n}\right)}{\log n + \log\left(1+\frac{2}{n}\right)}\right]^q \cdot \frac{1}{x} = \left[\frac{\log n + \frac{1}{n} - \frac{1}{2n^2} + \ldots}{\log n + \frac{2}{n} - \frac{1}{2}\cdot\frac{4}{n^2} + \ldots}\right]^q \cdot \frac{1}{x}$$

$$= \frac{\left(1+\frac{1}{n\log n} - \ldots\right)^q}{\left(1+\frac{2}{n\log n} - \ldots\right)^q}\cdot\frac{1}{x} = \left(1+\frac{1}{n\log n}+\ldots\right)^q \left(1+\frac{2}{n\log n}+\ldots\right)^{-q} \cdot \frac{1}{x}$$

[Expand by Binomial Theorem]

$$= \left(1+\frac{q}{n\log n}+\ldots\right)\left(1-\frac{2q}{n\log n}+\ldots\right)\cdot\frac{1}{x} = \left(1+\frac{q}{n\log n}-\frac{2q}{n\log n}+\ldots\right)\cdot\frac{1}{x}$$

$$= \left(1-\frac{q}{n\log n}+\ldots\right)\frac{1}{x} \qquad \ldots(A)$$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \frac{1}{x}.$$

Hence by Ratio test, the series Σu_n converges if $\frac{1}{x} > 1$ *i.e.,* if $x < 1$ and diverges if $\frac{1}{x} < 1$ or if $x > 1$.

If $x = 1$, $\quad \underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = 1 \quad \therefore$ Ratio test fails.

When $x = 1$, we have from (A) $\frac{u_n}{u_{n+1}} = 1 - \frac{q}{n\log n} + \ldots$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}}\, n\left(\frac{u_n}{u_{n+1}}-1\right) = \underset{n\to\infty}{\text{Lt}}\left(-\frac{q}{\log n}+\ldots\right) = 0 < 1.$$

$\therefore$ By Raabe's Test, the series diverges.

Hence the given series converges if $x < 1$ and diverges if $x \ge 1$.

Example 18. *Discuss the convergence of the series :* $\frac{1}{(\log 2)^p} + \frac{1}{(\log 3)^p} + \frac{1}{(\log 4)^p} + \ldots$

Sol. *Please try yourself.* **[Ans. Divergent]**

Example 19. *Discuss the convergence of the series :* $\frac{a}{b} + \frac{a(a+1)}{b(b+1)} + \frac{a(a+1)(a+2)}{b(b+1)(b+2)} + \ldots$

Sol.
$$u_n = \frac{a(a+1)(a+2)\ldots\ldots(a+n-1)}{b(b+1)(b+2)\ldots\ldots(b+n-1)}$$

and
$$u_{n+1} = \frac{a(a+1)(a+2)\ldots\ldots(a+n-1)(a+n)}{b(b+1)(b+2)\ldots\ldots(b+n-1)(b+n)}.$$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{b+n}{a+n} = \frac{1+\frac{b}{n}}{1+\frac{a}{n}}.$$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = 1 \quad \therefore \quad \text{Ratio test fails.}$$

Now expanding $\frac{u_n}{u_{n+1}}$ in powers of $\frac{1}{n}$, we have

$$\frac{u_n}{u_{n+1}} = \frac{1+\frac{b}{n}}{1+\frac{a}{n}} = \left(1+\frac{b}{n}\right)\left(1+\frac{a}{n}\right)^{-1} = \left(1+\frac{b}{n}\right)\left(1-\frac{a}{n}+\frac{a^n}{n^2}-\ldots\right)$$

$$= 1 + \frac{b}{n} - \frac{a}{n} + \frac{a^2}{n^2} - \frac{ab}{n^2} + \ldots\ldots = 1 + \frac{b-a}{n} + O\left(\frac{1}{n^2}\right).$$

$\therefore$ By Gauss test, the series Σu_n converges if $b - a > 1$ and diverges if $b - a \le 1$.

i.e., series converges or diverges according as $b > a + 1$ or $b \le a + 1$.

Example 20. *Discuss the convergence of the series :* $\frac{a}{b} + \frac{a(a+d)}{b(b+d)}x + \frac{a(a+d)(a+2d)}{b(b+d)(b+2d)}x^2 + \ldots$

Sol. Here
$$u_n = \frac{a(a+d)(a+2d)\ldots\ldots[a+(n-1)d]}{b(b+d)(b+2d)\ldots\ldots[b+(n-1)d]} \cdot x^{n-1}$$

and
$$u_{n+1} = \frac{a(a+d)(a+2d)\ldots\ldots[a+(n-1)d](a+nd)}{b(b+d)(b+2d)\ldots\ldots[b+(n-1)d](b+nd)}$$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{b+nd}{a+nd} \cdot \frac{1}{x} = \frac{1+\frac{b}{nd}}{1+\frac{a}{nd}} \cdot \frac{1}{x}$$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \frac{1}{x}.$$

$\therefore$ By D' Alembert's ratio test, the series Σu_n converges if $\frac{1}{x} > 1$ *i.e.,* if $x < 1$ and diverges if $\frac{1}{x} < 1$ *i.e.,* if $x > 1$.

If $x = 1$, then $\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = 1$ $\therefore$ Ratio test fails.

Putting $x = 1$, we have

$$\frac{u_n}{u_{n+1}} = \frac{1+\frac{b}{nd}}{1+\frac{a}{nd}} = \left(1+\frac{b}{nd}\right)\left(1+\frac{a}{nd}\right)^{-1}$$

$$= \left(1+\frac{b}{nd}\right)\left(1-\frac{a}{nd}+\frac{a^2}{n^2d^2}\ldots\ldots\right) = 1 + \frac{b}{nd} - \frac{a}{nd} + \frac{a^2}{n^2d^2} - \frac{ab}{n^2d^2} + \ldots\ldots$$

$$= 1 + \frac{b-a}{nd} + \frac{a^2-ab}{n^2d^2} + \ldots\ldots = 1 + \frac{1}{n}\left(\frac{b-a}{d}\right) + \frac{1}{n^2}\left(\frac{a^2-ab}{d^2}\right) + \ldots\ldots$$

$\therefore$ By Gauss test, the series Σu_n converges if $\frac{b-a}{d} > 1$ or if $b > a + d$ and diverges if $\frac{b-a}{d} \le 1$ or if $b \le a + d$.

Thus the series converges or diverges according as $b > a + d$ or $b \le a + d$.

Example 21. *Discuss the convergence of the series :*

$$1 + \frac{\alpha\beta}{1.\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}x^2 + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1.2.3.\gamma(\gamma+1)(\gamma+2)}x^3 + \ldots\ldots$$

Sol. Neglecting the first term,

$$u_n = \frac{\alpha(\alpha+1)\ldots\ldots(\alpha+n-1)\beta(\beta+1)\ldots\ldots(\beta+n-1)}{1.2.3\ldots n.\gamma(\gamma+1)\ldots\ldots(\gamma+n-1)} \cdot x^n$$

$$u_{n+1} = \frac{\alpha(\alpha+1)\ldots(\alpha+n-1)(\alpha+n)\beta(\beta+1)\ldots(\beta+n-1)(\beta+n)}{1.2.3\ldots n(n+1)\gamma(\gamma+1)(\gamma+2)\ldots(\gamma+n-1)(\gamma+n)} \cdot x^{n+1}$$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{(n+1)(\gamma+n)}{(\alpha+n)(\beta+n)} \cdot \frac{1}{x} = \frac{\left(1+\frac{1}{n}\right)\left(1+\frac{\gamma}{n}\right)}{\left(1+\frac{\alpha}{n}\right)\left(1+\frac{\beta}{n}\right)} \cdot \frac{1}{x}$$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \frac{1}{x}.$$

$\therefore$ By D' Alembert's Ratio test, the series Σu_n converges if $\frac{1}{x} > 1$ *i.e.,* if $x < 1$ and diverges if $\frac{1}{x} < 1$ or if $x > 1$.

If $x = 1$, $\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = 1$ $\therefore$ Ratio test fails.

Putting $x = 1$ in $\frac{u_n}{u_{n+1}}$, we have

$$\frac{u_n}{u_{n+1}} = \frac{\left(1+\frac{1}{n}\right)\left(1+\frac{\gamma}{n}\right)}{\left(1+\frac{\alpha}{n}\right)\left(1+\frac{\beta}{n}\right)}$$

$$= \left(1+\frac{1}{n}\right)\left(1+\frac{\gamma}{n}\right)\left(1+\frac{\alpha}{n}\right)^{-1}\left(1+\frac{\beta}{n}\right)^{-1}$$

[Expand by Binomial Theorem]

$$= \left(1+\frac{1}{n}\right)\left(1+\frac{\gamma}{n}\right)\left(1-\frac{\alpha}{n}+\frac{\alpha^2}{n^2}+\ldots\right)\left(1-\frac{\beta}{n}+\frac{\beta^2}{n^2}\right)$$

$$= \left(1+\frac{1}{n}+\frac{\gamma}{n}+\frac{\gamma}{n^2}\right)\left(1-\frac{\alpha}{n}-\frac{\beta}{n}+\frac{\alpha\beta}{n^2}+\frac{\alpha^2}{n^2}+\frac{\beta^2}{n^2}\ldots\right)$$

$$= 1 + \frac{1}{n}(1+\gamma-\alpha-\beta) + O\left(\frac{1}{n^2}\right).$$

$\therefore$ By Gauss test, the series Σu_n converges if $1+\gamma-\alpha-\beta > 1$ *i.e.,* if $\gamma > \alpha+\beta$ and diverges if $1+\gamma-\alpha-\beta \le 1$ *i.e.,* if $\gamma \le \alpha+\beta$.

Thus the given series *converges if $x < 1$ and diverges if $x > 1$. If $x = 1$, then the series converges if $\gamma > \alpha + \beta$ and diverges if $\gamma \le \alpha + \beta$.*

Example 22. *Discuss the convergence of the series :*

$$\sum \frac{n!}{x(x+1)(x+2)\dots(x+n-1)}$$

Sol. Here
$$u_n = \frac{n!}{x(x+1)(x+2)\dots(x+n-1)}$$
$$u_{n+1} = \frac{(n+1)!}{x(x+1)(x+2)\dots(x+n-1)(x+n)}$$

$\therefore$
$$\frac{u_n}{u_{n+1}} = \frac{x+n}{n+1} = \frac{1+\frac{x}{n}}{1+\frac{1}{n}}$$

$\therefore$
$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}} \frac{1+\frac{x}{n}}{1+\frac{1}{n}} = 1$$

$\therefore$ Ratio Test fails.

Now
$$\frac{u_n}{u_{n+1}} - 1 = \frac{x+n}{n+1} - 1 = \frac{x-1}{n+1}$$

$\therefore$
$$\underset{n\to\infty}{\text{Lt}}\, n\left(\frac{u_n}{u_{n+1}} - 1\right) = \underset{n\to\infty}{\text{Lt}} \frac{n(x-1)}{n+1} = \underset{n\to\infty}{\text{Lt}} \frac{x-1}{1+\frac{1}{n}} = x-1.$$

$\therefore$ By Raabe's test, Σu_n is convergent if $x - 1 > 1$ *i.e.,* $x > 2$ and divergent if $x - 1 < 1$ *i.e.,* $x < 2$.

When $x = 2$,
$$\frac{u_n}{u_{n+1}} = \frac{n+2}{n+1} = \frac{1+\frac{2}{n}}{1+\frac{1}{n}} = \left(1+\frac{2}{n}\right)\left(1+\frac{1}{n}\right)^{-1}$$
$$= \left(1+\frac{2}{n}\right)\left(1-\frac{1}{n}+\frac{1}{n^2}-\dots\right) = 1-\frac{1}{n}+\frac{2}{n}+\frac{1}{n^2}-\frac{2}{n^2}\dots\dots$$
$$= 1+\frac{1}{n}-\frac{1}{n^2}\dots\dots = 1+\frac{1}{n}+O\left(\frac{1}{n^2}\right)$$

By Gauss Test, (here $\lambda = 1$) Σu_n is divergent.

Hence Σu_n is convergent if $x > 2$ and divergent if $x \le 2$.

Example 23. *Discuss the convergence of the series :*

$$1+\frac{1}{2}x+\frac{1.3}{2.4}x^2+\frac{1.3.5}{2.4.6}x^3+\dots\dots (x>0).$$

Sol. Neglecting the first term
$$u_n = \frac{1.3.5\dots\dots(2n-1)}{2.4.6\dots\dots(2n)}x^n$$
$$u_{n+1} = \frac{1.3.5\dots\dots(2n-1)(2n+1)}{2.4.6\dots\dots(2n)(2n+2)}x^{n+1}$$
$$\frac{u_n}{u_{n+1}} = \frac{2n+2}{2n+1}\cdot\frac{1}{x} = \frac{1+\frac{1}{n}}{1+\frac{1}{2n}}\cdot\frac{1}{x} \to \frac{1}{x} \text{ as } n\to\infty.$$

$\therefore$ By Ratio test, Σu_n convergent if $\frac{1}{x} > 1$ *i.e.,* $x < 1$ and diverges if $\frac{1}{x} < 1$ *i.e.,* $x > 1$.

If $x = 1$, Ratio test fails.

When $x = 1$,
$$\frac{u_n}{u_{n+1}} = \frac{2n+2}{2n+1}$$
$$n\left(\frac{u_n}{u_{n+1}} - 1\right) = n\left(\frac{2n+2}{2n+1} - 1\right) = \frac{n}{2n+1} = \frac{1}{2+\frac{1}{n}}$$
$$\underset{n\to\infty}{\text{Lt}}\, n\left(\frac{u_n}{u_{n+1}} - 1\right) = \underset{n\to\infty}{\text{Lt}} \frac{1}{2+\frac{1}{n}} = \frac{1}{2} < 1.$$

$\therefore$ By Raabe's test, Σu_n is divergent.

Hence Σu_n is convergent if $x < 1$ and divergent if $x \ge 1$.

Example 24. *Discuss the convergence of the series :*

$$\sum \frac{1.3.5\dots\dots(2n-1)}{2.4.6\dots\dots(2n)}\cdot\frac{x^{2n}}{2n}.$$

Sol. Here
$$u_n = \frac{1.3.5\dots\dots(2n-1)}{2.4.6\dots\dots(2n)}\cdot\frac{x^{2n}}{2n}$$

$\therefore$
$$u_{n+1} = \frac{1.3.5\dots\dots(2n-1)(2n+1)}{2.4.6\dots\dots(2n)(2n+2)}\cdot\frac{x^{2n+2}}{2n+2}$$
$$\frac{u_n}{u_{n+1}} = \frac{(2n+2)(2n+2)}{2n(2n+1)}\cdot\frac{1}{x^2} = \frac{\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)}{1+\frac{1}{2n}}\cdot\frac{1}{x^2} \to \frac{1}{x^2} \text{ as } n\to\infty.$$

$\therefore$ By Ratio test, Σu_n converges if $\frac{1}{x^2} > 1$ *i.e.,* $x^2 < 1$ and diverges if $\frac{1}{x^2} < 1$ *i.e.,* $x^2 > 1$.

When $x^2 = 1$, Ratio test fails.

When $x^2 = 1$,
$$\frac{u_n}{u_{n+1}} = \frac{(2n+2)^2}{2n(2n+1)} = \frac{4n^2+8n+4}{4n^2+2n}$$

$$n\left(\frac{u_n}{u_{n+1}}-1\right)=n\left(\frac{4n^2+8n+4}{4n^2+2n}-1\right)=n\left(\frac{6n+4}{4n^2+2n}\right)=\frac{3n+2}{2n+1}=\frac{3+\frac{2}{n}}{2+\frac{1}{n}}$$

$$\underset{n\to\infty}{\text{Lt}}\ n\left(\frac{u_n}{u_{n+1}}-1\right)=\underset{n\to\infty}{\text{Lt}}\ \frac{3+\frac{2}{n}}{2+\frac{1}{n}}=\frac{3}{2}>1.$$

$\therefore$ By Raabe's test, Σu_n is convergent.

Hence Σu_n is convergent if $x^2 \le 1$ and divergent if $x^2 \ge 1$.

Example 25. *Discuss the convergence of the series :*

$$\sum \frac{(n!)^2}{(2n)!}x^n,\quad x>0.$$

Sol. Here $\quad u_n=\dfrac{(n!)^2}{(2n)!}x^n$

$$u_{n+1}=\frac{((n+1)!)^2}{(2n+2)!}\cdot x^{n+1}$$

$$\therefore\quad \frac{u_n}{u_{n+1}}=\frac{(n!)^2}{(2n)!}\cdot\frac{(2n+2)!}{((n+1)!)^2}\cdot\frac{1}{x}$$

$$=\frac{(n!)^2\cdot(2n+2)(2n+1)(2n)!}{(2n)!\,.(n+1)^2(n!)^2}\cdot\frac{1}{x}=\frac{2(2n+1)}{n+1}\cdot\frac{1}{x}=\frac{4+\frac{2}{n}}{1+\frac{1}{n}}\cdot\frac{1}{x}$$

$$\underset{n\to\infty}{\text{Lt}}\ \frac{u_n}{u_{n+1}}=\underset{n\to\infty}{\text{Lt}}\ \frac{4+\frac{2}{n}}{1+\frac{1}{n}}\cdot\frac{1}{x}=\frac{4}{x}.$$

$\therefore$ By Ratio test, Σu_n converges if $\dfrac{4}{x}>1$ *i.e.*, $x<4$ and diverges if $\dfrac{4}{x}<1$, *i.e.*, $x>4$.

Ratio test fails when $x = 4$.

When $x = 4$, $\quad \dfrac{u_n}{u_{n+1}}=\dfrac{4n+2}{4n+4}=\dfrac{2n+1}{2n+2}$

$$n\left(\frac{u_n}{u_{n+1}}-1\right)=n\left(\frac{2n+1}{2n+2}-1\right)\frac{-n}{2n+2}=\frac{-1}{2+\frac{2}{n}}$$

$$\underset{n\to\infty}{\text{Lt}}\ n\left(\frac{u_n}{u_{n+1}}-1\right)=\underset{n\to\infty}{\text{Lt}}\ \frac{-1}{2+\frac{2}{n}}=-\frac{1}{2}<1.$$

$\therefore$ By Raabe's test, Σu_n is divergent.

Hence Σu_n converges if $x < 4$ and diverges if $x \ge 4$.

Example 26. *Test the series* $\left(\frac{1}{3}\right)^2+\left(\frac{1.2}{3.5}\right)^2+\left(\frac{1.2.3}{3.5.7}\right)^2+\ldots\ldots$ *for convergence.*

Sol. *Please try yourself.* **[Ans.** Convergent]

Example 27. *Test for convergence the positive term series*

$$\frac{\alpha}{\beta}+\frac{1+\alpha}{1+\beta}+\frac{(1+\alpha)(2+\alpha)}{(1+\beta)(2+\beta)}+\ldots\ldots$$

Sol. Here $\quad u_n=\dfrac{(1+\alpha)(2+\alpha)\ldots\ldots(n+\alpha)}{(1+\beta)(2+\beta)\ldots\ldots(n+\beta)}$ after neglecting first term

$$u_{n+1}=\frac{(1+\alpha)(2+\alpha)\ldots\ldots(n+\alpha)(n+1+\alpha)}{(1+\beta)(2+\beta)\ldots\ldots(n+\beta)(n+1+\beta)}$$

$$\therefore\quad \frac{u_n}{u_{n+1}}=\frac{n+1+\beta}{n+1+\alpha}=\frac{1+\frac{1+\beta}{n}}{1+\frac{1+\alpha}{n}}$$

$$\underset{n\to\infty}{\text{Lt}}\ \frac{u_n}{u_{n+1}}=1.\quad\therefore\quad \text{The Ratio test fails}$$

Now $\quad \dfrac{u_n}{u_{n+1}}=\left(1+\dfrac{1+\beta}{n}\right)\left(1+\dfrac{1+\alpha}{n}\right)^{-1}$

$$=\left(1+\frac{1+\beta}{n}\right)\left(1-\frac{1+\alpha}{n}+\ldots\ldots\right)=1+\frac{\beta-\alpha}{n}+O\left(\frac{1}{n^2}\right)$$

$\therefore$ By Gauss test, the series is convergent if $\beta-\alpha>1$ *i.e.*, $\beta>\alpha+1$ and divergent if $\beta-\alpha\le 1$ *i.e.*, if $\beta\le\alpha+1$.

Example 28. *Show that the series*

$$1+\frac{a(1-a)}{1^2}+\frac{(1+a)\,a(1-a)(2-a)}{1^2\,.\,2^2}+\frac{(2+a)(1+a)\,a(1-a)(2-a)(3-a)}{1^2\,.\,2^2\,.\,3^2}+\ldots\ldots$$

where a is a proper fraction, is divergent.

Sol. Neglecting the first term, we have

$$u_n=\frac{(n-1+a)(n-2+a)\ldots\ldots(1+a)\,a(1-a)(2-a)\ldots(n-a)}{1^2\,.\,2^2\,.\,3^2\ldots n^2}$$

$$u_{n+1}=\frac{(n+a)(n-1+a)\ldots\ldots(1+a)\,a(1-a)\ldots(n-a)(n+1-a)}{1^2\,.\,2^2\,.\,3^2\ldots n^2(n+1)^2}$$

$$\therefore\quad \frac{u_n}{u_{n+1}}=\frac{(n+1)^2}{(n+a)(n+1-a)}=\frac{\left(1+\frac{1}{n}\right)^2}{\left(1+\frac{a}{n}\right)\left(1+\frac{1-a}{n}\right)}.$$

$$\lim_{n \to \infty} \frac{u_n}{u_{n+1}} = 1$$

$\therefore$ The Ratio test fails.

Now $\dfrac{u_n}{u_{n+1}} = \left(1+\dfrac{1}{n}\right)^2 \left(1+\dfrac{a}{n}\right)^{-1} \left(1+\dfrac{1-a}{n}\right)^{-1}$

$$= \left(1+\frac{2}{n}+\frac{1}{n^2}\right)\left(1-\frac{a}{n}+.....\right)\left(1-\frac{1-a}{n}-......\right) = \left(1-\frac{2}{n}+\frac{1}{n^2}\right)\left(1-\frac{1}{n}+.....\right)$$

$$= 1-\frac{3}{n}+O\left(\frac{1}{n^2}\right) = 1+\frac{-3}{n}+O\left(\frac{1}{n^2}\right)$$

$\therefore$ By Gauss Test, the series is divergent. (Here $\lambda = -3 < 1$)

Example 29. *Discuss the convergence of the series :*

(i) $\dfrac{1}{2} \cdot \dfrac{x^2}{4} + \dfrac{1.3.5}{2.4.6} \cdot \dfrac{x^4}{8} + \dfrac{1.3.5.7.9}{2.4.6.8.10} \cdot \dfrac{x^6}{12} + ...$

(ii) $1 + \dfrac{a}{2} + \dfrac{a}{3} + \dfrac{a^2}{4} + \dfrac{a^2}{5} + \dfrac{a^3}{6} + \dfrac{a^3}{7} + ...$ $(0 < a \le 1)$.

Sol. (i) Here $u_n = \dfrac{1.3.5.....(4n-5)(4n-3)}{2.4.6.......(4n-4)(4n-2)} \cdot \dfrac{x^{2n}}{4n}$

$$u_{n+1} = \frac{1.3.5.....(4n-5)(4n-3)(4n-1)(4n+1)}{2.4.6.......(4n-4)(4n-2)4n(4n+2)} \cdot \frac{x^{2n+2}}{4n+4}$$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{4n(4n+2)}{(4n-1)(4n+1)} \cdot \frac{4n+4}{4n} \cdot \frac{1}{x^2}$$

$$= \frac{(4n+2)(4n+4)}{(4n-1)(4n+1)} \cdot \frac{1}{x^2} = \frac{\left(1+\frac{1}{2n}\right)\left(1+\frac{1}{n}\right)}{\left(1-\frac{1}{4n}\right)\left(1+\frac{1}{4n}\right)} \cdot \frac{1}{x^2}$$

$$\lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \frac{1}{x^2}$$

$\therefore$ By Ratio test, the series is convergent if $\dfrac{1}{x^2} > 1$ *i.e.,* if $x^2 < 1$, and divergent if $\dfrac{1}{x^2} < 1$ *i.e.,* if $x^2 > 1$.

When $x^2 = 1$, the ratio test fails.

However, when $x^2 = 1$, we have

$$\frac{u_n}{u_{n+1}} = \frac{\left(1+\frac{1}{2n}\right)\left(1+\frac{1}{n}\right)}{\left(1-\frac{1}{4n}\right)\left(1+\frac{1}{4n}\right)} = \frac{1+\frac{3}{2n}+\frac{1}{2n^2}}{1-\frac{1}{16n^2}}$$

$$= \left(1+\frac{3}{2n}+\frac{1}{2n^2}\right)\left(1-\frac{1}{16n^2}\right)^{-1} = \left(1+\frac{3}{2n}+\frac{1}{2n^2}\right)\left(1+\frac{1}{16n^2}+.....\right)$$

$$= 1+\frac{3}{2n}+O\left(\frac{1}{n^2}\right) = 1+\frac{\frac{3}{2}}{n}+O\left(\frac{1}{n^2}\right)$$

$\therefore$ By Gauss test, $\sum u_n$ is convergent ($\lambda = \frac{3}{2} > 1$).

Hence $\sum u_n$ is convergent if $x^2 \le 1$ and divergent if $x^2 > 1$.

(ii) The given series is $1 + \dfrac{2+3}{2.3}a + \dfrac{4+5}{4.5}a^2 + \dfrac{6+7}{6.7}a^3 + ...$

Neglecting the first term

$$u_n = \frac{2n+(2n+1)}{2n(2n+1)}a^n = \frac{4n+1}{2n(2n+1)}a^n$$

$$\therefore \quad u_{n+1} = \frac{4(n+1)+1}{2(n+1)[2(n+1)+1]}a^{n+1} = \frac{4n+5}{(2n+2)(2n+3)}a^{n+1}$$

$$\frac{u_n}{u_{n+1}} = \frac{(2n+2)(2n+3)(4n+1)}{2n(2n+1)(4n+5)} \cdot \frac{1}{a}$$

$$= \frac{\left(1+\frac{1}{n}\right)\left(1+\frac{3}{2n}\right)\left(1+\frac{1}{4n}\right)}{\left(1+\frac{1}{2n}\right)\left(1+\frac{5}{4n}\right)} \cdot \frac{1}{a}$$

$$\lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \frac{1}{a}$$

When $0 < a < 1$, $\dfrac{1}{a} > 1$.

$\therefore$ By ratio test, the series is convergent when $0 < a < 1$.

When $a = 1$, the series becomes $1 + \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + ... = \sum \dfrac{1}{n}$ which is divergent.

Hence the series is convergent when $0 < a < 1$ and divergent when $a = 1$.

Example 30. *Test for convergence the series*

$$1 + \frac{a}{b}x + \frac{a(a+1)}{b(b+1)} \cdot \frac{x^2}{2!} + \frac{a(a+1)(a+2)}{b(b+1)(b+2)} \cdot \frac{x^3}{3!} + ... (a > 0, b > 0).$$

Sol. Neglecting the first term, we have

$$u_n = \frac{a(a+1)(a+2)......[a+(n-1)]}{b(b+1)(b+2)......[b+(n-1)]} \cdot \frac{x^n}{n!}$$

$$u_{n+1} = \frac{a(a+1)(a+2)......[a+(n-1)](a+n)}{b(b+1)(b+2)......[b+(n-1)](b+n)} \cdot \frac{x^{n+1}}{(n+1)!}$$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{b+n}{a+n} \cdot \frac{(n+1)!}{n!} \cdot \frac{x^n}{x^{n+1}} = \frac{b+n}{a+n} \cdot \frac{(n+1)}{x} = \frac{\frac{b}{n}+1}{\frac{a}{n}+1} \cdot \frac{n+1}{x}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{\frac{b}{n}+1}{\frac{a}{n}+1} \cdot \frac{n+1}{x} = \infty$$

$\therefore$ By ratio test, the series is convergent for every x.

Example 31. *Test for convergence the series* $1 + \left(\frac{2}{3}\right)^p + \left(\frac{2.4}{3.5}\right)^p + \left(\frac{2.4.6}{3.5.7}\right)^p + \ldots\ldots$

Sol. Neglecting the first term, we have

$$u_n = \left[\frac{2.4.6\ldots..(2n)}{3.5.7\ldots..(2n+1)}\right]^p$$

$$u_{n+1} = \left[\frac{2.4.6\ldots..(2n)(2n+2)}{3.5.7\ldots..(2n+1)(2n+3)}\right]^p$$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \left(\frac{2n+3}{2n+2}\right)^p = \left(\frac{1+\frac{3}{2n}}{1+\frac{1}{n}}\right)^p = \left(1+\frac{3}{2n}\right)^p\left(1+\frac{1}{n}\right)^{-p}$$

$$= \left[1 + p.\frac{3}{2n} + O\left(\frac{1}{n^2}\right)\right]\left[1 - \frac{p}{n} + O\left(\frac{1}{n}\right)\right]$$

$$= 1 + \left(\frac{3}{2} - 1\right)\frac{p}{n} + O\left(\frac{1}{n^2}\right) = 1 + \frac{\frac{1}{2}p}{n} + O\left(\frac{1}{n^2}\right)$$

$\therefore$ By Gauss test, the series is convergent if $\frac{p}{2} > 1$ *i.e.*, if $p > 2$ and divergent if $\frac{p}{2} \le 1$ *i.e.* if $p \le 2$.

Example 32. *Test for convergence the series :*

(*i*) $\sum \frac{1.3.5\ldots..(2n-1)}{2.4.6\ldots...(2n)} \cdot \frac{1}{2n+1}$

(*ii*) $1 + \frac{2!}{2^2}x + \frac{3!}{3^3}x^2 + \ldots\ldots, \quad x > 0$

(*iii*) $\sum \frac{2n!}{(n!)^2} . x^n, \ x > 0.$

Sol. (*i*) Here

$$u_n = \frac{1.3.5\ldots..(2n-1)}{2.4.6\ldots...(2n)} \cdot \frac{1}{2n+1}$$

$$u_{n+1} = \frac{1.3.5\ldots..(2n-1)(2n+1)}{2.4.6\ldots...(2n)(2n+2)} \cdot \frac{1}{2n+3}$$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{2n+2}{2n+1} \cdot \frac{2n+3}{2n+1} = \frac{\left(1+\frac{1}{n}\right)\left(1+\frac{3}{2n}\right)}{\left(1+\frac{1}{2n}\right)^2}$$

$$= \left(1+\frac{5}{2n}+\frac{3}{2n^2}\right)\left(1+\frac{1}{2n}\right)^{-2} = \left(1+\frac{5}{2n}+\frac{3}{2n^2}\right)\left(1-\frac{1}{n}+\ldots..\right)$$

$$= 1 + \frac{3}{2n} + O\left(\frac{1}{n^2}\right) = 1 + \frac{3/2}{n} + O\left(\frac{1}{n^2}\right) = 1 + \frac{\lambda}{n} + O\left(\frac{1}{n^2}\right)$$

$\therefore$ By Gauss test, the given series is convergent since $\lambda = \frac{3}{2} > 1$.

(*ii*) Here $\quad u_n = \frac{n!}{n^n} x^{n-1}$

$$u_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}} x^n$$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{n!}{(n+1)!} \cdot \frac{(n+1)^{n+1}}{n^n} \cdot \frac{1}{x}$$

$$= \frac{1}{n+1} \cdot \frac{(n+1)^{n+1}}{n^n} \cdot \frac{1}{x} = \frac{(n+1)^n}{n^n} \cdot \frac{1}{x} = \left(1+\frac{1}{n}\right)^n \cdot \frac{1}{x}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{e}{x}$$

$\therefore$ By ratio test, $\sum u_n$ is convergent if $\frac{e}{x} > 1$, *i.e.*, if $x < e$ and divergent if $\frac{e}{x} < 1$, *i.e.*, if $x > e$.

Ratio test fails when $x = e$.

Now, when $x = e$, $\quad \frac{u_n}{u_{n+1}} = \left(1+\frac{1}{n}\right)^n \cdot \frac{1}{e}$

$$\therefore \quad \log \frac{u_n}{u_{n+1}} = n \log\left(1+\frac{1}{n}\right) - \log e$$

$$= n\left[\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \ldots..\right] - 1 = -\frac{1}{2n} + \frac{1}{3n^2} \ldots\ldots$$

$$\Rightarrow \quad n \log \frac{u_n}{u_{n+1}} = -\frac{1}{2} + \frac{1}{3n} \ldots\ldots$$

$$\lim_{n\to\infty} n \log \frac{u_n}{u_{n+1}} = -\frac{1}{2} < 1$$

$\therefore$ By logarithmic test, $\sum u_n$ is divergent.

Hence $\sum u_n$ is convergent if $x < e$ and divergent if $x \ge e$.

(*iii*) Here
$$u_n = \frac{2n!}{(n!)^2} \cdot x^n$$
$$u_{n+1} = \frac{(2n+2)!}{((n+1)!)^2} \cdot x^{n+1}$$
$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{2n!}{(2n+2)!} \cdot \frac{((n+1)!)^2}{(n!)^2} \cdot \frac{1}{x} = \frac{1}{(2n+1)(2n+2)} \cdot (n+1)^2 \cdot \frac{1}{x}$$
$$= \frac{n+1}{2(2n+1)} \cdot \frac{1}{x} = \frac{1+\frac{1}{n}}{4\left(1+\frac{1}{2n}\right)} \cdot \frac{1}{x}$$
$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{4x}$$

$\therefore$ By ratio test, $\Sigma\, u_n$ is convergent if $\frac{1}{4x} > 1$, *i.e.*, if $x < \frac{1}{4}$ and divergent if $\frac{1}{4x} < 1$, *i.e.*, if $x > \frac{1}{4}$. When $x = \frac{1}{4}$, ratio test fails.

Now, when $x = \frac{1}{4}$, $\frac{u_n}{u_{n+1}} = \frac{1+\frac{1}{n}}{1+\frac{1}{2n}} = \left(1+\frac{1}{n}\right)\left(1+\frac{1}{2n}\right)^{-1}$
$$= \left(1+\frac{1}{n}\right)\left(1-\frac{1}{2n}+\ldots\right) = 1+\frac{1}{2n}+O\left(\frac{1}{n^2}\right)$$
$$= 1+\frac{1/2}{n}+O\left(\frac{1}{n^2}\right) = 1+\frac{\lambda}{n}+O\left(\frac{1}{n^2}\right)$$

$\therefore$ By Gauss test, $\Sigma\, u_n$ is divergent, since $\lambda = \frac{1}{2} < 1$.

Hence $\Sigma\, u_n$ is convergent if $x < \frac{1}{4}$ and divergent if $x \ge \frac{1}{4}$.

Article 20. Cauchy's Integral Test

Statement. *If for $x \ge 1$, $f(x)$ is a non-negative monotonically decreasing integrable function of x such that $f(n) = u_n$ for all positive integral values of n, then the series $\sum_{n=1}^{\infty} u_n$ and the improper integral $\int_1^{\infty} f(x)\,dx$ converge or diverge together.*

Proof. f is non-negative on $[1, \infty)$ $\Rightarrow$ $f(x) \ge 0 \;\; \forall\, x \ge 1$

$\Rightarrow$ $\sum_{n=1}^{\infty} f(n)$ is a series of non-negative terms

$\Rightarrow$ $\sum_{n=1}^{\infty} u_n$ is a series of non-negative terms.

Now let r be any positive integer. Choose a real number x such that $r+1 \ge x \ge r$

Since f is a monotonically decreasing function of x
$$\therefore \quad f(r+1) \le f(x) \le f(r)$$
Also f is integrable
$$\therefore \quad \int_r^{r+1} f(r+1)\,dx \le \int_r^{r+1} f(x)\,dx \le \int_r^{r+1} f(r)\,dx$$
$$\Rightarrow \quad f(r+1)\int_r^{r+1} dx \le \int_r^{r+1} f(x)\,dx \le f(r)\int_r^{r+1} dx$$
$$\Rightarrow \quad f(r+1)\Big[x\Big]_r^{r+1} \le \int_r^{r+1} f(x)\,dx \le f(r)\Big[x\Big]_r^{r+1}$$
$$\Rightarrow \quad f(r+1) \le \int_r^{r+1} f(x)\,dx \le f(r)$$
$$\Rightarrow \quad u_{r+1} \le \int_r^{r+1} f(x)\,dx \le u_r \qquad [\because\ f(n) = u_n\ \forall\, n \in \mathrm{N}]$$

Putting $r = 1, 2, 3, \ldots\ldots, (n-1)$ in succession, we have
$$u_2 \le \int_1^2 f(x)\,dx \le u_1$$
$$u_3 \le \int_2^3 f(x)\,dx \le u_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$u_n \le \int_{n-1}^n f(x)\,dx \le u_{n-1} \qquad \ldots(1)$$

Adding the above $(n-1)$ inequalities, we have
$$u_2 + u_3 + \ldots\ldots + u_n \le \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx + \ldots\ldots + \int_{n-1}^n f(x)\,dx$$
$$\le u_1 + u_2 + \ldots\ldots + u_{n-1}$$
$$\Rightarrow \quad \mathrm{S}_n - u_1 \le \int_1^n f(x)\,dx \le \mathrm{S}_n - u_n \quad \text{where } \mathrm{S}_n = \sum_1^n u_n = u_1 + u_2 + \ldots\ldots + u_n$$
$$\Rightarrow \quad \mathrm{S}_n - u_1 \le \mathrm{I}_n \le \mathrm{S}_n - u_n \quad \text{where } \mathrm{I}_n = \int_1^n f(x)\,dx$$
$$\Rightarrow \quad -u_1 \le \mathrm{I}_n - \mathrm{S}_n \le -u_n$$
$$\Rightarrow \quad u_1 \ge \mathrm{S}_n - \mathrm{I}_n \ge u_n \ge 0 \qquad [\because\ u_n \ge 0 \quad \forall\, n \in \mathrm{N}]$$
$$\Rightarrow \quad 0 \le \mathrm{S}_n - \mathrm{I}_n \le u_1 \qquad \ldots(2)$$

$\Rightarrow$ The sequence $< \mathrm{S}_n - \mathrm{I}_n >$ is bounded.

Also $(\mathrm{S}_n - \mathrm{I}_n) - (\mathrm{S}_{n-1} - \mathrm{I}_{n-1}) = (\mathrm{S}_n - \mathrm{S}_{n-1}) - (\mathrm{I}_n - \mathrm{I}_{n-1})$
$$= u_n - \left[\int_1^n f(x)\,dx - \int_1^{n-1} f(x)\,dx\right]$$

$$= u_n - \left[\int_1^n f(x)\,dx + \int_{n-1}^1 f(x)\,dx\right] = u_n - \int_{n-1}^n f(x)\,dx$$

$$\leq 0 \qquad [\because \text{ of } (1)]$$

$\Rightarrow \quad S_n - I_n \leq S_{n-1} - I_{n-1}$

$\Rightarrow$ The sequence $< S_n - I_n >$ is monotonically decreasing.

Since every bounded monotonic sequence converges, therefore, $< S_n - I_n >$ converges.

$\therefore$ From (2), $\quad 0 \leq \lim\limits_{n\to\infty} (S_n - I_n) \leq u_1$

$\Rightarrow \quad 0 \leq \lim\limits_{n\to\infty} S_n - \lim\limits_{n\to\infty} I_n \leq u_1$

$\Rightarrow \quad \lim\limits_{n\to\infty} I_n \leq \lim\limits_{n\to\infty} S_n \qquad ...(3)$

and $\quad \lim\limits_{n\to\infty} S_n \leq u_1 + \lim\limits_{n\to\infty} I_n \qquad ...(4)$

From (3) and (4), we conclude that $< S_n >$ and $< I_n >$ converge or diverge together ; and hence $\sum\limits_{n=1}^{\infty} u_n$ and $\int_1^{\infty} f(x)\,dx$ converge or diverge together.

Note 1. If $x \geq k$, then Σu_n and $\int_k^{\infty} f(x)\,dx$ converge or diverge together.

Note 2. Since $\int_1^n f(x)\,dx = I_n$

$\therefore \quad \int_1^{\infty} f(x)\,dx = \lim\limits_{n\to\infty} \int_1^n f(x)\,dx = \lim\limits_{n\to\infty} I_n.$

ILLUSTRATIVE EXAMPLES

Example 1. *Test for convergence the series :*

(i) $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2+1}$ (ii) $\sum\limits_{n=1}^{\infty} \dfrac{1}{2n+3}$ (iii) $\sum\limits_{n=1}^{\infty} \dfrac{1}{n(n+1)}$

(iv) $\sum\limits_{n=1}^{\infty} \dfrac{1}{\sqrt{n}}$ (v) $\sum\limits_{n=1}^{\infty} \dfrac{1}{(n+1)^2}$ (vi) $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^{2/3}}$.

Sol. (i) Here $\quad u_n = \dfrac{1}{n^2+1} = f(n)$

$\therefore \quad f(x) = \dfrac{1}{x^2+1}$

For $x \geq 1$, $f(x) \geq 0$ and a monotonically decreasing function of x.

$\therefore$ Cauchy's integral test is applicable.

$$I_n = \int_1^n f(x)\,dx = \int_1^n \frac{dx}{x^2+1} = \tan^{-1} x \Big]_1^n = \tan^{-1} n - \tan^{-1} 1 = \tan^{-1} n - \pi/4.$$

$\therefore \quad \int_1^{\infty} f(x)\,dx = \lim\limits_{n\to\infty} I_n = \lim\limits_{n\to\infty}\left(\tan^{-1} n - \dfrac{\pi}{4}\right) = \dfrac{\pi}{2} - \dfrac{\pi}{4} = \dfrac{\pi}{4} = \text{finite}$

$\Rightarrow \quad \int_1^{\infty} f(x)\,dx$ converges and, hence, by Integral Test, $\sum\limits_{n=1}^{\infty} u_n$ is convergent.

(ii) Here $\quad u_n = \dfrac{1}{2n+3} = f(n)$

$\therefore \quad f(x) = \dfrac{1}{2n+3}$

For $x \geq 1$, $f(x) \geq 0$ and a monotonically decreasing function of x.

$\therefore$ Cauchy's integral test is applicable.

$$I_n = \int_1^n f(x)\,dx = \int_1^n \frac{dx}{2x+3} = \frac{1}{2}\log(2x+3)\Big]_1^n = \tfrac{1}{2}[\log(2n+3) - \log 5]$$

$\therefore \quad \int_1^{\infty} f(x)\,dx = \lim\limits_{n\to\infty} I_n = \lim\limits_{n\to\infty} \dfrac{1}{2}[\log(2n+3) - \log 5] = \infty$

$\Rightarrow \quad \int_1^{\infty} f(x)\,dx$ diverges and, hence, by Integral Test, $\sum\limits_{n=1}^{\infty} u_n$ is divergent.

(iii) Here $\quad I_n = \int_1^n f(x)\,dx = \int_1^n \dfrac{dx}{x(x+1)} = \int_1^n \left(\dfrac{1}{x} - \dfrac{1}{x+1}\right)dx$

$$= \log x - \log(x+1)\Big]_1^n = \log\frac{x}{x+1}\Big]_1^n = \log\frac{n}{n+1} - \log\frac{1}{2}$$

$\therefore \quad \int_1^{\infty} f(x)\,dx = \lim\limits_{n\to\infty} I_n = \lim\limits_{n\to\infty}\left[\log\dfrac{1}{1+\frac{1}{n}} - \log\dfrac{1}{2}\right]$

$$= \log 1 - \log\tfrac{1}{2} = \log 2 = \text{finite}$$

$\Rightarrow \quad \int_1^{\infty} f(x)\,dx$ converges and, hence, by Integral Test, $\sum\limits_{n=1}^{\infty} u_n$ is convergent.

(iv) *Please try yourself.* **[Ans.** Divergent]

(v) *Please try yourself.* **[Ans.** Convergent]

(vi) *Please try yourself.* **[Ans.** Divergent]

Example 2. *Test for convergence the series :*

(i) $\sum_{n=1}^{\infty} \frac{2n^3}{n^4+3}$ (ii) $\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^2}$ (iii) $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$

(iv) $\sum_{n=1}^{\infty} ne^{-n^2}$ (v) $\sum_{n=1}^{\infty} \frac{1}{(n+1)\log(n+1)}$ (vi) $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^3}$

Sol. (i) Here $u_n = \frac{2n^3}{n^4+3} = f(n)$

$\therefore$ $f(x) = \frac{2n^3}{n^4+3}$

For $x \geq 1$, $f(x)$ is positive and decreasing.

$\therefore$ By Cauchy's Integral Test, $\int_1^{\infty} f(x)\,dx$ and $\sum_{n=1}^{\infty} u_n$ ~~converge or diverge~~ together.

$$I_n = \int_1^n f(x)\,dx = \int_1^n \frac{2x^3}{x^4+3}\,dx = \frac{1}{2}\int_1^n \frac{4x^3}{x^4+3}\,dx$$

$$= \frac{1}{2}\log(x^4+3)\Big]_1^n = \tfrac{1}{2}[\log(n^4+3) - \log 4]$$

$\therefore$ $$\int_1^{\infty} f(x)\,dx = \lim_{n\to\infty} I_n = \lim_{n\to\infty} \frac{1}{2}[\log(n^4+3) - \log 4] = \tfrac{1}{2}(\infty - \log 4) = \infty$$

$\Rightarrow$ $\int_1^{\infty} f(x)\,dx$ diverges $\Rightarrow$ $\sum_{n=1}^{\infty} u_n$ diverges.

(ii) Here $u_n = \frac{n}{(n^2+1)^2} = f(n)$ $\therefore$ $f(x) = \frac{x}{(x^2+1)^2}$

For $x \geq 1$, $f(x)$ is positive and decreasing.

$\therefore$ By Cauchy's Integral Test, $\int_1^{\infty} f(x)\,dx$ and $\sum_{n=1}^{\infty} u_n$ converge or diverge together.

$$I_n = \int_1^n f(x)\,dx = \int_1^n \frac{x}{(x^2+1)^2}\,dx = \frac{1}{2}\int_1^n (x^2+1)^{-2} \cdot 2x\,dx$$

$$= \frac{1}{2} \cdot \frac{(x^2+1)^{-1}}{-1}\Big]_1^n = -\frac{1}{2}\left[\frac{1}{n^2+1} - \frac{1}{2}\right] = \frac{1}{2}\left[\frac{1}{2} - \frac{1}{n^2+1}\right]$$

$\therefore$ $$\int_1^{\infty} f(x)\,dx = \lim_{n\to\infty} I_n = \lim_{n\to\infty} \frac{1}{2}\left[\frac{1}{2} - \frac{1}{n^2+1}\right] = \tfrac{1}{2}(\tfrac{1}{2} - 0) = \tfrac{1}{4} = \text{finite}$$

$\Rightarrow$ $\int_1^{\infty} f(x)\,dx$ converges $\Rightarrow$ $\sum_{n=1}^{\infty} u_n$ converges.

(iii) Here $u_n = \frac{1}{n\sqrt{n^2-1}} = f(n), n \geq 2$

$\therefore$ $f(x) = \frac{1}{x\sqrt{x^2-1}}$, $x \geq 2$

For $x \geq 2$, $f(x)$ is positive and decreasing.

$\therefore$ By Cauchy's Integral Test, $\int_1^{\infty} f(x)\,dx$ and $\sum_{n=2}^{\infty} u_n$ converge or diverge together.

$$I_n = \int_2^n f(x)\,dx = \int_2^n \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x\Big]_2^n = \sec^{-1} n - \sec^{-1} 2$$

$\therefore$ $$\int_2^{\infty} f(x)\,dx = \lim_{n\to\infty} I_n = \lim_{n\to\infty} (\sec^{-1} n - \sec^{-1} 2)$$

$$= \sec^{-1}\infty - \sec^{-1} 2 = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6} = \text{finite}$$

$\Rightarrow$ $\int_2^{\infty} f(x)\,dx$ converges $\Rightarrow$ $\sum_{n=2}^{\infty} u_n$ converges.

(iv) Here $u_n = ne^{-n^2} = f(n)$ $\therefore$ $f(x) = xe^{-x^2}$

For $x \geq 1$, $f(x)$ is positive and decreasing.

$\therefore$ By Cauchy's Integral Test, $\int_1^{\infty} f(x)\,dx$ and $\sum_{n=1}^{\infty} u_n$ converge or diverge together.

$$I_n = \int_1^n f(x)\,dx = \int_1^n xe^{-x^2}\,dx$$

$$= -\frac{1}{2}\int_1^n e^{-x^2}(-2x)\,dx = -\frac{1}{2}\int_{-1}^{-n^2} e^t\,dt \text{ where } t = -x^2$$

$$= -\frac{1}{2}\left[e^t\right]_{-1}^{-n^2} = -\frac{1}{2}(e^{-n^2} - e^{-1}) = \frac{1}{2}\left(\frac{1}{e} - e^{-n^2}\right)$$

$\therefore$ $$\int_1^{\infty} f(x)\,dx = \lim_{n\to\infty} I_n = \lim_{n\to\infty} \frac{1}{2}\left(\frac{1}{e} - e^{-n^2}\right) = \frac{1}{2}\left(\frac{1}{e} - 0\right) = \frac{1}{2e} = \text{finite}$$

$\Rightarrow$ $\int_1^{\infty} f(x)\,dx$ converges $\Rightarrow$ $\sum_{n=1}^{\infty} u_n$ converges.

(v) **Hint.** $$I_n = \int_1^n \frac{dx}{(x+1)\log(x+1)} = \int_1^n \frac{\frac{1}{x+1}}{\log(x+1)}\,dx$$

$$= \log\log(x+1)\Big]_1^n = \log\log(n+1) - \log\log 2$$

$\therefore$ $$\int_1^\infty f(x)\,dx = \log\log\infty - \log\log 2 = \infty - \log\log 2 = \infty$$

$\Rightarrow$ $\int_1^\infty f(x)\,dx$ diverges $\Rightarrow$ $\sum_{n=1}^{\infty} u_n$ diverges.

(*vi*) **Hint.** $$I_n = \int_2^n \frac{dx}{x(\log x)^3} = \int_2^n (\log x)^{-3} \cdot \frac{1}{x}\,dx = \frac{(\log x)^{-2}}{-2}\Big]_2^n = \frac{1}{2}\left[\frac{1}{(\log 2)^2} - \frac{1}{(\log n)^2}\right]$$

$\therefore$ $$\int_2^\infty f(x)\,dx = \frac{1}{2}\left[\frac{1}{(\log 2)^2} - 0\right] = \frac{1}{2(\log 2)^2} = \text{finite}$$

$\Rightarrow$ $\int_2^\infty f(x)\,dx$ converges $\Rightarrow$ $\sum_{n=2}^{\infty} u_n$ converges.

Example 3. *Using Integral test, show that the series* $\sum_{n=1}^{\infty} \frac{1}{n^p}$ *converges if* $p > 1$ *and diverges if* $0 < p \le 1$.

Sol. Here $\quad u_n = \frac{1}{n^p} = f(n) \quad \therefore \quad f(x) = \frac{1}{x^p}$

For $x \ge 1$ and $p > 0$, $f(x)$ is positive and decreasing.

$\therefore$ Cauchy's integral test is applicable.

Case I. When $p \ne 1$, $I_n = \int_1^n f(x)\,dx = \int_1^n \frac{dx}{x^p} = \int_1^n x^{-p}\,dx = \frac{x^{1-p}}{1-p}\Big]_1^n$

Sub-case (*i*). When $p > 1$, $p - 1$ is positive so that

$$I_n = -\frac{1}{p-1}\left[\frac{1}{x^{p-1}}\right]_1^n = -\frac{1}{p-1}\left[\frac{1}{n^{p-1}} - 1\right] = \frac{1}{p-1}\left[1 - \frac{1}{n^{p-1}}\right]$$

$\therefore$ $$\int_1^\infty f(x)\,dx = \lim_{n\to\infty} I_n$$

$$= \lim_{n\to\infty} \frac{1}{p-1}\left[1 - \frac{1}{n^{p-1}}\right] = \frac{1}{p-1}(1-0) = \frac{1}{p-1} = \text{finite}$$

$\Rightarrow$ $\int_1^\infty f(x)\,dx$ converges $\Rightarrow$ $\sum_{n=1}^{\infty} u_n$ converges.

Sub-case (*ii*). When $0 < p < 1$; $1 - p$ is positive so that $I_n = \frac{1}{1-p}[n^{1-p} - 1]$

$\therefore$ $$\int_1^\infty f(x)\,dx = \lim_{n\to\infty} I_n = \lim_{n\to\infty} \frac{1}{1-p}[n^{1-p} - 1] = \frac{1}{1-p}[\infty - 1] = \infty$$

$\Rightarrow$ $\int_1^\infty f(x)\,dx$ diverges $\Rightarrow$ $\sum_{n=1}^{\infty} u_n$ diverges.

Case II. When $p = 1$, $f(x) = \frac{1}{x}$

$$I_n = \int_1^n f(x)\,dx = \int_1^n \frac{1}{x}\,dx = \log x\Big]_1^n = \log n - \log 1 = \log n$$

$(\because \log 1 = 0)$

$\therefore$ $$\int_1^\infty f(x)\,dx = \lim_{n\to\infty} I_n = \lim_{n\to\infty} \log n = \infty$$

$\Rightarrow$ $\int_1^\infty f(x)\,dx$ diverges $\Rightarrow$ $\sum_{n=1}^{\infty} u_n$ diverges.

Hence $\sum_{n=1}^{\infty} u_n$ converges if $p > 1$ and diverges if $0 < p \le 1$.

Example 4. *Using integral test, discuss the convergence of the series :*

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p},\ p > 0.$$

Sol. Here $\quad u_n = \frac{1}{n(\log n)^p} = f(n) \quad \therefore \quad f(x) = \frac{1}{x(\log x)^p}$

For $x \ge 2$, $p > 0$, $f(x)$ is positive and decreasing.

$\therefore$ By Cauchy's integral test, $\sum_{n=2}^{\infty} u_n$ and $\int_2^\infty f(x)\,dx$ converge or diverge together.

Case I. When $p \ne 1$.

$$I_n = \int_2^n f(x)\,dx = \int_2^n \frac{dx}{x(\log x)^p} = \int_2^n (\log x)^{-p} \cdot \frac{1}{x}\,dx = \frac{(\log x)^{1-p}}{1-p}\Big]_2^n$$

Sub-case (*i*). When $p > 1$, $p - 1$ is positive so that

$$I_n = -\frac{1}{p-1}\left[\frac{1}{(\log x)^{p-1}}\right]_2^n = \frac{1}{p-1}\left[\frac{1}{(\log 2)^{p-1}} - \frac{1}{(\log n)^{p-1}}\right]$$

$\therefore$ $$\int_2^\infty f(x)\,dx = \lim_{n\to\infty} I_n = \lim_{n\to\infty} \frac{1}{p-1}\left[\frac{1}{(\log 2)^{p-1}} - \frac{1}{(\log n)^{p-1}}\right]$$

$$= \frac{1}{p-1}\left[\frac{1}{(\log 2)^{p-1}} - 0\right] = \frac{1}{(p-1)(\log 2)^{p-1}} = \text{finite}$$

$\Rightarrow$ $\int_2^\infty f(x)\,dx$ converges $\Rightarrow$ $\sum_{n=2}^{\infty} u_n$ converges.

Sub-case (*ii*). When $p < 1$, $1 - p$ is positive so that $I_n = \frac{1}{1-p}\left[(\log n)^{1-p} - (\log 2)^{1-p}\right]$

$$\therefore \quad \int_2^\infty f(x)\,dx = \lim_{n\to\infty} I_n$$

$$= \lim_{n\to\infty} \frac{1}{1-p}\left[(\log n)^{1-p} - (\log 2)^{1-p}\right] = \frac{1}{1-p}\left[\infty - (\log 2)^{1-p}\right] = \infty$$

$\Rightarrow \quad \int_2^\infty f(x)\,dx$ diverges $\Rightarrow \sum_{n=2}^\infty u_n$ diverges.

Case II. When $p = 1$, $f(x) = \frac{1}{x \log x}$

$$I_n = \int_2^n \frac{dx}{x \log x} = \int_2^n \frac{\frac{1}{x}}{\log x}\,dx\ \log\log x\Big]_2^n = \log\log n - \log\log 2$$

$$\therefore \quad \int_2^\infty f(x)\,dx = \lim_{n\to\infty} I_n = \lim_{n\to\infty} (\log\log n - \log\log 2) = \infty - \log\log 2 = \infty$$

$\Rightarrow \quad \int_2^\infty f(x)\,dx$ diverges $\Rightarrow \sum_{n=2}^\infty u_n$ diverges.

Hence $\sum_{n=2}^\infty u_n$ converges if $p > 1$ and diverges if $0 < p \le 1$.

Example 5. *Discuss the convergence of the series :*

$$\sum_{n=3}^\infty \frac{1}{n \log n (\log\log n)^p},\ p > 0.$$

Sol. $\qquad u_n = \frac{1}{n \log n (\log\log n)^p} = f(n),\ n \ge 3$

$$\therefore \quad f(x) = \frac{1}{x \log x (\log\log x)^p},\ x \ge 3.$$

For $x \ge 3$, $p > 0$, $f(x)$ is positive and decreasing.

$\therefore$ By Cauchy's integral test, $\sum_{n=3}^\infty u_n$ and $\int_3^\infty f(x)\,dx$ converge or diverge together.

Case I. When $p \ne 1$

$$I_n = \int_3^n f(x)\,dx = \int_3^n \frac{dx}{x \log x (\log\log x)^p}$$

$$= \int_3^n (\log\log x)^{-p} \cdot \frac{1}{x \log x}\,dx = \frac{(\log\log x)^{1-p}}{1-p}\Bigg]_3^n$$

Sub-case (*i*). When $p > 1$, $p - 1$ is positive so that

$$I_n = -\frac{1}{p-1}\left[\frac{1}{(\log\log x)^{p-1}}\right]_3^n = \frac{1}{p-1}\left[\frac{1}{(\log\log 3)^{p-1}} - \frac{1}{(\log\log n)^{p-1}}\right]$$

$$\therefore \quad \int_3^\infty f(x)\,dx = \lim_{n\to\infty} I_n$$

$$= \lim_{n\to\infty} \frac{1}{p-1}\left[\frac{1}{(\log\log 3)^{p-1}} - \frac{1}{(\log\log n)^{p-1}}\right]$$

$$= \frac{1}{p-1}\left[\frac{1}{(\log\log 3)^{p-1}} - 0\right] = \frac{1}{(p-1)(\log\log 3)^{p-1}} = \text{finite}$$

$\Rightarrow \quad \int_3^\infty f(x)\,dx$ converges $\Rightarrow \sum_{n=3}^\infty u_n$ converges.

Sub-Case (*ii*) When $p < 1$, $1 - p$ is positive so that

$$I_n = \frac{1}{1-p}\left[(\log\log n)^{1-p} - (\log\log 3)^{1-p}\right]$$

$$\therefore \quad \int_3^\infty f(x)\,dx = \lim_{n\to\infty} I_n$$

$$= \lim_{n\to\infty} \frac{1}{1-p}\left[(\log\log n)^{1-p} - (\log\log 3)^{1-p}\right]$$

$$= \frac{1}{1-p}\left[\infty - (\log\log 3)^{1-p}\right] = \infty$$

$\Rightarrow \quad \int_3^\infty f(x)\,dx$ diverges $\Rightarrow \sum_{n=3}^\infty u_n$ diverges.

Case II. When $p = 1$, $f(x) = \frac{1}{x \log x (\log\log x)}$

$$I_n = \int_3^n f(x)\,dx = \int_3^n \frac{dx}{x \log x (\log\log x)} = \int_3^n \frac{\frac{1}{x \log x}}{\log\log x}\,dx$$

$$= \log(\log\log x)\Big]_3^n = \log(\log\log n) - \log(\log\log 3)$$

$$\therefore \quad \int_3^\infty f(x)\,dx = \lim_{n\to\infty} I_n = \lim_{n\to\infty} [\log(\log\log n) - \log(\log\log 3)]$$

$$= \infty - \log(\log\log 3) = \infty$$

$\Rightarrow \quad \int_3^\infty f(x)\,dx$ diverges $\Rightarrow \sum_{n=3}^\infty u_n$ diverges.

Hence $\sum_{n=3}^\infty u_n$ converges if $p > 1$ and diverges if $0 < p \le 1$.

Article 21. Cauchy's Condensation Test

Statement. *If $<f(x)>$ is a decreasing sequence of positive terms and 'a' is positive integer greater than 1, then $\Sigma f(n)$ and $\Sigma a^n f(a^n)$ converge or diverge together.*

Proof. Let S_n and σ_n denote the nth partial sums of the series $\sum_{n=1}^{\infty} f(n)$ and $\sum_{n=1}^{\infty} a^n f(a^n)$ respectively.

Then $$S_n = f(1) + f(2) + \ldots\ldots + f(n) = \sum_{m=1}^{n} f(m)$$

and $$\sigma_n = af(a) = a^2 f(a^2) + \ldots\ldots + a^n f(a^n) = \sum_{m=1}^{n} a^m f(a^m).$$

Since both the series are series of positive terms, the sequences $<S_n>$ and $<\sigma_n>$ are monotonically increasing.

The series $\Sigma f(n)$ can be re-written as

$$\begin{aligned} \Sigma f(n) = {} & [f(1) + f(2) + \ldots\ldots\ldots\ldots + f(a)] \\ & + [f(a+1) + f(a+2) + \ldots\ldots\ldots + f(a^2)] \\ & + [f(a^2+1) + f(a^2+2) + \ldots\ldots\ldots + f(a^3)] \\ & + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & + [f(a^{k-1}+1) + f(a^{k-1}+2) + \ldots\ldots\ldots + f(a^k)] \\ & + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

The terms in the kth group are $\quad f(a^{k-1}+1) + f(a^{k-1}+2) + \ldots\ldots\ldots + f(a^k)$

Number of terms in the kth group $\quad = a^k - a^{k-1} = a^{k-1}(a-1)$

Since $<f(n)>$ is a decreasing sequence, each term in the kth group $\geq f(a^k)$ and $\leq f(a^{k-1})$.

$$\therefore \quad a^{k-1}(a-1) f(a^k) \leq f(a^{k-1}+1) + f(a^{k-1}+2) + \ldots\ldots + f(a^k) \leq a^{k-1}(a-1) f(a^{k-1})$$

$$\Rightarrow \quad a^k \left(\frac{a-1}{a}\right) f(a^k) \leq f(a^{k-1}+1) + f(a^{k-1}+2) + \ldots\ldots + f(a^k) \leq a^{k-1}(a-1) f(a^{k-1})$$

Putting $k = 1, 2, 3, \ldots\ldots, n$, we get

$$a\left(\frac{a-1}{a}\right) f(a) \leq f(2) + f(3) + \ldots\ldots + f(a) \leq (a-1) f(1)$$

$$a^2\left(\frac{a-1}{a}\right) f(a^2) \leq f(a+1) + f(a+2) + \ldots\ldots + f(a^2) \leq (a-1)\, a f(a)$$

$$a^3\left(\frac{a-1}{a}\right) f(a^3) \leq f(a^2+1) + f(a^2+2) + \ldots\ldots + f(a^3) \leq (a-1)\, a^2 f(a^2)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a^n\left(\frac{a-1}{a}\right) f(a^n) \leq f(a^{n-1}+1) + f(a^{n-1}+2) + \ldots\ldots\ldots + f(a^n) \leq (a-1)\, a^{n-1} f(a^{n-1})$$

Adding the above n inequalities,

$$\left(\frac{a-1}{a}\right) \sum_{m=1}^{n} a^m f(a^m) \leq \sum_{m=2}^{a^n} f(m) \leq (a-1) \sum_{m=1}^{n} a^{m-1} f(a^{m-1})$$

Adding $f(1)$ throughout, we have

$$f(1) + \left(\frac{a-1}{a}\right) \sum_{m=1}^{n} a^m f(a^m) \leq \sum_{m=1}^{a^n} f(m) \leq f(1) + (a-1) \sum_{m=1}^{n} a^{m-1} f(a^{m-1})$$

$$\Rightarrow \quad f(1) + \left(\frac{a-1}{a}\right) \sigma_n \leq S_{a^n} \leq f(1) + (a-1)\, \sigma_{n-1} \qquad \ldots(1)$$

Case (*i*) Suppose $\Sigma a^n f(a^n)$ is convergent. Then the sequence $<\sigma_n>$ is convergent and, therefore, bounded.

Since $<S_n>$ is increasing and $\quad a^n > n \quad \forall\, n$

$\therefore \qquad S_n < S_{a^n}$

Using (1), we have $\quad S_n < S_{a^n} \leq f(1) + (a-1)\, \sigma_{n-1}$

$\Rightarrow \quad <S_n>$ is bounded above.

$\Rightarrow \quad <S_n>$ is convergent. $\Rightarrow \quad \sum_{n=1}^{\infty} f(n)$ is convergent.

Case (*ii*) Suppose $\Sigma f(n)$ is convergent. Then the sequence $<S_n>$ is convergent and, therefore, bounded.

Since $<S_{a^n}>$ is a subsequence of $<S_n>$, therefore, $<S_{a^n}>$ is bounded.

Using (1), we have $\quad \sigma_n \leq \dfrac{a}{a-1}(S_{a^n} - f(1))$

$\Rightarrow \quad <\sigma_n>$ is bounded above.

$\Rightarrow \quad <\sigma_n>$ is convergent. $\Rightarrow \quad \sum_{n=1}^{\infty} a^n f(a^n)$ is convergent.

Since the convergence of $\Sigma f(n) \;\Leftrightarrow\;$ convergence of $\Sigma a^n f(a^n)$, therefore, both the series diverge also together.

Article 22. (A Particular Case of Article 21 with a = 2)

Statement. *If $<u_n>$ is a decreasing sequence of non-negative terms, then the two series $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} 2^n u_{2^n}$ converge or diverge together.*

Proof. Let S_n and σ_n denote the nth partial sums of the series $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} 2^n u_{2^n}$ respectively.

Then $$S_n = u_1 + u_2 + \ldots\ldots + u_n = \sum_{m=1}^{n} u_m$$

and $$\sigma_n = 2u_2 + 2^2 u_{2^2} + \ldots\ldots + 2^n u_{2^n} = \sum_{m=1}^{n} 2^m u_{2^m}.$$

Since both the series are series of non-negative terms, the sequences $<S_n>$ and $<\sigma_n>$ are monotonically increasing.

The series Σu_n can be re-written as

$$\Sigma u_n = (u_1 + u_2) = (u_3 + u_4) + (u_5 + u_6 + u_7 + u_8) + \ldots\ldots$$
$$+ (u_{2^{k-1}+1} + u_{2^{k-1}+2} + \ldots\ldots + u_{2^k}) + \ldots\ldots$$

The terms in the kth group are

$$u_{2^{k-1}+1}+u_{2^{k-1}+2}+\ldots\ldots+u_{2^k}$$

Number of terms in the kth group $= 2^k - 2^{k-1} = 2^{k-1}(2-1) = 2^{k-1}$.

Since $<u_n>$ is a decreasing sequence, each term in the kth group $\geq u_{2^k}$ and $\leq u_{2^{k-1}}$.

$\therefore$ $$2^{k-1}u_{2^k} \leq u_{2^{k-1}+1}+u_{2^{k-1}+2}+\ldots\ldots+u_{2^{k-1}} \leq 2^k u_{2^{k-1}}$$

Putting $k = 1, 2, 3, \ldots\ldots, n$, we get

$$u_2 \leq u_2 \leq u_1$$

$$2u_{2^2} \leq u_3 \leq u_4 \leq 2u_2$$

$$2^2 u_{2^3} \leq u_5+u_6+u_7+u_8 \leq 2^2 u_{2^2}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$2^{n-1}u_{2^n} \leq u_{2^{n-1}+1}+u_{2^{n-1}+2}+\ldots\ldots+u_{2^n} \leq 2^{n-1}u_{2^{n-1}}$$

Adding the above n inequalities, $\frac{1}{2}\sum_{m=1}^{\infty} 2^m u_{2^m} \leq \sum_{m=2}^{2^n} u_m \leq \sum_{m=1}^{n} 2^{m-1} u_{2^{m-1}}$

$\Rightarrow$ $$\tfrac{1}{2}\sigma_n \leq S_{2^n} - u_1 \leq \sigma_{n-1} \qquad \ldots(1)$$

Case (I) Suppose $\Sigma\, 2^n u_{2^n}$ is convergent. Then the sequence $<\sigma_n>$ is convergent, and therefore, bounded.

Since $<S_n>$ is increasing and $2^n > n\ \forall\ n$

$\therefore$ $$S_n \leq S_{2^n}$$

Using (1), we have $\quad S_n \leq S_{2^n} \leq u_1 + \sigma_{n-1}$

$\Rightarrow$ $<S_n>$ is bounded above.

$\Rightarrow$ $<S_n>$ is convergent. $\Rightarrow$ $\sum_{n=1}^{\infty} u_n$ is convergent.

Case (II) Suppose Σu_n is convergent. Then the sequence $<S_n>$ is convergent and therefore, bounded.

Since $<S_{2^n}>$ is a subsequence of $<S_n>$,

therefore, $<S_{2^n}>$ is bounded.

Using (1), we have $\quad \sigma_n \leq 2(S_{2^n} - n_1)$

$\Rightarrow$ $<\sigma_n>$ is bounded above

$\Rightarrow$ $<\sigma_n>$ is convergent. $\Rightarrow$ $\sum_{n=1}^{\infty} 2^n u_{2^n}$ is convergent.

Since the convergence of $\Sigma u_n \Leftrightarrow$ convergence of $\Sigma 2^n u_{2^n}$, therefore both the series diverge also together.

ILLUSTRATIVE EXAMPLES

Example 1. *Using Cauchy's condensation test, discuss the convergence of the following series :*

(*i*) $\sum_{n=2}^{\infty} \frac{1}{\log n}$ (*ii*) $\sum_{n=2}^{\infty} \frac{1}{n \log n}$ (*iii*) $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\log n}}$

(*iv*) $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^{3/2}}$ (*v*) $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ (*vi*) $\sum_{n=2}^{\infty} \frac{1}{(\log n)^p}$.

Sol. (*i*) Here $\quad u_n = \frac{1}{\log n}$

Since $<\log n>$ is an increasing sequence, $<u_n>$ is a decreasing sequence.

i.e., $$u_n > u_{n+1} > 0\ \forall\ n \geq 2$$

$\therefore$ By Cauchy's condensation test, the series $\sum_{n=2}^{\infty} u_n$ and $\sum_{n=2}^{n} 2^n u_{2^n}$ converge or diverge together.

Now $$\sum_{n=2}^{\infty} 2^n u_{2^n} = \sum_{n=2}^{\infty} 2^n \cdot \frac{1}{\log 2^n} = \sum_{n=2}^{\infty} \frac{2^n}{n \log 2} = \frac{1}{\log 2}\sum_{n=2}^{\infty} \frac{2^n}{n}$$

Consider $v_n = \frac{2^n}{n}$ so that $v_n^{1/n} = \frac{2}{n^{1/n}}$

$$\lim_{n\to\infty} v_n^{1/n} = \lim_{n\to\infty} \frac{2}{n^{1/n}} = \frac{2}{1} = 2 > 1$$

$\therefore$ By Cauchy's root test, Σv_n is divergent.

$\Rightarrow$ $\sum_{n=2}^{\infty} 2^n u_{2^n}$ is divergent $\Rightarrow$ $\sum_{n=2}^{\infty} u_n$ is divergent.

(*ii*) Here $\quad u_n = \frac{1}{n \log n}$

Since $<n \log n>$ is an increasing sequence, $<u_n>$ is a decreasing sequence.

i.e., $$u_n > u_{n+1} > 0 \quad \forall\ n \geq 2$$

$\therefore$ By Cauchy's condensation test, the series $\sum_{n=2}^{\infty} u_n$ and $\sum_{n=2}^{\infty} 2^n u_{2^n}$ converge or diverge together.

Now $$\sum_{n=2}^{\infty} 2^n u_{2^n} = \sum_{n=2}^{\infty} 2^n \cdot \frac{1}{2^n \log 2^n} = \sum_{n=2}^{\infty} \frac{1}{n \log 2} = \frac{1}{\log 2}\sum_{n=2}^{\infty} \frac{1}{n}$$

Since $\sum_{n=2}^{\infty} \frac{1}{n}$ is divergent, $\sum_{n=2}^{\infty} 2^n u_{2^n}$ is divergent $\Rightarrow$ $\sum_{n=2}^{\infty} u_n$ is divergent.

(*iii*) Here $u_n = \dfrac{1}{n\sqrt{\log n}}$.

Since $< n\sqrt{\log n} >$ is an increasing sequence, $< u_n >$ is a decreasing sequence.

i.e., $u_n > u_{n+1} > 0 \quad \forall\, n \geq 2$

$\therefore$ By Cauchy's condensation test, the series $\sum_{n=2}^{\infty} u_n$ and $\sum_{n=2}^{\infty} 2^n u_{2^n}$ converge or diverge together.

Now $$\sum_{n=2}^{\infty} 2^n u_{2^n} = \sum_{n=2}^{\infty} 2^n \cdot \frac{1}{2^n\sqrt{\log 2^n}} = \sum_{n=2}^{\infty} \frac{1}{\sqrt{n \log 2}} = \frac{1}{\sqrt{\log 2}} \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$$

Since $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ is divergent $\quad (\because\ p = \frac{1}{2} < 1)$, $\sum_{n=2}^{\infty} 2^n u_{2^n}$ is divergent.

$\Rightarrow \sum_{n=2}^{\infty} u_n$ is divergent.

(*iv*) *Please try yourself.* **[Ans.** Convergent]

(*v*) Here $u_n = \dfrac{1}{n(\log n)^p}$.

Case (I) When $p \leq 0$

$$\frac{1}{n(\log n)^p} \geq \frac{1}{n} \quad \forall\, n \geq 2$$

Since $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges, by comparison test $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ diverges.

Case (II) When $p > 0$.

Since $< n\,(\log n)^p >$ is an increasing sequence, $< u_n >$ is a decreasing sequence

i.e. $u_n > u_{n+1} > 0 \quad \forall\, n \geq 2$

$\therefore$ By Cauchy's condensation test, the series $\sum_{n=2}^{\infty} u_n$ and $\sum_{n=2}^{\infty} 2^n u_{2^n}$ converge or diverge together.

Now $$\sum_{n=2}^{\infty} 2^n u_{2^n} = \sum_{n=2}^{\infty} 2^n \cdot \frac{1}{2^n (\log 2^n)^p} = \sum_{n=2}^{\infty} \frac{1}{(n \log 2)^p} = \frac{1}{(\log 2)^p} \sum_{n=2}^{\infty} \frac{1}{n^p}.$$

Since $\sum_{n=2}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$

$\therefore \sum_{n=2}^{\infty} 2^n u_{2^n}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

$\Rightarrow \sum_{n=2}^{\infty} u_n$ is convergent if $p > 1$ and divergent if $p \leq 1$.

Hence $\sum_{n=2}^{\infty} u_n$ is convergent if $p > 1$ and divergent if $p \leq 1$.

(*vi*) Here $u_n = \dfrac{1}{(\log n)^p}$.

Case I. When $p = 0$, $u_n = 1$

Since $\lim_{n\to\infty} u_n = 1 \neq 0$, $\sum_{n=2}^{\infty} u_n$ diverges.

Case II. When $p < 0$, let $p = -q$ where $q > 0$.

Since $$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{(\log n)^{-q}} = \lim_{n\to\infty} (\log n)^q = \infty \neq 0$$

$\therefore \sum_{n=2}^{\infty} u_n$ diverges.

Case-III. When $p > 0$.

Since $< (\log n)^p >$ is an increasing sequence, $< u_n >$ is a decreasing sequence.

i.e. $u_n > u_{n+1} \quad \forall\, n \geq 2$

$\therefore$ By Cauchy's condensation test, the series $\sum_{n=2}^{\infty} u_n$ and $\sum_{n=2}^{\infty} 2^n u_{2^n}$ converge or diverge together.

Now $$\sum_{n=2}^{\infty} 2^n u_{2^n} = \sum_{n=2}^{\infty} 2^n \cdot \frac{1}{(\log 2^n)^p} = \sum_{n=2}^{\infty} 2^n \cdot \frac{1}{(n \log 2)^p} = \frac{1}{(\log 2)^p} \sum_{n=2}^{\infty} \frac{2^p}{n^p}$$

Consider $v_n = \dfrac{2^n}{n^p}$ so that $v_n^{1/n} = \dfrac{2}{(n^{1/n})}$

$$\lim_{n\to\infty} v_n^{1/n} = \lim_{n\to\infty} \frac{2}{(n^{1/n})} = 2 > 1$$

$\therefore$ By Cauchy's root test, Σv_n is divergent.

$\Rightarrow \sum_{n=2}^{\infty} 2^n u_{2^n}$ is divergent. $\Rightarrow \sum_{n=2}^{\infty} u_n$ is divergent.

Hence $\sum_{n=2}^{\infty} u_n$ is divergent for all values of p.

Example 2. *Using Cauchy's condensation test, discuss the convergence of the following series :*

(*i*) $\sum_{n=1}^{\infty} \dfrac{1}{n^p}$ (*ii*) $\sum_{n=1}^{\infty} \dfrac{\log n}{n}$ (*iii*) $\sum_{n=1}^{\infty} \left(\dfrac{\log n}{n}\right)^2$

(*iv*) $\sum_{n=3}^{\infty} \dfrac{1}{n \log n\,(\log \log n)}$ (*v*) $\sum_{n=2}^{\infty} \dfrac{1}{(n \log n)^p}$.

Sol. (*i*) Here $u_n = \dfrac{1}{n^p}$

Case I. When $p = 0$, $u_n = 1$

Since $\lim\limits_{n\to\infty} u_n = 1 \neq 0$, Σu_n diverges.

Case II. When $p < 0$, let $p = -q$ where $q > 0$

$$u_n = \frac{1}{n^p} = \frac{1}{n^{-q}} = n^q$$

$\therefore$ $u_n \to \infty$ as $n \to \infty$

$$\lim_{n\to\infty} u_n \neq 0$$

$\therefore$ Σu_n diverges.

Case III. When $p > 0$

Since $\dfrac{1}{n^p} > \dfrac{1}{(n+1)^p}$ $\forall\, n$

$< u_n >$ is a decreasing sequence.

$\therefore$ By Cauchy's condensation test, the series $\sum\limits_{n=1}^{\infty} u_n$ and $\sum\limits_{n=1}^{\infty} 2^n u_{2^n}$ converge or diverge together.

Now $$\sum_{n=1}^{\infty} 2^n u_{2^n} = \sum_{n=1}^{\infty} 2^n \cdot \frac{1}{(2^n)^p} = \sum_{n=1}^{\infty} \left(\frac{1}{2^{p-1}}\right)^n$$

is a geometric series with common ratio $\dfrac{1}{2^{p-1}}$ and is, therefore, convergent if $\dfrac{1}{2^{p-1}} < 1$ *i.e.,* if $2^{p-1} > 1$ *i.e.,* if $p > 1$ and divergent if $\dfrac{1}{2^{p-1}} \geq 1$ *i.e.* if $p \leq 1$.

Hence the given series is convergent if $p > 1$ and divergent if $p \leq 1$.

(*ii*) Here $u_n = \dfrac{\log n}{n} \geq 0\ \forall\, n$

Consider $f(x) = \dfrac{\log x}{x}, x > 0$

$$f'(x) = \frac{x \cdot \frac{1}{x} - \log x}{x^2} = \frac{1 - \log x}{x^2}$$

$$f'(x) < 0 \Rightarrow 1 - \log x < 0 \Rightarrow \log x > 1$$

$\Rightarrow$ $x > e$ $[\because \log e = 1]$

$\therefore$ $f(x)$ is a decreasing function when $x > e$

$\Rightarrow$ $u_n > u_{n+1}$ $\forall\, n > 2$ $(\because 2 < e < 3)$

$\Rightarrow$ $< u_n >$ is a decreasing sequence of positive terms.

$\therefore$ By Cauchy's condensation test, the series $\sum\limits_{n=1}^{\infty} u_n$ and $\sum\limits_{n=1}^{\infty} 2^n u_{2^n}$ converge or diverge together.

Now $$\sum_{n=1}^{\infty} 2^n u_{2^n} = \sum_{n=1}^{\infty} 2^n \cdot \frac{\log 2^n}{2^n} = \sum_{n=1}^{\infty} n \log 2 = \log 2 \sum_{n=1}^{\infty} n$$

Since $\sum\limits_{n=1}^{\infty} n$ is divergent, $\sum\limits_{n=1}^{\infty} 2^n u_{2^n}$ is divergent.

$\Rightarrow$ $\sum\limits_{n=1}^{\infty} u_n$ is divergent.

(*iii*) Here $u_n = \left(\dfrac{\log n}{n}\right)^2 \geq 0\ \ \forall\, n$

Consider $f(x) = \left(\dfrac{\log x}{x}\right)^2, x > 0$

$$f'(x) = 2\left(\frac{\log x}{x}\right)\left(\frac{1 - \log x}{x^2}\right)$$

$$f'(x) < 0 \Rightarrow 1 - \log x < 0 \Rightarrow \log x > 1 \Rightarrow x > e \quad (\because \log e = 1)$$

$\therefore$ $f(x)$ is a decreasing function when $x > e$.

$\Rightarrow$ $u_n < u_{n+1}$ $\forall\, n > 2$ $(\because 2 < e > 3)$

$\Rightarrow$ $< u_n >$ is a decreasing sequence of positive terms.

$\therefore$ By Cauchy's condensation test, the series $\sum\limits_{n=1}^{\infty} u_n$ and $\sum\limits_{n=1}^{\infty} 2^n u_{2^n}$ converge or diverge together.

Now $$\sum_{n=1}^{\infty} 2^n u_{2^n} = \sum_{n=1}^{\infty} 2^n \cdot \left(\frac{\log 2^n}{2^n}\right)^2 = \sum_{n=1}^{\infty} 2^n \cdot \frac{(n \log 2)^2}{2^{2n}} = (\log 2)^2 \sum_{n=1}^{\infty} \frac{n^2}{2^n}$$

Consider $v_n = \dfrac{n^2}{2^n}$ so that $v_n^{1/n} = \dfrac{(n^{1/n})^2}{2}$

$$\lim_{n\to\infty} v_n^{1/n} = \frac{1}{2} < 1$$

$\therefore$ By Cauchy's root test, $\sum\limits_{n=1}^{\infty} v_n$ converges.

$\Rightarrow$ $\sum\limits_{n=1}^{\infty} 2^n u_{2^n}$ converges. $\Rightarrow$ $\sum\limits_{n=1}^{\infty} u_n$ converges.

Hence the given series is convergent.

(*iv*) Here $u_n = \dfrac{1}{n \log n (\log \log n)}, n \geq 3$

Since $<u_n>$ is a decreasing sequence, by Cauchy's condensation test, the series $\sum_{n=3}^{\infty} u_n$ and $\sum_{n=3}^{\infty} 2^n u_{2^n}$ converge or diverge together.

Now $$\sum_{n=3}^{\infty} 2^n u_{2^n} = \sum_{n=3}^{\infty} 2^n \cdot \frac{1}{2^n \log 2^n (\log\log 2^n)} = \sum_{n=3}^{\infty} \frac{1}{n \log 2[\log (n \log 2)]}$$

Since $\log 2 < 1 \quad \therefore \quad n \log 2 < n$

$$\frac{1}{n \log 2[\log (n \log 2)]} > \frac{1}{n \log n}$$

But the series $\sum_{n=3}^{\infty} \frac{1}{n (\log n)^p}$ is divergent when $p = 1$.

$\therefore \sum_{n=3}^{\infty} \frac{1}{n \log 2[\log (n \log 2)]}$ is divergent.

$\Rightarrow \sum_{n=3}^{\infty} 2^n u_{2^n}$ is divergent. $\Rightarrow \sum_{n=3}^{\infty} u_n$ is divergent.

(v) Here $$u_n = \frac{1}{(n \log n)^p}, \; n \geq 2$$

Case I. When $p = 0$, $u_n = 1$

Since $\lim\limits_{n \to \infty} u_n = 1 \neq 0$, $\sum_{n=2}^{\infty} u_n$ is divergent.

Case II. When $p < 0$, let $p = -q$ where $q > 0$.

$$u_n = \frac{1}{(n \log n)^{-q}} = (n \log n)^q$$

$\therefore \; u_n \to \infty$ as $n \to \infty$

Since $\lim\limits_{n \to \infty} u_n \neq 0$, $\sum u_n$ is divergent.

Case III. When $p > 0$, $<u_n>$ is a decreasing sequence.

$\therefore$ By Cauchy's condensation test, the series $\sum_{n=2}^{\infty} u_n$ and $\sum_{n=2}^{\infty} 2^n u_{2^n}$ converge or diverge together.

Now $$\sum_{n=2}^{\infty} 2^n u_{2^n} = \sum_{n=2}^{\infty} 2^n \cdot \frac{1}{(2^n \log 2^n)^p} = \sum_{n=2}^{\infty} 2^n \cdot \frac{1}{(2^n . n \log 2)^p}$$

$$= \sum_{n=2}^{\infty} \cdot \frac{1}{2^{n(p-1)} . n^p (\log 2)^p} = \frac{1}{(\log 2)^p} \sum_{n=2}^{\infty} \frac{1}{2^{n(p-1)} n^p}$$

If $p > 1$, then $2^{n(p-1)} > 1$ so that $\frac{1}{2^{n(p-1)}} < 1$ and $\frac{1}{2^{n(p-1)} . n^p} < \frac{1}{n^p}$

But the series $\sum \frac{1}{n^p}$ is convergent for $p > 1$.

$\therefore \sum_{n=2}^{\infty} \frac{1}{2^{n(p-1)} . n^p}$ is convergent. $\Rightarrow \sum_{n=2}^{\infty} 2^n u_{2^n}$ is convergent for $p > 1$.

If $p = 1$, then $\sum_{n=2}^{\infty} 2^n u_{2^n} = \frac{1}{\log 2} \sum_{n=2}^{\infty} \frac{1}{n}$

But $\sum \frac{1}{n}$ is divergent.

$\therefore \sum_{n=2}^{\infty} 2^n u_{2^n}$ is divergent for $p = 1$.

If $p < 1$, then $1 - p > 0$ so that $2^{n(1-p)} > 1$ *i.e.*, $\frac{1}{2^{n(p-1)}} > 1$ and $\frac{1}{2^{n(p-1)} . n^p} > \frac{1}{n^p}$

But the series $\sum \frac{1}{n^p}$ is divergent for $p < 1$.

$\therefore \sum_{n=2}^{\infty} \frac{1}{2^{n(p-1)} . n^p}$ is divergent. $\Rightarrow \sum_{n=2}^{\infty} 2^n u_{2^n}$ is divergent for $p < 1$.

Hence $\sum_{n=2}^{\infty} u_n$ is convergent for $p > 1$ and divergent for $p \leq 1$.

Article 23. Series of Arbitrary Terms

So far we have been discussing series of positive terms only which either converge or diverge but never oscillate. A series with all terms negative can also be discussed, since the removal of the factor (-1) from each term does not affect the behaviour of the series. A series with finitely many terms of one sign and the remaining terms of other sign can also be discussed, since the behaviour of the series is not affected by omitting finitely many terms. With the exception of series for which the limit of the nth term is not zero, we are not in a position, at present, to discuss series having infinitely many terms of both signs (positive an negative). Such series are called '*series of arbitrary terms*' or '*series with mixed signs*'.

e.g., $$1 - 1 + 1 - 1 + 1 - 1 + \ldots\ldots$$

$$1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} - \frac{1}{6} - \frac{1}{7} + \frac{1}{8} \ldots\ldots$$

In details, we shall study series of arbitrary terms in Chapter 8. Here we intend to study the simplest type among the series of arbitrary terms.

Article 24. Alternating Series

A series with terms alternately positive and negative is called an alternating series.

Thus the series $u_1 - u_2 + u_3 - u_4 + \ldots\ldots$ where $u_n > 0$

for each n, is an alternating series and is briefly written as $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$.

Article 25. Leibnitz's Test on Alternating Series

Statement. *The alternating series*

$$\Sigma(-1)^{n-1} u_n = u_1 - u_2 + u_3 - u_4 + \ldots\ldots (u_n > 0 \;\forall\; n) \text{ converges if}$$

(i) $u_n \geq u_{n+1} \;\forall\; n$ *and* *(ii)* $\underset{n\to\infty}{\text{Lt}}\; u_n = 0$.

Proof. Let S_n denote the nth partial sum of the series $\Sigma(-1)^{n-1} u_n$.

$$\begin{aligned} S_{2n} &= u_1 - u_2 + u_3 - u_4 + u_5 - \ldots\ldots - u_{2n-2} + u_{2n-1} - u_{2n} \\ &= u_1 - (u_2 - u_3) - (u_4 - u_5) \ldots\ldots - (u_{2n-2} - u_{2n-1}) - u_{2n} \\ &= u_1 - [(u_2 - u_3) + (u_4 - u_5) + \ldots\ldots + (u_{2n-2} - u_{2n-1}) + u_{2n}] \\ &\leq u_1 \qquad [\because\; u_n > u_{n+1} \text{ and } u_n > 0 \text{ for all } n] \end{aligned}$$

$\Rightarrow$ The sequence $\{S_{2n}\}$ is bounded above.

Also $\quad S_{2n+2} = S_{2n} + S_{2n+1} - u_{2n+2}$

$\Rightarrow \quad S_{2n+2} - S_{2n} = u_{2n+1} - u_{2n+2} \geq 0$ for all n

$\Rightarrow \quad S_{2n+2} \geq S_{2n}$

$\Rightarrow$ The sequence $\{S_{2n}\}$ is monotonically increasing.

Since every monotonically increasing sequence which is bounded above converges, therefore, the sequence $\{S_{2n}\}$ converges. Let it converges to S, then $\underset{n\to\infty}{\text{Lt}}\; S_{2n} = S$

Now $\quad \underset{n\to\infty}{\text{Lt}}\; S_{2n+1} = \underset{n\to\infty}{\text{Lt}}\; (S_{2n} + u_{2n+1}) = \underset{n\to\infty}{\text{Lt}}\; S_{2n} + \underset{n\to\infty}{\text{Lt}}\; u_{2n+1}$

$\quad = S + 0 \qquad [\because\; \underset{n\to\infty}{\text{Lt}}\; u_n = 0]$

$\quad = S$

$\therefore$ The sequences $\{S_{2n}\}$ and $\{S_{2n+1}\}$ converge to the same real number S.

$\therefore$ Given $\varepsilon > 0$ there exist positive integers m_1 and m_2 such that

$|S_{2n} - S| < \varepsilon \quad \forall\; 2n > m_1$

and $\quad |S_{2n+1} - S| < \varepsilon \quad \forall\; 2n + 1 > m_2$

Let $m = \max. \{m_1, m_2\}$, then

$|S_{2n} - S| < \varepsilon \quad \forall\; 2n > m$

and $\quad |S_{2n+1} - S| < \varepsilon \quad \forall\; 2n + 1 > m$

$\Rightarrow \quad |S_n - S| < \varepsilon \quad \forall\; n > m$

$\therefore$ The sequence $\{S_n\}$ converges to S.

Hence the given series is convergent.

Note. The alternating series will not be convergent if any one of the two conditions is not satisfied.

Article 25A. Absolute and Conditional Convergence

Definition 1. A series $\sum_{n=1}^{\infty} u_n$ is said to be **absolutely convergent** if the series $\sum_{n=1}^{\infty} |u_n|$ is convergent.

Definition 2. If $\sum_{n=1}^{\infty} u_n$ converges but not absolutely, *i.e.*, $\sum_{n=1}^{\infty} |u_n|$ diverges, then the series $\sum_{n=1}^{\infty} u_n$ is called **conditionally convergent** (or *semi-convergent* or *non-absolutely convergent*).

Note 1. Since $\sum_{n=1}^{\infty} |u_n|$ is a series of positive terms, it either converges or diverges.

All the tests established for testing the convergence of a series of positive terms can be used for testing the convergence of the series $\sum_{n=1}^{\infty} |u_n|$. However these tests cannot give any information about the conditional convergence of the series.

Note 2. If Σu_n is a series of positive terms, then $\Sigma u_n = \Sigma |u_n|$. Therefore, for a series of positive terms, the concepts of convergence and absolute convergence are the same.

Article 26. Theorem. *Every absolutely convergent series is convergent.*

Proof. Let $\sum_{n=1}^{\infty} u_n$ be an absolutely convergent series. Then $\sum_{n=1}^{\infty} |u_n|$ is convergent.

$\therefore$ By Cauchy's general principle of convergence, given $\varepsilon > 0$, $\exists$ a positive integer m such that

$$||u_{m+1}| + |u_{m+2}| + \ldots\ldots + |u_n|| < \varepsilon \quad \forall\; n > m$$

i.e.,

$$|u_{m+1}| + |u_{m+2}| + \ldots\ldots + |u_n| < \varepsilon \quad \forall\; n > m \qquad \ldots(1)$$

Now, by triangle inequality, we have

$$|u_{m+1} + u_{m+2} + \ldots\ldots + u_n| \leq |u_{m+1}| + |u_{m+2}| + \ldots\ldots + |u_n| < \varepsilon \;\forall\; n > m \qquad \text{[Using (1)]}$$

$\therefore$ By Cauchy's general principle of convergence, the series $\sum_{n=1}^{\infty} u_n$ is convergent.

Hence $\Sigma |u_n|$ is convergent $\Rightarrow u_n$ is convergent.

Note 1. Absolute convergence $\Rightarrow$ convergence, but convergence need not imply absolute convergence *i.e.*, the converse of above theorem need not be true.

For example, consider the series

$$\Sigma \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\ldots$$

It is an alternating series.

Here $u_n = \frac{1}{n}$. Clearly $u_n > 0 \ \forall\, n$.

Since $\quad \frac{1}{n} > \frac{1}{n+1}, u_n > u_{n+1} \ \forall\, n$

Also $\quad \lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n} = 0$

$\therefore$ By Leibnitz's test, $\sum \frac{(-1)^{n-1}}{n}$ converges.

But the series $\sum \left| \frac{(-1)^{n-1}}{n} \right| = \sum \frac{1}{n}$ is divergent.

Note 2. The divergence of $\Sigma \mid u_n \mid$ does not imply the divergence of Σu_n.

For example $\sum \left| \frac{(-1)^{n-1}}{n} \right| = \sum \frac{1}{n}$ is divergent whereas $\sum \frac{(-1)^{n-1}}{n}$ is convergent.

Article 27. Theorem. *If $\sum_{n=1}^{\infty} u_n$ is an absolutely convergent series, then the series of its positive terms and the series of its negative terms are both convergent.*

Proof. Let S_n and σ_n be the nth partial sums of the series $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} \mid u_n \mid$ respectively.

Then $\quad S_n = u_1 + u_2 + \ldots\ldots + u_n$ and $\sigma_n = \mid u_1 \mid + \mid u_2 \mid + \ldots\ldots + \mid u_n \mid$

Let P_n and $-Q_n$ denote the sum of positive and negative terms in S_n.

Then $\quad S_n = P_n - Q_n$ and $\sigma_n = P_n + Q_n$

$$\Rightarrow \quad P_n = \tfrac{1}{2}(\sigma_n + S_n) \qquad \ldots(1)$$

$$\text{and} \quad Q_n = \tfrac{1}{2}(\sigma_n - S_n)$$

Since the series $\sum_{n=1}^{\infty} u_n$ is absolutely convergent, both $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} \mid u_n \mid$ are convergent.

$\Rightarrow$ Both $< S_n >$ and $< \sigma_n >$ are convergent.

Let $\quad \lim_{n\to\infty} S_n = l$ and $\lim_{n\to\infty} \sigma_n = l'$

$\therefore$ From (1), $\quad \lim_{n\to\infty} P_n = \lim_{n\to\infty} \frac{1}{2}(\sigma_n + S_n) = \frac{1}{2}(l' + l)$

and $\quad \lim_{n\to\infty} Q_n = \lim_{n\to\infty} \frac{1}{2}(\sigma_n - S_n) = \frac{1}{2}(l' - l)$

$\Rightarrow$ Both $< P_n >$ and $< Q_n >$ are convergent.

Hence the series of positive terms and the series of negative terms are separately convergent.

Corollary 1. *If $\sum_{n=1}^{\infty} u_n$ is conditionally convergent, then the series of its positive terms and the series of its negative terms are both divergent.*

Since $\sum_{n=1}^{\infty} u_n$ is conditionally convergent, $\sum_{n=1}^{\infty} u_n$ is convergent and $\sum_{n=1}^{\infty} \mid u_n \mid$ is divergent.

$\Rightarrow \ < S_n >$ is convergent and $< \sigma_n >$ is divergent.

$\Rightarrow \ \lim_{n\to\infty} S_n = l$ (say) and $\lim_{n\to\infty} \sigma_n = \infty$

$\therefore$ From (1), $\lim_{n\to\infty} P_n = \infty$ and $\lim_{n\to\infty} Q_n = \infty$

$\Rightarrow$ Both $< P_n >$ and $< Q_n >$ are divergent.

Hence the series of positive terms and the series of negative terms are separately divergent.

Corollary 2. *A series with mixed signs cannot converge if the series of its positive terms is convergent (divergent) and the series of its negative terms is divergent (convergent).*

ILLUSTRATIVE EXAMPLES

Example 1. *Test the convergence and absolute convergence of the series :*

(*i*) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\ldots$ $\qquad$ (*ii*) $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots$

(*iii*) $\frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots\ldots$ $\qquad$ (*iv*) $\frac{1}{1.3} - \frac{1}{2.4} + \frac{1}{3.5} - \frac{1}{4.6} + \ldots\ldots$

(*v*) $\frac{1}{\log 2} - \frac{1}{\log 3} + \frac{1}{\log 4} - \frac{1}{\log 5} + \ldots\ldots$

Sol. (*i*) The given series is

$$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \sum_{n=1}^{\infty} (-1)^{n-1} v_n$$

It is an alternating series.

Here $\quad v_n = \frac{1}{n} > 0 \ \forall\, n, \ v_{n+1} = \frac{1}{n+1}$

Since $\quad \frac{1}{n} > \frac{1}{n+1} \ \forall\, n, \ \therefore \ v_n > v_{n+1} \ \forall\, n$

Also $\quad \lim_{n\to\infty} v_n = \lim_{n\to\infty} \frac{1}{n} = 0$

$\therefore$ By Leibnitz's test, the series is convergent.

Now $\sum_{n=1}^{\infty} |u_n| = \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Hence the given series is conditionally convergent.

(*ii*) The given series is

$$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} = \sum_{n=1}^{\infty} (-1)^{n-1} v_n$$

It is an alternating series.

Here $\quad v_n = \frac{1}{2n-1} > 0 \quad \forall\, n, v_{n+1} = \frac{1}{2n+1}$

Since $\quad 2n - 1 < 2n + 1 \quad \forall\, n$

$\therefore \quad \frac{1}{2n-1} > \frac{1}{2n+1} \quad \forall\, n$

$\Rightarrow \quad v_n > v_{n+1} \quad \forall\, n$

Also $\quad \lim_{n\to\infty} v_n = \lim_{n\to\infty} \frac{1}{2n-1} = 0$

$\therefore$ By Leibnitz's test, the series is convergent.

Now $\quad |u_n| = \frac{1}{2n-1} > \frac{1}{2n} \quad \forall\, n$

Since $\sum \frac{1}{2n} = \frac{1}{2} \sum \frac{1}{n}$ diverges, by comparison test, $\Sigma\, |u_n|$ also diverges.

Hence the given series is conditionally convergent.

OR

$$|u_n| = \frac{1}{2n-1} \text{ Take } V_n = \frac{1}{n}$$

$$\lim_{n\to\infty} \frac{|u_n|}{V_n} = \lim_{n\to\infty} \frac{n}{2n-1} = \lim_{n\to\infty} \frac{1}{2-\frac{1}{n}} = \frac{1}{2} \text{ which is non-zero and finite.}$$

$\therefore$ By comparison test, $\Sigma\, |u_n|$ and ΣV_n converge or diverge together. Since $\sum V_n = \sum \frac{1}{n}$ diverges, $\sum |u_n|$ also diverges.

(*iii*) The given series is

$$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{5+(n-1)\times 2} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n+3} = \sum_{n=1}^{\infty} (-1)^{n-1} v_n$$

It is an alternating series.

Here $\quad v_n = \frac{1}{2n+3} > 0 \quad \forall\, n, \quad v_{n+1} = \frac{1}{2n+5}$

Since $\quad \frac{1}{2n+3} > \frac{1}{2n+5} \quad \forall\, n \quad \therefore \quad v_n > v_{n+1} \quad \forall\, n$

Also $\quad \lim_{n\to\infty} v_n = \lim_{n\to\infty} \frac{1}{2n+3} = 0$

$\therefore$ By Leibnitz's test, the series is convergent.

Now $\quad |u_n| = \frac{1}{2n+3}$ Take $V_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{|u_n|}{V_n} = \lim_{n\to\infty} \frac{n}{2n+3} = \lim_{n\to\infty} = \frac{1}{2+\frac{3}{n}} = \frac{1}{2} \text{ which is non-zero and finite.}$$

$\therefore$ By comparison test $\Sigma\, |u_n|$ and ΣV_n converge or diverge together. Since $\sum V_n = \sum \frac{1}{n}$ diverges, $\sum |u_n|$ also diverges.

Hence the given series is conditionally convergent.

(*iv*) The given series is $\quad \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(n+2)} = \sum_{n=1}^{\infty} (-1)^{n-1} v_n$

It is an alternating series.

Here $\quad v_n = \frac{1}{n(n+2)} > 0 \quad \forall\, n, \quad v_{n+1} = \frac{1}{(n+1)(n+3)}$

Since $\quad \frac{1}{n} > \frac{1}{n+1}$ and $\frac{1}{n+2} > \frac{1}{n+3} \quad \forall\, n$

$\therefore \quad v_n > v_{n+1} \quad \forall\, n$

Also $\quad \lim_{n\to\infty} v_n = \lim_{n\to\infty} \frac{1}{n(n+2)} = 0$

$\therefore$ By Leibnitz's test, the series is convergent.

Now $\quad |u_n| = \frac{1}{n(n+2)}$ Take $V_n = \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{|u|}{V_n} = \lim_{n\to\infty} \frac{n^2}{n(n+2)} = \lim_{n\to\infty} \frac{1}{1+\frac{2}{n}} = 1 \text{ which is non-zero and finite.}$$

$\therefore$ By comparison test, $\Sigma\, |u_n|$ and ΣV_n converge or diverge together. Since $\sum V_n = \sum \frac{1}{n^2}$ converges ($\because \; p = 2 > 1$), $\sum |u_n|$ also converges.

Hence the given series is absolutely convergent.

(*v*) The given series is $\quad \sum_{n=2}^{\infty} u_n = \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\log n} = \sum_{n=2}^{\infty} (-1)^{n-1} v_n$

It is an alternating series.

Here $\quad v_n = \dfrac{1}{\log n} > 0 \quad \forall\, n \geq 2$

$$v_{n+1} = \frac{1}{\log (n+1)}$$

Since $\quad \log n < \log (n+1) \qquad$ [$\because$ $\log x$ is an increasing function]

$\therefore \quad \dfrac{1}{\log n} > \dfrac{1}{\log (n+1)} \Rightarrow v_n > v_{n+1} \quad \forall\, n \geq 2$

Also $\quad \lim\limits_{n\to\infty} v_n = \lim\limits_{n\to\infty} \dfrac{1}{\log n} = 0$

$\therefore$ By Leibnitz's test, the series is convergent.

Now $\quad |\, u_n \,| = \dfrac{1}{\log n} > \dfrac{1}{n} \quad \forall\, n \geq 2 \qquad (\because \log n < n \quad \forall\, n \geq 2)$

Since $\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$ diverges, by comparison test, $\Sigma\, |\, u_n \,|$ also diverges.

Hence the given series is conditionally convergent.

Example 2. *Test the convergence and absolute convergence of the series :*

(*i*) $1 - \dfrac{1}{2^p} + \dfrac{1}{3^p} - \dfrac{1}{4^p} + \ldots\ldots (p > 0)$

(*ii*) $1 - \dfrac{1}{3!} + \dfrac{1}{5!} - \dfrac{1}{7!} + \ldots\ldots$ $\qquad$ (*iii*) $1 - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2} + \ldots\ldots$

(*iv*) $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{\sqrt{n}}$ $\qquad$ (*v*) $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{n\sqrt{n}}$.

Sol. (*i*) The given series is

$$\sum u_n = \sum \frac{(-1)^{n-1}}{n^p} = \sum (-1)^{n-1} v_n$$

It is an alternating series.

Here $\quad v_n = \dfrac{1}{n^p} > 0 \quad \forall\, n, \quad v_{n+1} = \dfrac{1}{(n+1)^p}$

Since $\quad n^p < (n+1)^p \quad \forall\, n, \quad p > 0$

$\therefore \quad v_n > v_{n+1} \quad \forall\, n$

Also $\quad \lim\limits_{n\to\infty} v_n = \lim\limits_{n\to\infty} \dfrac{1}{n^p} = 0$

$\therefore$ By Leibnitz's test, the series is convergent.

Now $\quad |\, u_n \,| = \dfrac{1}{n^p}$

$\therefore \quad \sum |\, u_n \,| = \sum \dfrac{1}{n^p}$ is convergent if $p > 1$ and divergent if $0 < p \leq 1$.

Hence the given series is absolutely convergent if $p > 1$ and conditionally convergent if $0 < p \leq 1$.

(*ii*) The given series is $\sum u_n = \sum \dfrac{(-1)^{n-1}}{(2n-1)!} = \Sigma (-1)^{n-1} v_n$

It is an alternating series.

Here $\quad v_n = \dfrac{1}{(2n-1)!} > 0 \quad \forall\, n$

$$v_{n+1} = \frac{1}{(2n+1)!}$$

Since $\quad (2n-1)! < (2n+1)! \quad \therefore \quad v_n > n_{n+1} \quad \forall\, n$

Also $\quad \lim\limits_{n\to\infty} v_n = \lim\limits_{n\to\infty} \dfrac{1}{(2n-1)!} = 0$

$\therefore$ By Leibnitz's test, the series is convergent.

Now $\quad |\, u_n \,| = \dfrac{1}{(2n-1)!}, \quad |\, u_{n+1} \,| = \dfrac{1}{(2n+1)!}$

$\Rightarrow \quad \dfrac{|\, u_n \,|}{|\, u_{n+1} \,|} = \dfrac{(2n+1)!}{(2n-1)!} = (2n+1)(2n)$

$$\lim_{n\to\infty} \frac{|\, u_n \,|}{|\, u_{n+1} \,|} = \lim_{n\to\infty} 2n(2n+1) = \infty$$

$\therefore$ By ratio test, $\Sigma\, |\, u_n \,|$ is convergent.

Hence the given series is absolutely convergent.

(*iii*) *Please try yourself.* **[Ans.** Absolutely convergent]

(*iv*) *Please try yourself.* **[Ans.** Conditionally convergent]

(*v*) *Please try yourself.* **[Ans.** Absolutely convergent]

Note. Parts (*iii*), (*iv*) and (*v*) of above example are particular cases of part (*i*) with $p = 2, \dfrac{1}{2}$ and 3/2 respectively.

Example 3. *Test the convergence and absolute convergence of the series :*

(*i*) $1 - \dfrac{1}{2^3}(1+2) + \dfrac{1}{3^3}(1+2+3) - \ldots\ldots$

(*ii*) $\dfrac{1}{2^3} - \dfrac{1}{3^3}(1+2) + \dfrac{1}{4^4}(1+2+3) - \ldots\ldots$

(*iii*) $\sum\limits_{n=1}^{\infty} (-1)^{n-1} \sin \dfrac{1}{n}$.

Sol. (*i*) The given series is

$$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1+2+3+\ldots\ldots+n}{n^3}$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{n(n+1)}{2n^3} = \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{n+1}{2n^2} = \sum_{n=1}^{\infty} (-1)^{n-1} \cdot v_n$$

It is an alternating series.

Here $\quad v_n = \dfrac{n+1}{2n^2} > 0 \quad \forall\, n$

$$v_{n+1} = \frac{n+2}{2(n+1)^2}$$

$$\therefore \quad v_n - v_{n+1} = \frac{n+1}{2n^2} - \frac{n+2}{2(n+1)^2}$$

$$= \frac{(n+1)^3 - n^2(n+2)}{2n^2(n+1)^2} = \frac{n^2+3n+1}{2n^2(n+1)^2} > 0 \quad \forall\, n$$

$$\Rightarrow \quad v_n > v_{n+1} \quad \forall\, n$$

Also $\quad \lim\limits_{n\to\infty} v_n = \lim\limits_{n\to\infty} \dfrac{n+1}{2n^2} = \lim\limits_{n\to\infty} \dfrac{1}{2}\left(\dfrac{1}{n} + \dfrac{1}{n^2}\right) = 0$

$\therefore$ By Leibnitz's test, the series is convergent.

Now $\quad |u_n| = \dfrac{n+1}{2n^2} = \dfrac{n\left(1+\dfrac{1}{n}\right)}{2n^2} = \dfrac{1+\dfrac{1}{n}}{2n}$

Take $\quad V_n = \dfrac{1}{n}$

$\therefore \quad \lim\limits_{n\to\infty} \dfrac{|u_n|}{V_n} = \lim\limits_{n\to\infty} \dfrac{1}{2}\left(1+\dfrac{1}{n}\right) = \dfrac{1}{2}$ which is non-zero and finite.

$\therefore$ By comparison test, $\Sigma\,|u_n|$ and $\Sigma\, V_n$ converge or diverge together. Since $\sum V_n = \sum \dfrac{1}{n}$ diverges, $\Sigma\,|u_n|$ also diverges.

Hence the given series is conditionally convergent.

(*ii*) The given series is

$$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{(n+1)^3} \cdot (1+2+3+\ldots\ldots+n)$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{(n+1)^3} \cdot \frac{n(n+1)}{2}$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{n}{2(n+1)^2} = \sum_{n=1}^{\infty} (-1)^{n-1} v_n$$

It is an alternating series.

Here $\quad v_n = \dfrac{n}{2(n+1)^2} > 0 \quad \forall\, n$

$$v_{n+1} = \frac{n+1}{2(n+2)^2}$$

$$\therefore \quad v_n - v_{n+1} = \frac{n}{2(n+1)^2} - \frac{n+1}{2(n+2)^2}$$

$$= \frac{n(n+2)^2 - (n+1)^3}{2(n+1)^2(n+2)^2} = \frac{n^2+n-1}{2(n+1)^2(n+2)^2} > 0 \quad \forall\, n$$

$$\Rightarrow \quad v_n > v_{n+1} \quad \forall\, n$$

Also $\quad \lim\limits_{n\to\infty} v_n = \lim\limits_{n\to\infty} \dfrac{n}{2(n+1)^2} = \lim\limits_{n\to\infty} \dfrac{\dfrac{1}{n}}{2\left(1+\dfrac{1}{n}\right)^2} = 0$

$\therefore$ By Leibnitz's test, the series is convergent.

Now $\quad |u_n| = \dfrac{n}{2(n+1)^2}$ Take $V_n = \dfrac{1}{n}$

$$\lim_{n\to\infty} \frac{|u_n|}{V_n} = \lim_{n\to\infty} \frac{n^2}{2(n+1)^2} = \lim_{n\to\infty} \frac{1}{2\left(1+\dfrac{1}{n}\right)^2} = \frac{1}{2}$$

which is non-zero and finite.

$\therefore$ By comparison test, $\Sigma\,|u_n|$ and $\Sigma\, V_n$ converge or diverge together. Since $\sum V_n = \sum \dfrac{1}{n}$ diverges, $\Sigma\,|u_n|$ diverges.

Hence the given series is conditionally convergent.

(*iii*) The given series is $\sum\limits_{n=1}^{\infty} u_n = \sum\limits_{n=1}^{\infty} (-1)^{n-1} \sin\dfrac{1}{n} = \sum\limits_{n=1}^{\infty} (-1)^{n-1} v_n$

It is an alternating series.

$\left[\text{Note that } 0 < \dfrac{1}{n} \le 1 \;\; \forall\, n, \;\; \therefore \;\; \sin\dfrac{1}{n} > 0\right]$

Here $\quad v_n = \sin\dfrac{1}{n} > 0 \quad \forall\, n$

$$v_{n+1} = \sin\frac{1}{n+1}$$

$$v_n - v_{n+1} = \sin\frac{1}{n} - \sin\frac{1}{n+1} > 0 \quad \forall\, n$$

$\left[\because \; \sin x \text{ is an increasing function when } 0 < x < \dfrac{\pi}{2}\right]$

$$\Rightarrow \quad v_n > v_{n+1}$$

Also $\quad \lim\limits_{n\to\infty} v_n = \lim\limits_{n\to\infty} \sin\dfrac{1}{n} = 0$

$\therefore$ By Leibnitz's test, the series is convergent.

Now $\quad |u_n| = \sin\dfrac{1}{n} \quad$ Take $V_n = \dfrac{1}{n}$

$$\lim_{n\to\infty}\frac{|u_n|}{V_n}=\lim_{n\to\infty}\frac{\sin\frac{1}{n}}{\frac{1}{n}}=1 \text{ which is finite and non-zero.}$$

$\therefore$ By comparison test, $\Sigma\,|\,u_n\,|$ and $\Sigma\,V_n$ converge or diverge together. Since $\Sigma V_n = \sum\frac{1}{n}$ diverges, $\Sigma\,|\,u_n\,|$ also diverges.

Hence the given series is conditionally convergent.

Example 4. *Test the convergence and absolute convergence of the series :*

(*i*) $2-\frac{3}{2}+\frac{4}{3}-\frac{5}{4}+\ldots\ldots$ (*ii*) $\frac{1}{a}-\frac{1}{a+b}+\frac{1}{a+2b}-\frac{1}{a+3b}+\ldots\ldots (a>0, b>0)$

(*iii*) $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n}{2n-1}$ (*iv*) $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n}{10n-1}$

(*v*) $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n}{n+2}$ (*vi*) $\sum_{n=1}^{\infty}(-1)^{n-1}\cdot\frac{n}{5n-7}$.

Sol. (*i*) The given series is

$$\sum_{n=1}^{\infty}u_n=\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{n+1}{n}\right)=\sum_{n=1}^{\infty}(-1)^{n-1}v_n$$

It is an alternating series.

Here $\quad v_n=\frac{n+1}{n}>\ \forall\, n$

$$v_{n+1}=\frac{n+2}{n+1}$$

$$v_n-v_{n+1}=\frac{n+1}{n}-\frac{n+2}{n+1}=\frac{(n+1)^2-n(n+2)}{n(n+1)}=\frac{1}{n(n+1)}>0\ \forall\, n$$

$\therefore \quad v_n>v_{n+1}\quad \forall\, n$

Also $\quad \lim_{n\to\infty}v_n=\lim_{n\to\infty}\frac{n+1}{n}=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)=1\neq 0.$

Since the second condition of Leibnitz's test is not satisfied, the series is not convergent.

Note. The condition $u_n\to 0$ as $n\to\infty$ is absolutely essential for the convergence of any series Σu_n.

(*ii*) The given series is

$$\sum_{n=1}^{\infty}u_n=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{a+(n-1)b}=\sum_{n=1}^{\infty}(-1)^{n-1}v_n.$$

It is an alternating series.

Here $\quad v_n=\frac{1}{a+(n-1)b}>0\quad\forall\, n$

$$v_{n+1}=\frac{1}{a+nb}$$

Since $\quad a+(n-1)\,b<a+nb$

$\therefore \quad \frac{1}{a+(n-1)b}>\frac{1}{a+nb}\quad\Rightarrow\quad v_n>v_{n+1}\quad\forall\, n$

Also $\quad \lim_{n\to\infty}v_n=\lim_{n\to\infty}\frac{1}{a+(n-1)b}=0$

$\therefore$ By Leibnitz's test, the series is convergent.

Now $\quad |\,u_n\,|=\frac{1}{a+(n-1)b}=\frac{1}{n\left[\frac{a}{n}+\left(1-\frac{1}{n}\right)b\right]}$

Take $\quad V_n=\frac{1}{n}$

$$\lim_{n\to\infty}\frac{|u_n|}{V_n}=\lim_{n\to\infty}\frac{1}{\frac{a}{n}+\left(1-\frac{1}{n}\right)b}=\frac{1}{b} \text{ which is non-zero and finite.}$$

$\therefore$ By comparison test, $\Sigma\,|\,u_n\,|$ and ΣV_n converge or diverge together. Since $\Sigma V_n=\sum\frac{1}{n}$ diverges, $\Sigma\,|\,u_n\,|$ also diverges.

Hence the given series is conditionally convergent.

(*iii*) The given series is $\quad \sum_{n=1}^{\infty}u_n=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}.n}{2n-1}=\sum_{n=1}^{\infty}(-1)^{n-1}v_n$

~~It is an alternating series.~~

Here $\quad v_n=\frac{n}{2n-1}>0\quad\forall\, n$

$$v_{n+1}=\frac{n+1}{2n+1}$$

$$v_n-v_{n+1}=\frac{n}{2n-1}-\frac{n+1}{2n+1}=\frac{(2n^2+n)-(2n^2+n-1)}{(2n-1)(2n+1)}$$

$$=\frac{1}{4n^2-1}>0\quad\forall\, n$$

$\Rightarrow \quad v_n>v_{n+1}\quad\forall\, n$

Also $\quad \lim_{n\to\infty}v_n=\lim_{n\to\infty}\frac{2}{2n-1}=\lim_{n\to\infty}\frac{1}{2-\frac{1}{n}}=\frac{1}{2}\neq 0$

Since the second condition of Leibnitz's test is not satisfied, the series is not convergent.

(*iv*) *Please try yourself.* **[Ans.** Not convergent]

(*v*) *Please try yourself.* **[Ans.** Not convergent]

(*vi*) *Please try yourself.* **[Ans.** Not convergent]

Example 5. *Test the convergence and absolute convergence of the series :*

(i) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!}$ (ii) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n+1}$ (iii) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3n-2}$

(iv) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n+1}}$ (v) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}.n}{5^n}$.

Sol. (i) The given series is

$$\sum u_n = \sum \frac{(-1)^{n-1}}{n!} = \Sigma(-1)^{n-1} v_n$$

It is an alternating series.

Here $v_n = \frac{1}{n!} > 0 \quad \forall\, n, v_{n+1} = \frac{1}{(n+1)!}$

Since $\frac{1}{n!} > \frac{1}{(n+1)!} \quad \forall\, n$

$\therefore$ $v_n > v_{n+1} \quad \forall\, n$

Also $\lim_{n\to\infty} v_n = \lim_{n\to\infty} \frac{1}{n!} = 0$

$\therefore$ By Leibnitz's test, the series is convergent.

Now $|u_n| = \frac{1}{n!} \quad \therefore \quad |u_{n+1}| = \frac{1}{(n+1)!}$

$\therefore$ $\lim_{n\to\infty} \frac{|u_n|}{|u_{n+1}|} = \lim_{n\to\infty} \frac{(n+1)!}{n!} = \lim_{n\to\infty}(n+1) = \infty$

By ratio test, $\Sigma\,|u_n|$ is convergent.

Hence the given series is absolutely convergent.

(ii) *Please try yourself.* **[Ans.** Conditionally convergent**]**

(iii) *Please try yourself.* **[Ans.** Conditionally convergent**]**

(iv) *Please try yourself.* **[Ans.** Conditionally convergent**]**

(v) The given series is

$$\sum u_n = \sum \frac{(-1)^{n-1}.n}{5^n} = \Sigma(-1)^{n-1} v_n$$

It is an alternating series.

Here $v_n = \frac{n}{5^n}, \quad v_{n+1} = \frac{n+1}{5^{n+1}}$

$$v_n - v_{n+1} = \frac{n}{5^n} - \frac{n+1}{5^{n+1}} = \frac{5n-(n+1)}{5^{n+1}} = \frac{4n-1}{5^{n+1}} > 0 \quad \forall\, n$$

$\Rightarrow$ $v_n > v_{n+1} \quad \forall\, n$

Also $\lim_{n\to\infty} v_n = \lim_{n\to\infty} \frac{n}{5^n}$ | Form $\frac{\infty}{\infty}$

$= \lim_{n\to\infty} \frac{1}{5^n \log 5}$ | Hospital's Rule

$= 0$

$\therefore$ By Leibnitz's test, the series is convergent.

Now $|u_n| = \frac{n}{5^n}, \; |u_{n+1}| = \frac{n+1}{5^{n+1}}$

$$\lim_{n\to\infty} \frac{|u_n|}{|u_{n+1}|} = \lim_{n\to\infty} \frac{5n}{n+1} = \lim_{n\to\infty} \frac{5}{1+\frac{1}{n}} = 5 > 1$$

$\therefore$ By ratio test, $\Sigma\,|u_n|$ is convergent.

Hence the given series is absolutely convergent.

Example 6. *Test the convergence and absolute convergence of the series :*

(i) $\frac{1}{\sqrt{2}+1} - \frac{1}{\sqrt{3}+1} + \frac{1}{\sqrt{4}+1} - \frac{1}{\sqrt{5}+1} + \ldots\ldots$

(ii) $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{n+2}{2^n+5}$ (iii) $1 - \frac{1}{4.3} + \frac{1}{4^2.5} - \frac{1}{4^3.7} + \ldots\ldots$

(iv) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}.n}{n^2+1}$.

Sol. (i) The given series is

$$\Sigma u_n = \sum \frac{(-1)^{n-1}}{\sqrt{n+1}+1} = \Sigma(-1)^{n-1} v_n$$

It is an alternating series.

Here $v_n = \frac{1}{\sqrt{n+1}+1} > 0 \quad \forall\, n$

$v_{n+1} = \frac{1}{\sqrt{n+2}+1}$

Clearly, $v_n > v_{n+1} \quad \forall\, n$

Also $\lim_{n\to\infty} v_n = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}+1} = 0$

$\therefore$ By Leibnitz's test, the series is convergent.

Now $|u_n| = \frac{1}{\sqrt{n+1}+1} = \frac{1}{\sqrt{n}\left(\sqrt{1+\frac{1}{n}}+\frac{1}{\sqrt{n}}\right)}$

Take $V_n = \frac{1}{\sqrt{n}}$

$$\lim_{n\to\infty} \frac{|u_n|}{V_n} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}+\frac{1}{\sqrt{n}}} = \frac{1}{\sqrt{1+0}+0} = 1 \text{ which is non-zero and finite.}$$

$\therefore$ By comparison test, $\Sigma\,|u_n|$ and ΣV_n converge or diverge together. Since

$\Sigma V_n = \sum \frac{1}{n^{1/2}} \left(p = \frac{1}{2} < 1\right)$ diverges, $\Sigma\,|u_n|$ diverges.

Hence the given series is conditionally convergent.

(*ii*) The given series is $\Sigma u_n = \Sigma(-1)^n \cdot \dfrac{n+2}{2^n+5} = \Sigma(-1)^n v_n$

It is an alternating series.

Here $\quad v_n = \dfrac{n+2}{2^n+5} > 0 \quad \forall\, n$

$$v_{n+1} = \frac{n+3}{2^{n+1}+5}$$

$$\therefore \quad v_n - v_{n+1} = \frac{n+2}{2^n+5} + \frac{n+3}{2^{n+1}+5} = \frac{(n+2)(2^{n+1}+5)-(n+3)(2^n+5)}{(2^n+5)(2^{n+1}+5)}$$

$$= \frac{(n+2)2^{n+1}+5n+10-(n+3)2^n-5n-15}{(2^n+5)(2^{n+1}+5)} = \frac{(2n+4-n-3)2^n-5}{(2^n+5)(2^{n+1}+5)}$$

$$= \frac{(n+1)2^n-5}{(2^n+5)(2^{n+1}+5)} > 0 \ \forall\, n \geq 2$$

$\Rightarrow \quad v_n > v_{n+1} \quad \forall\, n \geq 2$

Also $\quad \lim\limits_{n\to\infty} v_n = \lim\limits_{n\to\infty} \dfrac{n+2}{2^n+5} \qquad \left| \text{Form } \dfrac{\infty}{\infty} \right.$

$$= \lim_{n\to\infty} \frac{1}{2^n \log 2} = 0$$

$\therefore$ By Leibnitz's test, the series is convergent.

Now $\quad |u_n| = \dfrac{n+2}{2^n+5}, \ |u_{n+1}| = \dfrac{n+3}{2^{n+1}+5}$

$$\lim_{n\to\infty} \frac{|u_n|}{|u_{n+1}|} = \lim_{n\to\infty} \frac{n+2}{n+3} \cdot \frac{2^{n+1}+5}{2^n+5}$$

$$= \lim_{n\to\infty} \frac{1+\frac{2}{n}}{1+\frac{3}{n}} \cdot \frac{2^n\left(2+\frac{5}{2^n}\right)}{2^n\left(1+\frac{5}{2^n}\right)} = \lim_{n\to\infty} \frac{1+\frac{2}{n}}{1+\frac{3}{n}} \cdot \frac{2+\frac{5}{2^n}}{1+\frac{5}{2^n}} = 1 \times 2 = 2 > 1$$

$\therefore$ By ratio test, $\Sigma\, |u_n|$ is convergent. Hence the given series is absolutely convergent.

(*iii*) The given series is $\quad \Sigma u_n = \sum \dfrac{(-1)^{n-1}}{4^{n-1}.(2n-1)} = \Sigma(-1)^{n-1} v_n$

It is an alternating series.

Here $\quad v_n = \dfrac{1}{4^{n-1}(2n-1)}, \ v_{n+1} = \dfrac{1}{4^n(2n+1)}$

$$\therefore \quad v_n - v_{n+1} = \frac{1}{4^{n-1}(2n-1)} - \frac{1}{4^n(2n+1)} = \frac{4(2n+1)-(2n-1)}{4^n(2n-1)(2n+1)}$$

$$= \frac{6n+5}{4^n(2n-1)(2n+1)} > 0 \qquad \forall\, n$$

$\Rightarrow \quad v_n > v_{n+1} \ \forall\, n$

Also $\quad \lim\limits_{n\to\infty} v_n = \lim\limits_{n\to\infty} \dfrac{1}{4^{n-1}(2n-1)} = 0$

$\therefore$ By Leibnitz's test, the series is convergent.

Now $\quad |u_n| = \dfrac{1}{4^{n-1}(2n-1)}, \ |u_{n+1}| = \dfrac{1}{4^n(2n+1)}$

$$\lim_{n\to\infty} \frac{|u_n|}{|u_{n+1}|} = \lim_{n\to\infty} \frac{4(2n+1)}{2n-1} = \lim_{n\to\infty} \frac{4\left(1+\frac{1}{2n}\right)}{1-\frac{1}{2n}} = 4 > 1$$

$\therefore$ By ratio test, $\Sigma\, |u_n|$ is convergent. Hence the given series is absolutely convergent.

(*iv*) The given series is $\Sigma u_n = \sum \dfrac{(-1)^{n+1}.n}{n^2+1} = \Sigma(-1)^{n+1} v_n$

It is an alternating series.

Here $\quad v_n = \dfrac{n}{n^2+1} > 0 \quad \forall\, n$

$$v_{n+1} = \frac{n+1}{(n+1)^2+1}$$

$$\therefore \quad v_n - v_{n+1} = \frac{n}{n^2+1} - \frac{n+1}{(n+1)^2+1} = \frac{n(n^2+2n+2)-(n+1)(n^2+1)}{(n^2+1)(n^2+2n+2)}$$

$$= \frac{n^2+n-1}{(n^2+1)(n^2+2n+2)} > 0 \quad \forall\, n$$

$\Rightarrow \quad v_n > v_{n+1} \quad \forall\, n$

Also $\quad \lim\limits_{n\to\infty} v_n = \lim\limits_{n\to\infty} \dfrac{n}{n^2+1} = \lim\limits_{n\to\infty} \dfrac{\frac{1}{n}}{1+\frac{1}{n^2}} = 0$

$\therefore$ By Leibnitz's test, the series is convergent.

Now $\quad |u_n| = \dfrac{n}{n^2+1} = \dfrac{n}{n^2\left(1+\frac{1}{n^2}\right)} = \dfrac{1}{n\left(1+\frac{1}{n^2}\right)}$

Take $\quad V_n = \dfrac{1}{n}$

$$\lim_{n\to\infty} \frac{|u_n|}{V_n} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^2}} = 1 \text{ which is finite and non-zero.}$$

$\therefore$ By comparison test, $\Sigma\, |u_n|$ and ΣV_n converge or diverge together. Since $\Sigma V_n = \sum \dfrac{1}{n}$ diverges, $\Sigma\, |u_n|$ also diverges.

Hence the given series is conditionally convergent.

Example 7. *Show that the following series are convergent :*

(*i*) $\frac{\log 2}{2^2} - \frac{\log 3}{3^2} + \frac{\log 4}{4^2} - \ldots\ldots$ (*ii*) $\frac{\log 2}{2^3} - \frac{\log 3}{3^3} + \frac{\log 4}{4^3} - \ldots\ldots$

Sol. (*i*) The given series is

$$\sum_{n=2}^{\infty} (-1)^n . v_n = \sum_{n=2}^{\infty} (-1)^n . \frac{\log n}{n^2}$$

It is an alternating series.

Here $$v_n = \frac{\log n}{n^2}, n \geq 2$$

Consider the function $f(x) = \frac{\log x}{x^2}, x > 0$

then $$f'(x) = \frac{x^2 . \frac{1}{x} - 2x \log x}{x^4} = \frac{x(1 - 2\log x)}{x^4}$$

$$= \frac{1 - 2\log x}{x^3} < 0 \text{ whenever } 1 - 2\log x < 0$$

i.e., whenever $2 \log x > 1$ *i.e.,* $\log x > \frac{1}{2}$ or $x > e^{1/2}$

$\therefore$ $f(x)$ is monotonically decreasing for $x > \sqrt{e}$

Since $2 < e < 3 \sqrt{e} > \sqrt{2} = 1.414$

$\therefore$ $f(n) > f(n+1) \quad \forall\, n \geq 2$

or $v_n > v_{n+1} \quad \forall\, n \geq 2$

Also $$\lim_{n\to\infty} v_n = \lim_{n\to\infty} \frac{\log n}{n^2} \qquad \left| \text{Form } \frac{\infty}{\infty} \right.$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n}}{2n} = \lim_{n\to\infty} \frac{1}{2n^2} = 0$$

$\therefore$ By Leibnitz's test, the series is convergent.

(*ii*) *Please try yourself.*

Example 8. (*a*) *Test for convergence and absolute convergence the series :*

(*i*) $\sum_{n=1}^{\infty} (-1)^{n+1}(\sqrt{n^2+1} - n)$ (*ii*) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n} + \sqrt{a}}$

(*iii*) $\frac{1}{2(\log 2)^p} - \frac{1}{3(\log 3)^p} + \frac{1}{4(\log 4)^p} - \ldots\ldots (p > 0)$

(*iv*) $\sum_{n=1}^{\infty} (-1)^{n-1}\left[\frac{1}{n^2} + \frac{1}{(n+1)^2}\right]$.

(*b*) *Use Leibnitz test to show that* $\sum_{n=1}^{\infty} \frac{(-1)^n (n+5)}{n(n+1)}$ *is convergent.*

Sol. (*a*) (*i*) The given series is

$$\Sigma u_n = \Sigma(-1)^{n+1} (\sqrt{n^2+1} - n) = \Sigma(-1)^{n+1} v_n$$

It is an alternating series.

Here $$v_n = \sqrt{n^2+1} - n = (\sqrt{n^2+1} - n) \times \frac{\sqrt{n^2+1}+n}{\sqrt{n^2+1}+n}$$

$$= \frac{(n^2+1) - n^2}{\sqrt{n^2+1}+n} = \frac{1}{\sqrt{n^2+1}+n} > 0 \;\forall\, n$$

$$v_{n+1} = \frac{1}{\sqrt{(n+1)^2+1}+n}$$

Clearly $v_n > v_{n+1} \;\forall\, n$

Also $$\lim_{n\to\infty} v_n = \lim_{n\to\infty} \frac{1}{\sqrt{n^2+1}+n} = 0$$

$\therefore$ By Leibnitz's test, the series is convergent.

Now $$|u_n| = \frac{1}{\sqrt{n^2+1}+n}$$

Take $$V_n = \frac{1}{n}$$

$\therefore$ $$\lim_{n\to\infty} \frac{|u_n|}{V_n} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}+n} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n^2}}+1}$$

$$= \frac{1}{2} \text{ which is finite and non-zero.}$$

$\therefore$ By comparison test, $\Sigma\, |u_n|$ and ΣV_n converge or diverge together. Since $\Sigma V_n = \sum \frac{1}{n}$ diverges, $\Sigma\, |u_n|$ also diverges.

Hence the given series is conditionally convergent.

(*ii*) *Please try yourself.* **[Ans.** Conditionally convergent]

(*iii*) The given series is $\sum_{n=2}^{\infty} u_n = \sum_{n=2}^{\infty} \frac{(-1)^n}{n(\log n)^p} = \sum_{n=2}^{\infty} (-1)^n v_n$

It is an alternating series.

Here $$v_n = \frac{1}{n(\log n)^p} > 0 \;\forall\, n \geq 2$$

$$v_{n+1} = \frac{1}{(n+1)[\log(n+1)]^p}$$

Since $n(\log n)^p < (n+1)[\log(n+1)]^p \quad \forall\, n \geq 2$

$\therefore$ $v_n > v_{n+1}$

Also $$\lim_{n\to\infty} v_n = \lim_{n\to\infty} \frac{1}{n(\log n)^p} = 0$$

$\therefore$ By Leibnitz's test, the series is convergent.

Now $$|u_n| = v_n = \frac{1}{n(\log n)^p}$$

Since $\quad v_n > v_{n+1} \quad \forall\, n \geq 2$

$\therefore$ By Cauchy's condensation test, the series $\sum_{n=2}^{\infty} v_n$ and $\sum_{n=2}^{\infty} 2^n v_{2^n}$ converge or diverge together.

$$\sum_{n=2}^{\infty} 2^n v_{2^n} = \sum_{n=2}^{\infty} 2^n \cdot \frac{1}{2^n (\log 2^n)^p} = \sum_{n=2}^{\infty} \frac{1}{(n \log 2)^p} = \frac{1}{(\log 2)^p} \sum_{n=2}^{\infty} \frac{1}{n^p}$$

which is convergent if $p > 1$ and divergent if $p \leq 1$.

$\Rightarrow$ The series $\Sigma \mid u_n \mid$ is convergent if $p > 1$ and divergent if $p \leq 1$.

Hence the given series is absolutely convergent if $p > 1$ and conditionally convergent if $0 < p \leq 1$.

(*iv*) The given series is

$$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} (-1)^{n-1}\left[\frac{1}{n^2} + \frac{1}{(n+1)^2}\right] = \sum_{n=1}^{\infty} (-1)^{n-1} v_n$$

It is an alternating series.

Here $\quad v_n = \dfrac{1}{n^2} + \dfrac{1}{(n+1)^2} > 0 \;\forall\, n$

$$v_{n+1} = \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2}$$

$\therefore \quad v_n - v_{n+1} = \dfrac{1}{n^2} - \dfrac{1}{(n+2)^2} = \dfrac{(n+2)^2 - n^2}{n^2 (n+2)^2} = \dfrac{4n+4}{n^2 (n+2)^2} > 0 \qquad \forall\, n$

$\Rightarrow \quad v_n > v_{n+1} \quad \forall\, n$

Also $\quad \lim\limits_{n\to\infty} v_n = \lim\limits_{n\to\infty}\left[\dfrac{1}{n^2} + \dfrac{1}{(n+1)^2}\right] = 0$

$\therefore$ By Leibnitz's test, the series is convergent.

Now $\quad \mid u_n \mid = \dfrac{1}{n^2} + \dfrac{1}{(n+1)^2} < \dfrac{1}{n^2} + \dfrac{1}{n^2} = \dfrac{2}{n^2} \;\forall\, n$

Since the series $\sum \dfrac{2}{n^2} = 2\sum \dfrac{1}{n^2}$ is convergent, by comparison test, $\Sigma \mid u_n \mid$ is also convergent.

Hence the given series is absolutely convergent.

(*b*) The given series is $\sum_{n=1}^{\infty} (-1)^n u_n$ where $u_n = \dfrac{n+5}{n(n+1)} > 0.$

$$u_{n+1} = \frac{n+6}{(n+1)(n+2)}$$

$\therefore \quad u_n - u_{n+1} = \dfrac{n+5}{n(n+1)} - \dfrac{n+6}{(n+1)(n+2)} = \dfrac{(n+5)(n+2) - n(n+6)}{n(n+1)(n+2)}$

$$= \frac{n+10}{n(n+1)(n+2)} > 0 \quad \forall\, n \in \mathrm{N}.$$

$\Rightarrow \quad u_n > u_{n+1} \quad \forall\, n \in \mathrm{N}$

Also $\quad \lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty}$

$$\frac{n+5}{n(n+1)} = \lim_{n\to\infty} \frac{1+\frac{5}{n}}{n\left(1+\frac{1}{n}\right)} = 0$$

$\therefore$ By Leibnitz's test on alternating series, $\sum_{n=1}^{\infty} (-1)^n u_n$ is convergent.

Example 9. *Examine the convergence of the series* $\sum_{n=1}^{\infty} (-1)^n \cdot \left(\dfrac{1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots+\frac{1}{n}}{n}\right)$

Sol. The given series is $\sum_{n=1}^{\infty} (-1)^n \cdot \left(\dfrac{1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots+\frac{1}{n}}{n}\right) = \sum_{n=1}^{\infty} (-1)\, v_n$

Here $\quad v_n = \dfrac{1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots+\frac{1}{n}}{n} > 0 \;\forall\, n$

$$v_n - v_{n+1} = \frac{1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots+\frac{1}{n}}{n} - \frac{1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots+\frac{1}{n}+\frac{1}{n+1}}{n+1}$$

$$= \frac{(n+1)\left(1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots+\frac{1}{n}\right) - n\left(1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots+\frac{1}{n}+\frac{1}{n+1}\right)}{n(n+1)}$$

$$= \frac{\left(1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots+\frac{1}{n}\right)(n+1-n) - \frac{n}{n+1}}{n(n+1)}$$

$$= \frac{\left(1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots+\frac{1}{n}\right) - \left(\frac{1}{n+1}+\frac{1}{n+1}+\ldots\ldots+\frac{1}{n+1}\right)}{n(n+1)}$$

$$= \frac{\left(1-\frac{1}{n+1}\right)+\left(\frac{1}{2}-\frac{1}{n+1}\right)+\ldots\ldots+\left(\frac{1}{n}-\frac{1}{n+1}\right)}{n(n+1)}$$

$$> 0 \quad \forall\, n$$

$\Rightarrow \quad v_n > v_{n+1} \quad \forall\, n$

Since $\lim\limits_{n\to\infty} \dfrac{1}{n} = 0$, by Cauchy's first theorem on limits,

$$\lim_{n\to\infty} \frac{1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots+\frac{1}{n}}{n} = 0 \Rightarrow \lim_{n\to\infty} v_n = 0$$

$\therefore$ By Leibnitz's test, the given series is convergent.

Example 10. *Test the convergence and absolute convergence of the series :*

(i) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \sin n\alpha}{n^2}$, α *real* (ii) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cos^2 n\alpha}{n\sqrt{n}}$, α *real*

(iii) $\sum (-1)^n \cdot \frac{\sin nx}{n^3}$.

Sol. (i) The given series is $\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \sin n\alpha}{n^2}$

Now $|u_n| = \left|\frac{\sin n\alpha}{n^2}\right| = \frac{|\sin n\alpha|}{n^2} \le \frac{1}{n^2}$ $\forall n$ and $\sum \frac{1}{n^2}$ converges.

$\therefore$ By comparison test, the series $\Sigma |u_n|$ converges.

$\Rightarrow$ The given series is absolutely convergent.

Since every absolutely convergent series is convergent, therefore, the given series is convergent.

(ii) The given series is $\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cos^2 n\alpha}{n\sqrt{n}}$

Now $|u_n| = \left|\frac{\cos^2 n\alpha}{n\sqrt{n}}\right| = \frac{\cos^2 n\alpha}{n\sqrt{n}} \le \frac{1}{n^{3/2}}$ $\forall n$ and $\sum \frac{1}{n^{3/2}}$ converges.

$\therefore$ By comparison test, the series $\Sigma |u_n|$ converges.

$\Rightarrow$ The given series is absolutely convergent.

Since every absolutely convergent series is convergent, therefore, the given series is convergent.

(iii) *Please try yourself.* **[Ans. Absolutely convergent]**

Example 11. *Show that the series* $\sum_{n=1}^{\infty} \left(1 - \cos \frac{\pi}{n}\right)$ *converges.*

Sol. The given series is $\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \left(1 - \cos\frac{\pi}{n}\right) = \sum_{n=1}^{\infty} 2\sin^2 \frac{\pi}{2n}$

$$u_n = 2\sin^2 \frac{\pi}{2n} > 0 \quad \forall n$$

$\Rightarrow$ It is a series of positive terms.

Take $v_n = \frac{1}{n^2}$

Then $\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{2\sin^2 \frac{\pi}{2n}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\pi^2}{2} \cdot \left(\frac{\sin \frac{\pi}{2n}}{\frac{\pi}{2n}}\right)^2 = \frac{\pi^2}{2} \times 1^2 = \frac{\pi^2}{2}$

which is non-zero and finite.

$\therefore$ By comparison test, Σu_n and Σv_n converge or diverge together. Since $\Sigma v_n = \sum \frac{1}{n^2}$ converges, Σu_n also converges.

Hence the given series is convergent.

Note. If Σu_n is a series of positive terms, then $\Sigma u_n = \Sigma |u_n|$. Therefore, for a series of positive terms, the concepts of convergence and absolute convergence are the same.

Example 12. *Test for absolute convergence the series :*

(i) $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{2^n}{n!}$ (ii) $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{n^{100}}{2n!}$ (iii) $\sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{n^3}{(n+1)!}$.

Sol. (i) The given series is

$$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{n^n}{n!}.$$

Here $|u_n| = \frac{2^n}{n!}, \; |u_{n+1}| = \frac{2^{n+1}}{(n+1)!}$

$\therefore$ $\lim_{n\to\infty} \frac{|u_n|}{|u_{n+1}|} = \lim_{n\to\infty} \left(\frac{2^n}{n!} \cdot \frac{(n+1)!}{2^{n+1}}\right) = \lim_{n\to\infty} \frac{n+1}{2} = \infty$

$\therefore$ By ratio test, the series $\Sigma |u_n|$ is convergent.

Hence the given series is absolutely convergent.

(ii) The given series is $\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} (-1)^n \cdot \frac{n^{100}}{2n!}$

Here $|u_n| = \frac{n^{100}}{2n!}, \; |u_{n+1}| = \frac{(n+1)^{100}}{(2n+2)!}$

$\therefore$ $\lim_{n\to\infty} \frac{|u_n|}{|u_{n+1}|} = \lim_{n\to\infty} \frac{n^{100}}{2n!} \cdot \frac{(2n+2)!}{(n+1)^{100}} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{100} \cdot (2n+2)(2n+1)$

$$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^{100}} \cdot (2n+2)(2n+1) = \infty$$

$\therefore$ By ratio test, the series $\Sigma |u_n|$ is convergent.

Hence the given series is absolutely convergent.

(iii) The given series is $\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{n^3}{(n+1)!}$

Here $|u_n| = \frac{n^3}{(n+1)!}, \; |u_{n+1}| = \frac{(n+1)^3!}{(n+2)!}$

$\therefore$ $\lim_{n\to\infty} \frac{|u_n|}{|u_{n+1}|} = \lim_{n\to\infty} \frac{n^3}{(n+1)!} \cdot \frac{(n+2)!}{(n+1)^3}$

$$= \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^3 \cdot (n+2) = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^3}\cdot(n+2) = \infty$$

$\therefore$ By ratio test, the series $\Sigma \mid u_n \mid$ is convergent. Hence the given series is absolutely convergent.

Example 13. (*a*) *Discuss the convergence of the series* $1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots\ldots$ *for all values of* x.

(*b*) *Show that the series* $x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\ldots$ *converges absolutely for all values of* x.

Sol. (*a*) The given series is

$$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty}\frac{x^{n-1}}{(n-1)!}$$

Here $\quad u_n = \frac{x^{n-1}}{(n-1)!}$ and $u_{n+1} = \frac{x^n}{n!}$

$\therefore \quad \frac{|u_n|}{|u_{n+1}|} = \frac{|x|^{n-1}}{(n-1)!}\times\frac{n!}{|x|^n} = \frac{n}{|x|}$ for $x \neq 0$

$$\lim_{n\to\infty}\frac{|u_n|}{|u_{n+1}|} = \lim_{n\to\infty}\frac{n}{|x|} = \infty \text{ for } x \neq 0$$

$\therefore$ By ratio test, the series $\sum_{n=1}^{\infty} \mid u_n \mid$ is convergent for $x \neq 0$.

When $x = 0$, the series becomes $1 + 0 + 0 + \ldots\ldots$ and is convergent.

Thus $\Sigma \mid u_n \mid$ is convergent for all x.

$\Rightarrow$ The given series is absolutely convergent for all x.

• Since every absolutely convergent series is convergent, therefore, the given series is convergent for all x.

Note. In the above example, Σu_n is convergent for all x

$\Rightarrow \quad \lim_{n\to\infty} u_{n+1} = 0 \Rightarrow \lim_{n\to\infty}\frac{x^n}{n!} = 0$ for all x.

(*b*) *Please try yourself.*

Example 14. *Discuss the convergence and absolute convergence of the series* $x-\frac{x^3}{3}+\frac{x^5}{5}-\ldots\ldots$, *x being real.*

Sol. The given series is

$$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty}(-1)^{n-1}\cdot\frac{x^{2n-1}}{2n-1}$$

Here $\quad u_n = (-1)^{n-1}\cdot\frac{x^{2n-1}}{2n-1}$ and $u_{n+1} = (-1)^n\cdot\frac{x^{2n+1}}{2n+1}$

$$\therefore \quad \frac{|u_n|}{|u_{n+1}|} = \frac{|x|^{2n-1}}{2n-1}\times\frac{2n+1}{|x|^{2n+1}} = \frac{2n+1}{2n-1}\cdot\frac{1}{|x|^2} = \frac{1+\frac{1}{2n}}{1-\frac{1}{2n}}\cdot\frac{1}{x^2}$$

$(\because |x|^2 = x^2)$

$$\lim_{n\to\infty}\frac{|u_n|}{|u_{n+1}|} = \lim_{n\to\infty}\frac{1+\frac{1}{2n}}{1-\frac{1}{2n}}\cdot\frac{1}{x^2} = \frac{1}{x^2}$$

$\therefore$ By ratio test, the series $\sum_{n=1}^{\infty} \mid u_n \mid$ is convergent if $\frac{1}{x^2} > 1$ *i.e.*, if $x^2 < 1$ *i.e.*, if $-1 < x < 1$, and divergent if $\frac{1}{x^2} < 1$ *i.e.*, if $x^2 > 1$ *i.e.*, if $x > 1$ or $x < -1$.

Ratio test fail when $x^2 = 1$.

When $x^2 = 1$, we have

$$\frac{|u_n|}{|u_{n+1}|} = \frac{1+\frac{1}{2n}}{1-\frac{1}{2n}} = \left(1+\frac{1}{2n}\right)\left(1-\frac{1}{2n}\right)^{-1}$$

$$= \left(1+\frac{1}{2n}\right)\left[1+\frac{1}{2n}+O\left(\frac{1}{n^2}\right)\right] = 1+\frac{1}{n}+O\left(\frac{1}{n^2}\right)$$

$\therefore$ By Gauss test, the series $\Sigma \mid u_n \mid$ is divergent.

Hence the given series is absolutely convergent only when $-1 < x < 1$.

When x = 1, the series becomes

$$\Sigma u_n = \Sigma(-1)^{n-1}\cdot\frac{1}{2n-1} = \Sigma(-1)^{n-1} v_n$$

It is an alternating series.

Here $\quad v_n = \frac{1}{2n-1},\ v_{n+1} = \frac{1}{2n+1}$

Since $\quad 2n-1 < 2n+1 \quad \forall n$

$\therefore \quad \frac{1}{2n-1} > \frac{1}{2n+1} \quad \forall n$

or $\quad v_n > v_{n+1} \quad \forall n$

Also $\quad \lim_{n\to\infty} v_n = \lim_{n\to\infty}\frac{1}{2n-1} = 0$

$\therefore$ By Leibnitz's test, the series is convergent.

When x = – 1, the series becomes

$$\Sigma u_n = \Sigma(-1)^{n-1} \cdot \frac{(-1)^{2n-1}}{2n-1} = \Sigma(-1)^{n-1} \cdot \frac{-1}{2n-1} = -\Sigma(-1)^{n-1} \cdot \frac{1}{2n-1}$$

Since $\Sigma(-1)^{n-1} \cdot \frac{1}{2n-1}$ is convergent, Σu_n is convergent.

When $x^2 > 1$ *i.e.,* $|x| > 1$, $\lim\limits_{n\to\infty} \frac{x^{2n-1}}{2n-1} \neq 0$

$\therefore$ Σu_n does not converge for $|x| > 1$.

Hence Σu_n is convergent if $-1 \le x \le 1$ and absolutely convergent if $-1 < x < 1$.

Example 15. *Test convergence of the series* $\sum\limits_{n=1}^{\infty} \frac{(-1)^n (x+1)^n}{2^n \cdot n^2}$.

Sol. Here $\quad u_n = \frac{(-1)^n (x+1)^n}{2^n \cdot n^2}$ and $u_{n+1} = \frac{(-1)^{n+1} \cdot (x+1)^{n+1}}{2^{n+1} \cdot (n+1)^2}$

$$\frac{|u_n|}{|u_{n+1}|} = \frac{|x+1|^n}{2^n \cdot n^2} \times \frac{2^{n+1} \cdot (n+1)^2}{|x+1|^{n+1}}$$

$$= 2\left(\frac{n+1}{n}\right)^2 \cdot \frac{1}{|x+1|} = 2\left(1+\frac{1}{n}\right)^2 \cdot \frac{1}{|x+1|}$$

$\therefore$
$$\lim_{n\to\infty} \frac{|u_n|}{|u_{n+1}|} = \lim_{n\to\infty} 2\left(1+\frac{1}{n}\right)^2 \cdot \frac{1}{|x+1|} = \frac{2}{|x+1|}$$

$\therefore$ By ratio test, the series $\Sigma\,|u_n|$ is convergent

if $\quad \frac{2}{|x+1|} > 1$ *i.e.* if $|x+1| < 2$

i.e., if $\quad -2 < x+1 < 2$ *i.e.* if $-3 < x < 1$

Also $\Sigma\,|u_n|$ is divergent if $\frac{2}{|x+1|} < 1$ *i.e.* if $|x+1| > 2$ *i.e.* if $x+1 > 2$ or $x+1 < -2$ *i.e.* if $x > 1$ or $x < -3$.

Ratio test fails when $x = 1$ or -3.

When x = 1 $\quad \Sigma u_n = \sum \frac{(-1)^n \cdot 2^n}{2^n \cdot n^2} = \sum \frac{(-1)^n}{n^2} = \Sigma(-1)^n \cdot v_n$

It is an alternating series.

Here $v_n = \frac{1}{n^2}$, $\quad v_{n+1} = \frac{1}{(n+1)^2}$.

Clearly $\quad v_n > v_{n+1}\ \forall\ n$

Also $\quad \lim\limits_{n\to\infty} v_n = \lim\limits_{n\to\infty} \frac{1}{n^2} = 0$

$\therefore$ By Leibnitz's test, Σu_n is convergent.

When x = – 3 $\quad \Sigma u_n = \sum \frac{(-1)^n \cdot (-2)^n}{2^n \cdot n^2} = \sum \frac{(-1)^{2n} \cdot 2^n}{2^n \cdot n^2}$

$$= \sum \frac{1}{n^2} \text{ which is convergent.}$$

When $\quad |x+1| > 2$, $\lim\limits_{n\to\infty} \frac{(x+1)^n}{2^n \cdot n^2} \neq 0$

$\therefore$ Σu_n does not converge for $|x+1| > 2$.

Hence the given series is convergent if $-3 \le x \le 1$.

6

Limit and Continuity of Functions

6.1. EXPLAIN THE CONCEPT OF A FUNCTION, ITS DOMAIN AND RANGE

A **function** consists of two non-empty sets X and Y and a rule which assigns to each element of the set X **one and only one** element of the set Y.

The set X is called the **domain** of the function. If x is an element of X, then the element of Y which corresponds to it is called the value of the function at x (or the image of x) and is denoted by $f(x)$.

The **range** of a function is the set of all those elements of Y which are the values of the function.

$\therefore$ Range of $f(x) = \{f(x) : x \in X\}$, clearly range of $f \subseteq Y$.

6.2. What are Bounded and Unbounded Functions ? Give Various Examples of Functions Which are (*i*) Bounded (*ii*) Unbounded

A function is said to be bounded if its range is bounded, otherwise it is unbounded. Thus a function $f(x)$ is bounded in the domain D, if there exist two real numbers k and K such that

$$k \le f(x) \le K \text{ for all } x \in D.$$

Again, the bounds of the range of a bounded function are called the bounds of the function.

Example 1. The function f defined by $f(x) = \sin x$ for all $x \in R$ is a bounded function, because its range is the closed interval $[-1, 1]$ which is a bounded set. Clearly supremum or l.u.b. of f is 1 and infimum or g.l.b. of f is -1.

Example 2. The function $f(x) = \log x$ for all $x \in (0, \infty)$ has its range $(-\infty, \infty)$ which is not bounded. Thus the function f is unbounded in the domain $(0, \infty)$.

6.3. Define (*i*) Monotonically Increasing Function (*ii*) Monotonically Decreasing Function (*iii*) One-one Function (*iv*) Onto Function

Let $f : X \to Y$ (*i.e.*, f is a function whose domain is X and range $f(X) \subseteq Y$, the co-domain)

(*i*) f is called a monotonically increasing function if $x_1, x_2 \in X$ with $x_1 < x_2 \Rightarrow f(x_1) \le f(x_2)$.

(*ii*) f is called a monotonically decreasing function if $x_1, x_2 \in X$ with $x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2)$.

(*iii*) f is called a one-one function if $x_1, x_2 \in X$ with $x_1 \ne x_2 \Rightarrow f(x_1) \ne f(x_2)$.

(*iv*) f is called an onto function if to each $y \in Y$, $\exists$ at least one $x \in X$ s.t. $f(x) = y$.

6.4. Define The Limit of a Function at a Point. What are Left-hand and Right-hand Limits ? When Does The Limit of a Function Exist at a Point ?

A function $f(x)$ is said to tend to a limit l as x tends to a if to each given $\varepsilon > 0$, there exists a positive number δ (depending on ε) such that

$$| f(x) - l | < \varepsilon \text{ whenever } 0 < | x - a | < \delta,$$

i.e., $f(x) \in (l - \varepsilon, l + \varepsilon)$ for all those values of x (except at $x = a$) which belong to $(a - \delta, a + \delta)$. This is denoted by $\underset{x \to a}{\text{Lt}} f(x) = l$.

Left-hand and right-hand limits

$f(x)$ is said to tend to l as x tends to a through values less than a, if to each $\varepsilon > 0, \exists\, \delta > 0$, such that

$$| f(x) - l | < \varepsilon \text{ when } a - \delta < x < a,$$

so that $f(x) \in (l - \varepsilon, l + \varepsilon)$ whenever $x \in (a - \delta, a)$.

The limit in this case is called the **left-hand limit** (L.H.L.) and is denoted by $f(a - 0)$. Thus

$$f(a - 0) = \lim_{x \to a - 0} f(x)$$

Similarly, if $f(x)$ tends to l as x tends to a through values which are greater than a *i.e.*, if given $\varepsilon > 0, \exists\, \delta > 0$ such that

$$| f(x) - l | < \varepsilon \text{ when } a < x < a + \delta,$$

then $f(x)$ is said to tend to l from the right and the limit so obtained is called **the right hand limit** (R.H.L.) and is denoted by $f(a + 0)$.

We write $\quad f(a + 0) = \lim_{x \to a + 0} f(x).$

Existence of a limit at a point. *$f(x)$ is said to tend to a limit as x tends to 'a' if both the left and right hand limits exist and are equal, and their common value is called the limit of the function.*

Note. How to find the left-hand and right-hand limits ?

(*i*) To find $f(a - 0)$ or $\lim_{x \to a - 0} f(x)$, we first put $x = a - h$, $h > 0$ in $f(x)$ and then take the limit as $h \to 0+$. Thus

$$\lim_{x \to a - 0} f(x) = \lim_{h \to 0+} f(a - h).$$

(*ii*) To find $f(a + 0)$ or $\lim_{x \to a + 0} f(x)$, we first put $x = a + h$, $h > 0$ in $f(x)$ and then take the limit as $h \to 0+$. Thus

$$\lim_{x \to a + 0} f(x) = \lim_{h \to 0+} f(a + h).$$

6.5. LIMITS AT INFINITY AND INFINITE LIMITS

(*i*) $\lim_{x \to \infty} f(x) = l$

A function $f(x)$ is said to tend to l as $x \to \infty$ if given $\varepsilon > 0$, however small, $\exists$ a +ve number k (depending on ε) s.t.

$$| f(x) - l | < \varepsilon \quad \forall\, x \ge k \quad i.e., \quad l - \varepsilon < f(x) < l + \varepsilon \quad \forall\, x \ge k.$$

(*ii*) $\lim_{x \to -\infty} f(x) = l.$

A function $f(x)$ is said to tend to l as $x \to -\infty$ if given $\varepsilon > 0$, however small, $\exists$ a +ve number k (depending on ε) s.t.

$$| f(x - l) | < \varepsilon \quad \forall\, x \le -k \quad i.e., \quad l - \varepsilon < f(x) < l + \varepsilon \quad \forall\, x \le -k.$$

(*iii*) $\lim_{x \to a} f(x) = \infty.$

A function $f(x)$ is said to tend to ∞ as x tends to a, if given $k > 0$, however large, $\exists$ a +ve number δ s.t.

$$f(x) > k \text{ for } 0 < |x - a| < \delta.$$

(*iv*) $\lim\limits_{x \to a} f(x) = -\infty$.

A function $f(x)$ is said to tend to $-\infty$ as x tends to a, if given $k > 0$, however large, $\exists$ a +ve number δ s.t.

$$f(x) < -k \text{ for } 0 < |x - a| < \delta.$$

(*v*) $\lim\limits_{x \to \infty} f(x) = \infty$.

A function $f(x)$ is said to tend to ∞ as $x \to \infty$ if given $k > 0$, however large, $\exists$ a number $k' > 0$ s.t.

$$f(x) > k \quad \forall\, x \geq k'.$$

(*vi*) $\lim\limits_{x \to \infty} f(x) = -\infty$.

A function $f(x)$ is said to tend to $-\infty$ as $x \to \infty$, if given $k > 0$, however large, $\exists$ a number $k' > 0$ s.t.

$$f(x) < -k \quad \forall\, x \geq k'.$$

(*vii*) $\lim\limits_{x \to -\infty} f(x) = \infty$.

A function $f(x)$ is said to tend to ∞ as $x \to -\infty$, if given $k > 0$, however large, $\exists$ a number $k' > 0$ s.t.

$$f(x) > k \quad \forall\, x \leq -k'.$$

(*viii*) $\lim\limits_{x \to -\infty} f(x) = -\infty$.

A function $f(x)$ is said to tend to $-\infty$ and $x \to -\infty$, if given $k > 0$, however large, $\exists$ a number $k' > 0$ s.t.

$$f(x) < -k \;\forall\, x \leq -k'.$$

6.6. THE LIMIT OF A FUNCTION AT A POINT, WHEN IT EXISTS, IS UNIQUE

Suppose $\lim\limits_{x \to a} f(x)$ exists and is not unique.

Let $\quad \lim\limits_{x \to a} f(x) = l$ and $\lim\limits_{x \to a} f(x) = l'$, where $l \neq l'$.

Now $\quad l \neq l' \quad \Rightarrow \quad |l - l'| > 0.$

If we take $\varepsilon = \frac{1}{2}\,|l - l'| > 0$, then

$\lim\limits_{x \to a} f(x) = l \quad \Rightarrow \quad$ given $\varepsilon > 0, \exists\, \delta_1 > 0$ s.t.

$|f(x) - l| < \varepsilon$ whenever $0 < |x - a| < \delta_1$ ...(*i*)

Again $\quad \lim\limits_{x \to a} f(x) = l' \quad \Rightarrow \quad$ given $\varepsilon > 0, \exists\, \delta_2 > 0$ s.t.

$|f(x) - l'| < \varepsilon$ whenever $0 < |x - a| < \delta_2$ ...(*ii*)

Let $\delta = \min.\{\delta_1, \delta_2\}$, then from (*i*) and (*ii*), we have

$|f(x) - l| < \varepsilon$ and $|f(x) - l'| < \varepsilon$ whenever $0 < |x - a| < \delta$.

Now $\quad |l - l'| = |l - f(x) + f(x) - l'|$

$\leq |l - f(x)| + |f(x) - l'| = |f(x) - l| + |f(x) - l'|$

$< \varepsilon + \varepsilon = 2\varepsilon$ whenever $0 < |x - a| < \delta$

or $\quad |l - l'| < |l - l'|$ whenever $0 < |x - a| < \delta$

which is absurd, therefore, our supposition is wrong. Hence $l = l'$ which proves that $\lim\limits_{x \to a} f(x)$, if it exists, is unique.

6.7. ALGEBRA OF LIMITS

Let f and g be two functions and a be a point of their common domain.

If $\lim\limits_{x \to a} f(x) = l$ and $\lim\limits_{x \to a} g(x) = m$, then

(*I*) $\lim\limits_{x \to a} [f(x) + g(x)] = l + m$. $\qquad$ (*II*) $\lim\limits_{x \to a} [f(x) - g(x)] = l - m$.

(*III*) $\lim\limits_{x \to a} [f(x) \cdot g(x)] = lm$. $\qquad$ (*IV*) $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{l}{m}$, *provided* $m \neq 0$.

Proof. (I) $\quad \lim\limits_{x \to a} f(x) = l \quad \Rightarrow \quad$ given $\varepsilon > 0, \exists\, \delta_1 > 0$ s.t.

$|f(x) - l| < \dfrac{\varepsilon}{2} \quad$ for $0 < |x - a| < \delta_1$ ...(*i*)

$\lim\limits_{x \to a} g(x) = m \quad \Rightarrow \quad$ given $\varepsilon > 0, \exists\, \delta_2 > 0$ s.t.

$|g(x) - m| < \dfrac{\varepsilon}{2} \quad$ for $0 < |x - a| < \delta_2$ ...(*ii*)

Let $\delta = \min\{\delta_1, \delta_2\}$, then from (*i*) and (*ii*), we have

$|f(x) - l| < \dfrac{\varepsilon}{2}$ and $|g(x) - m| < \dfrac{\varepsilon}{2}$ for $0 < |x - a| < \delta$

Now $|f(x) + g(x) - (l + m)| = |f(x) - l + g(x) - m| \leq |f(x) - l| + |g(x) - m|$

$< \dfrac{\varepsilon}{2} + \dfrac{\varepsilon}{2} = \varepsilon \quad$ for $0 < |x - a| < \delta$

$\Rightarrow \quad \lim\limits_{x \to a} [f(x) + g(x)] = l + m$.

(II) Proceeding as in (I) above

$|f(x) - g(x) - (l - m)| = |f(x) - l + m - g(x)| \leq |f(x) - l| + |m - g(x)|$

$= |f(x) - l| + |g(x) - m| < \dfrac{\varepsilon}{2} + \dfrac{\varepsilon}{2} = \varepsilon$ for $0 < |x - a| < \delta$

$\Rightarrow \quad \lim\limits_{x \to a} [f(x) - g(x)] = l - m$.

(III) $\quad \lim\limits_{x \to a} f(x) = l \quad \Rightarrow \quad$ given $\varepsilon_1 > 0, \exists\, \delta_1 > 0$ s.t.

$|f(x) - l| < \varepsilon_1$ for $0 < |x - a| < \delta_1$ ...(*i*)

$\lim\limits_{x \to a} g(x) = m \quad \Rightarrow \quad$ given $\varepsilon_2 > 0, \exists\, \delta_2 > 0$ s.t.

$|g(x) - m| < \varepsilon_2$ for $0 < |x - a| < \delta_2$ ...(*ii*)

Let $\delta = \min.\{\delta_1, \delta_2\}$, then from (*i*) and (*ii*), we get

$|f(x) - l| < \varepsilon_1$ and $|g(x) - m| < \varepsilon_2$, for $0 < |x - a| < \delta$

Also $\quad |g(x)| = |g(x) - m + m| \leq |g(x) - m| + |m|$

$< \varepsilon_2 + |m| < 1 + |m| \qquad | \because \varepsilon_2 < 1$

Now $\quad |f(x)g(x) - lm| = |f(x)g(x) - lg(x) + lg(x) - lm| = |g(x)(f(x) - l) + l(g(x) - m)|$

$\leq |g(x)(f(x) - l)| + |l(g(x) - m)| = |g(x)||f(x) - l| + |l||g(x) - m|$

$< (1 + |m|)\varepsilon_1 + |l|\varepsilon_2$ for $0 < |x - a| < \delta$.

Taking $\varepsilon_1 = \dfrac{\varepsilon}{2(1+|m|)}$ and $\varepsilon_2 = \dfrac{\varepsilon}{2(1+|l|)}$, we have

$|f(x)g(x) - lm| < (1 + |m|)\varepsilon_1 + |l|\varepsilon_2$

$= (1 + |m|)\cdot\dfrac{\varepsilon}{2(1+|m|)} + |l|\cdot\dfrac{\varepsilon}{2(1+|l|)}$

$= \dfrac{\varepsilon}{2} + \dfrac{|l|}{1+|l|}\cdot\dfrac{\varepsilon}{2} < \dfrac{\varepsilon}{2} + \dfrac{\varepsilon}{2} = \varepsilon$ for $0 < |x - a| < \delta \qquad \left|\ \because \dfrac{|l|}{1+|l|} < 1\right.$

$\Rightarrow \quad \lim\limits_{x \to a} f(x)g(x) = lm.$

(IV) Let us first prove that

$$\lim_{x \to a} g(x) = m\ (\neq 0) \quad \Rightarrow \quad \lim_{x \to a} \frac{1}{g(x)} = \frac{1}{m}$$

$\lim\limits_{x \to a} g(x) = m \quad \Rightarrow \quad$ given $\varepsilon > 0, \exists\, \delta > 0$ s.t.

$|g(x) - m| < \varepsilon$ for $0 < |x - a| < \delta$

$\therefore \quad |m| = |m - g(x) + g(x)| \leq |m - g(x)| + |g(x)|$

$= |g(x) - m| + |g(x)| < \varepsilon + |g(x)|$ for $0 < |x - a| < \delta$

$\Rightarrow \quad |g(x)| > |m| - \varepsilon$ for $0 < |x - a| < \delta$

Taking $\quad \varepsilon > \dfrac{|m|}{2}$, we get $|g(x)| > \dfrac{|m|}{2}$ for $0 < |x - a| < \delta$

$\Rightarrow \quad \dfrac{1}{|g(x)|} < \dfrac{2}{|m|}$ for $0 < |x - a| < \delta \qquad \ldots(i)$

Again $\quad \lim\limits_{x \to a} g(x) = m \Rightarrow$ given $\varepsilon_1 = \dfrac{|m|^2}{2}\varepsilon > 0, \exists\, \delta > 0$ s.t.

$|g(x) - m| < \varepsilon_1$ for $0 < |x - a| < \delta \qquad \ldots(ii)$

$\therefore \quad \left|\dfrac{1}{g(x)} - \dfrac{1}{m}\right| = \left|\dfrac{m - g(x)}{m\,g(x)}\right| = \dfrac{|g(x) - m|}{|m||g(x)|}$

$< \dfrac{\varepsilon_1}{|m|}\cdot\dfrac{2}{|m|}$ for $0 < |x - a| < \delta \qquad |$ using (i) and (ii)

$= \dfrac{2}{|m|^2}\cdot\dfrac{|m|^2}{2}\varepsilon = \varepsilon$ *i.e.,* $\left|\dfrac{1}{g(x)} - \dfrac{1}{m}\right| < \varepsilon$ for $0 < |x - a| < \delta$

$\therefore \quad \lim\limits_{x \to a} \dfrac{1}{g(x)} = \dfrac{1}{m}$

Now $\quad \lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to a} f(x)\cdot\dfrac{1}{g(x)} = \lim\limits_{x \to a} f(x)\cdot\lim\limits_{x \to a}\dfrac{1}{g(x)} = l\cdot\dfrac{1}{m} = \dfrac{l}{m}$.

ILLUSTRATIVE EXAMPLES

Example 1. *Do the following limits exist? If yes, find them:*

(a) $\lim\limits_{x \to 1} \sin \dfrac{1}{x-1}$ $\qquad$ (b) $\lim\limits_{x \to 0} x \sin \dfrac{1}{x}$ $\qquad$ (c) $\lim\limits_{x \to 1} 2^{\frac{1}{x-1}}$

(d) $\lim\limits_{x \to 0} \dfrac{e^{1/x}}{e^{1/x} + 1}$ $\qquad$ (e) $\lim\limits_{x \to 0} \dfrac{1}{1 + e^{1/x}}$

(f) $\lim\limits_{x \to 1} f(x)$ *where* $f(x) = \begin{cases} 3x - 2 & \text{when } x < 1 \\ 4x^2 - 3x & \text{when } x > 1. \end{cases}$

Sol. (a) $\lim\limits_{x \to 1} \sin \dfrac{1}{x-1}$.

L.H.L. $= \lim\limits_{x \to 1-0} \sin \dfrac{1}{x-1}$ $\qquad$ **[Put x = 1 – h, h > 0]**

$= \lim\limits_{h \to 0} \sin \dfrac{1}{1-h-1} = \lim\limits_{h \to 0} -\sin \dfrac{1}{h}$.

Now as $h \to 0$, $\sin \dfrac{1}{h}$ is finite and oscillates between -1 and 1; so it does not tend to any unique and definite value as $h \to 0$. Hence L.H.L. does not exist.

Similarly the right hand limit also does not exist as $x \to 1$.

Thus $\lim\limits_{x \to 1} \sin \dfrac{1}{x-1}$ does not exist.

(b) $\lim\limits_{x \to 0} x \sin \dfrac{1}{x}$.

L.H.L. $= \lim\limits_{x \to 0-0} x \sin \dfrac{1}{x}$ $\qquad$ **[Put x = 0 – h, h > 0]**

$= \lim\limits_{h \to 0} (0 - h) \sin \dfrac{1}{0-h} = \lim\limits_{h \to 0} h \sin \dfrac{1}{h}$

$= 0 \times$ a finite quantity between -1 and $1 = 0$

Similarly, $\quad$ R.H.L. $= \lim\limits_{x \to 0+0} x \sin \dfrac{1}{x}$ $\qquad$ **[Put x = 0 + h, h > 0]**

$= \lim\limits_{h \to 0} (0 + h) \sin \dfrac{1}{0+h} = \lim\limits_{h \to 0} h \sin \dfrac{1}{h}$

$= 0 \times$ a finite quantity between -1 and $1 = 0$.

Thus L.H.L. and R.H.L. both exist and are equal, and hence $\lim\limits_{x \to 0} x \sin \dfrac{1}{x}$ exists and is equal to zero.

$\therefore \quad \lim\limits_{x \to 0} x \sin \dfrac{1}{x} = 0.$

(c) Let $\quad f(x) = 2^{\frac{1}{x-1}}$

$$\text{L.H.L.} = f(1-0) = \lim_{x \to 1-0} f(x) = \lim_{x \to 1-0} 2^{\frac{1}{x-1}} \qquad [\mathbf{Put\ x = 1 - h, h > 0}]$$

$$= \lim_{h \to 0} 2^{\frac{1}{1-h-1}} = \lim_{h \to 0} 2^{\frac{-1}{h}} = 2^{-\infty} = \frac{1}{2^{\infty}} = \frac{1}{\infty} = 0.$$

$$\text{R.H.L.} = \lim_{x \to 1+0} 2^{\frac{1}{x-1}} \qquad [\mathbf{Put\ x = 1 + h, h > 0}]$$

$$= \lim_{h \to 0} 2^{\frac{1}{1+h+1}} = \lim_{h \to 0} 2^{\frac{1}{h}} = 2^{\infty} = \infty.$$

Since L.H.L. $\neq$ R.H.L., $\therefore$ $\lim_{x \to 1} 2^{\frac{1}{x-1}}$ does not exist.

(*d*) $\lim_{x \to 0} \dfrac{e^{1/x}}{e^{1/x}+1}$

$$\text{L.H.L.} = \lim_{x \to 0-0} \frac{e^{1/x}}{e^{1/x}+1} \qquad [\mathbf{Put\ x = 0 - h, h > 0}]$$

$$= \lim_{h \to 0} \frac{e^{\frac{1}{0-h}}}{e^{\frac{1}{0-h}}+1} = \lim_{h \to 0} \frac{e^{-\frac{1}{h}}}{e^{-\frac{1}{h}}+1} = \frac{0}{0+1} = 0. \qquad \left[\because \lim_{h \to 0} e^{-\frac{1}{h}} = e^{-\infty} = \frac{1}{\infty} = 0\right]$$

$$\text{R.H.L.} = \lim_{x \to 0+0} \frac{e^{1/x}}{e^{1/x}+1} \qquad [\mathbf{Put\ x = 0 + h, h > 0}]$$

$$= \lim_{h \to 0} \frac{e^{\frac{1}{0+h}}}{e^{\frac{1}{0+h}}+1} = \lim_{h \to 0} \frac{e^{\frac{1}{h}}}{e^{\frac{1}{h}}+1}$$

$$\left[\text{Divide the numerator and denominator by } e^{\frac{1}{h}}\right]$$

$$= \lim_{h \to 0} \frac{1}{1+e^{-\frac{1}{h}}} = \frac{1}{1+e^{-\infty}} = \frac{1}{1+0} = 1.$$

Since L.H.L. $\neq$ R.H.L. $\therefore$ $\lim_{x \to 0} \dfrac{e^{1/x}}{e^{1/x}+1}$ does not exist.

(*e*) *Please try yourself.*

(*f*) $$\text{L.H.L.} = \lim_{h \to 1-0} f(x) = \lim_{x \to 1-0} (3x-2) \qquad [\mathbf{Put\ x = 1 - h, h > 0}]$$

$$= \lim_{h \to 0} (3-3h-2) = \lim_{h \to 0} (1-3h) = 1 - 0 = 1.$$

$$\text{R.H.L.} = \lim_{h \to 1+0} f(x) = \lim_{h \to 1+0} (4x^2-3x) \qquad [\mathbf{Put\ x = 1 + h, h > 0}]$$

$$= \lim_{h \to 0} [4(1+h)^2 - 3(1+h)] = \lim_{h \to 0} (1+5h+4h^2) = 1$$

$\therefore$ L.H.L. = R.H.L. = 1.

Hence $\lim_{x \to 0} f(x) = 1.$

Example 2. *Using the definition of limit, prove that*

(*i*) $\lim_{x \to a} \dfrac{x^2-a^2}{x-a} = 2a$ (*ii*) $\lim_{x \to 0} x \sin \dfrac{1}{x} = 0$

(*iii*) $\lim_{x \to 0} x^2 \sin \dfrac{1}{x} = 0.$

Sol. (*i*) Here $f(x) = \dfrac{x^2-a^2}{x-a}, x \neq a$

We must show that for any $\varepsilon > 0$, $\exists\ \delta > 0$ s.t.

$$| f(x) - 2a | < \varepsilon \text{ for } 0 < | x - a | < \delta$$

Now $$| f(x) - 2a | = \left| \frac{x^2-a^2}{x-a} - 2a \right| = \left| \frac{x^2-a^2-2ax+2a^2}{x-a} \right|$$

$$= \left| \frac{x^2-2ax+a^2}{x-a} \right| = \left| \frac{(x-a)^2}{x-a} \right| = | x - a |$$

$\therefore$ $| f(x) - 2a | < \varepsilon$ whenever $0 < | x - a | < \varepsilon.$

Choosing $\delta = \varepsilon$

$$| f(x) - 2a | < \varepsilon \text{ whenever } 0 < | x - a | < \delta$$

Hence $\lim_{x \to a} \dfrac{x^2-a^2}{x-a} = 2a.$

(*ii*) Here $f(x) = x \sin \dfrac{1}{x}$

$$| f(x) - 0 | = \left| x \sin \frac{1}{x} \right| = | x | \left| \sin \frac{1}{x} \right| \leq | x | \qquad \left(\because \left| \sin \frac{1}{x} \right| \leq 1\right)$$

$\therefore$ $| f(x) - 0 | < \varepsilon$ whenever $0 < | x | < \varepsilon$

Choosing $\delta = \varepsilon$

$$| f(x) - 0 | < \varepsilon \text{ whenever } 0 < | x | < \delta$$

Hence $\lim_{x \to 0} x \sin \dfrac{1}{x} = 0.$

(*iii*) Here $f(x) = x^2 \sin \dfrac{1}{x}$

$$| f(x) - 0 | = \left| x^2 \sin \frac{1}{x} \right| = | x^2 | \left| \sin \frac{1}{x} \right| \leq | x |^2 \qquad \left(\because \left| \sin \frac{1}{x} \right| \leq 1\right)$$

$\therefore$ $| f(x) - 0 | < \varepsilon$ whenever $0 < | x |^2 < \varepsilon$ *i.e.*, whenever $0 < | x | < \sqrt{\varepsilon}$

Choosing $\delta = \sqrt{\varepsilon}$

$$| f(x) - 0 | < \varepsilon \text{ whenever } 0 < | x | < \delta$$

Hence $\lim_{x \to 0} x^2 \sin \dfrac{1}{x} = 0.$

Example 3. *If $f(x) = [x]$ where $[x]$ denotes the greatest integer not greater than x, show that $\lim_{x \to 1} f(x)$ does not exist.*

Sol. $\text{L.H.L.} = \lim_{x \to 1-0} f(x) = \lim_{x \to 1-0} [x]$ [Put $x = 1 - h, h > 0$]

$= \lim_{h \to 0} [1 - h] = \lim_{h \to 0} (0) = 0$

$\text{R.H.L.} = \lim_{x \to 1+0} f(x) = \lim_{x \to 1+0} [x]$ [Put $x = 1 + h, h > 0$]

$= \lim_{h \to 0} [1 + h] = \lim_{h \to 0} (1) = 1$

$\because \lim_{x \to 1-0} f(x) \neq \lim_{x \to 1+0} f(x)$

$\therefore \lim_{x \to 1} f(x)$ does not exist.

Example 4. *Find* $\underset{x \to a}{Lt}\ f(x)$ *where* $f(x) = \begin{cases} \dfrac{x^2}{a} - a & \text{for } 0 < x < a \\ 0 & \text{for } x = a \\ a - \dfrac{a^3}{x^2} & \text{for } x > a \end{cases}$

Sol. $\text{L.H.L.} = \lim_{x \to a-0} f(x) = \lim_{x \to a-0} \left(\frac{x^2}{a} - a\right)$ [Put $x = a - h, h > 0$]

$= \lim_{h \to 0} \left[\frac{(a-h)^2}{a} - a\right] = \frac{a^2}{a} - a = a - a = 0$

$\text{R.H.L.} = \lim_{x \to a+0} f(x) = \lim_{x \to a+0} \left(a - \frac{a^3}{x^2}\right)$ [Put $x = a + h, h > 0$]

$= \lim_{h \to 0} \left[a - \frac{a^3}{(a+h)^2}\right] = a - \frac{a^3}{a^2} = a - a = 0$

$\therefore \lim_{x \to a-0} f(x)$ and $\lim_{x \to a+0} f(x)$ both exist and each is equal to 0.

$\therefore \lim_{x \to a} f(x) = 0.$

Example 5. *Let* $f(x) = \dfrac{x^2+2}{x^2+1}$, *then given* $\varepsilon > 0$, *find a real number* $\delta > 0$ *such that* $|f(x) - 2| < \varepsilon$ *for* $0 < |x| < \delta$.

Sol. $|f(x) - 2| = \left|\frac{x^2+2}{x^2+1} - 2\right| < \varepsilon$

if $\left|\frac{x^2+2-2x^2-2}{x^2+1}\right| < \varepsilon$ or if $\left|\frac{-x^2}{x^2+1}\right| < \varepsilon$

or if $\left|\frac{x^2}{x^2+1}\right| < \varepsilon$ $(\because |-x| = |x|)$

or if $\frac{x^2}{x^2+1} < \varepsilon$ $\left(\because \frac{x^2}{x^2+1} \geq 0\right)$

or if $x^2 < \varepsilon(x^2 + 1)$ $(\because x^2 + 1 > 0)$

or if $(1 - \varepsilon)x^2 < \varepsilon$ or if $x^2 < \frac{\varepsilon}{1 - \varepsilon}$ (if $1 - \varepsilon > 0$ *i.e.*, $\varepsilon < 1$)

or if $|x| < \sqrt{\frac{\varepsilon}{1-\varepsilon}}$

$\therefore$ Choosing $\delta = \sqrt{\frac{\varepsilon}{1-\varepsilon}}$, $0 < \varepsilon < 1$, we have $|f(x) - 2| < \varepsilon$ for $0 < |x| < \delta$.

Example 6. *Let* $f(x) = \dfrac{1}{x}$, $x \neq 0$. *Prove from definition* (ε, δ *method*) *that* $\lim_{x \to 2} f(x) = \dfrac{1}{2}$.

Sol. To prove that $\lim_{x \to 2} f(x) = \frac{1}{2}$, we have to show that for any $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon) > 0$ s.t.

$|f(x) - \frac{1}{2}| < \varepsilon$ when $0 < |x - 2| < \delta$

Now $|f(x) - \frac{1}{2}| = \left|\frac{1}{x} - \frac{1}{2}\right| = \left|\frac{2-x}{2x}\right| = \frac{|x-2|}{2|x|}$ ...(*i*)

Choosing $\delta \leq 1$ and $0 < |x - 2| < \delta$, we have

$0 < |x - 2| < 1 \Rightarrow |x - 2| > 0$ and $|x - 2| < 1$

$\Rightarrow x \neq 2$ and $2 - 1 < x < 2 + 1 \Rightarrow x \neq 2$ and $1 < x < 3$

$\Rightarrow x \neq 2$ and $1 > \frac{1}{x} > \frac{1}{3} \Rightarrow x \neq 2$ and $\frac{1}{3} < \frac{1}{x} < 1$

$\Rightarrow x \neq 2$ and $\frac{1}{|x|} < 1$ $\left[\because \frac{1}{x} > \frac{1}{3} > 0 \therefore \frac{1}{x} = \frac{1}{|x|}\right]$

$\therefore$ From (*i*), $|f(x) - \frac{1}{2}| = \frac{|x-2|}{2} \cdot \frac{1}{|x|} < \frac{\delta}{2} \cdot 1$

Let us choose δ s.t. $\frac{\delta}{2} < \varepsilon$ *i.e.*, $\delta < 2\varepsilon$

Also $\delta \leq 1$ $\therefore$ Choosing $\delta = \min. \{1, 2\varepsilon\}$, we have

$\left|f(x) - \frac{1}{2}\right| < \frac{\delta}{2} < \varepsilon$ when $0 < |x - 2| < \delta$

$\therefore \lim_{x \to 2} f(x) = \frac{1}{2}$.

Example 7. *If* $\lim_{x \to a} f(x)$ *exists and* $\lim_{x \to a} g(x)$ *does not exist, can* $\lim_{x \to a} [f(x) + g(x)]$ *exist ? Prove your assertion.*

Sol. $\because \lim_{x \to a} f(x)$ exists, let $\lim_{x \to a} f(x) = l$

$\Rightarrow \lim_{x \to a-0} f(x) = l = \lim_{x \to a+0} f(x)$

$\because \lim_{x \to a} g(x)$ does not exist, let $\lim_{x \to a-0} g(x) = m_1$ and $\lim_{x \to a+0} g(x) = m_2$, where $m_1 \neq m_2$

Now $\lim_{x \to a-0} [f(x) + g(x)] = \lim_{x \to a-0} f(x) + \lim_{x \to a-0} g(x) = l + m_1$

$\lim_{x \to a+0} [f(x) + g(x)] = \lim_{x \to a+0} f(x) + \lim_{x \to a+0} g(x) = l + m_2$

Since $l + m_1 \neq l + m_2$ $[\because m_1 \neq m_2]$

$\therefore \lim_{x \to a} [f(x) + g(x)]$ does not exist.

Example 8. *If $\lim_{x \to a} f(x)$ and $\lim_{x \to a} f(x)g(x)$ both exist, then does it follow that $\lim_{x \to a} g(x)$ exists?*

Sol. Let $f(x) = x,\ g(x) = \dfrac{|x|}{x},\ x \neq 0 = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$

$$f(x)g(x) = x \cdot \frac{|x|}{x} = |x|$$

$$\lim_{x \to 0} f(x) = 0 \text{ exists}; \ \lim_{x \to 0} f(x)g(x) = 0 \text{ exists}$$

But $\lim_{x \to 0-0} g(x) = \lim_{x \to 0-0} -1 = -1$

$$\lim_{x \to 0+0} g(x) = \lim_{x \to 0+0} 1 = 1$$

$\Rightarrow$ $\lim_{x \to 0} g(x)$ does not exist.

Thus $\lim_{x \to 0} f(x)$ and $\lim_{x \to 0} f(x)g(x)$ both exist does not necessarily imply that $\lim_{x \to 0} g(x)$ also exists.

Example 9. *If $\lim_{x \to a} f(x) = l$, then show that $\lim_{x \to a} |f(x)| = |l|$. Is its converse true?*

Sol. $\lim_{x \to 0} f(x) = l \ \Rightarrow$ for any given $\varepsilon > 0, \exists\ \delta > 0$ s.t.

$$|f(x) - l| < \varepsilon \text{ when } 0 < |x - a| < \delta \qquad ...(i)$$

Since $\quad |a - b| \geq ||a| - |b||$

$\therefore \quad |f(x) - l| \geq ||f(x)| - |l||$

$\Rightarrow \quad ||f(x)| - |l|| \leq |f(x) - l| < \varepsilon \quad$ when $0 < |x - a| < \delta \quad$ | Using (i)

$\Rightarrow \quad \lim_{x \to 0} |f(x)| = |l|$

The converse of this statement is not always true.

For example, consider $f(x) = \begin{cases} -1 & \text{if } x < a \\ 1 & \text{if } x \geq a \end{cases}$

then $\quad \lim_{x \to a-0} f(x) = \lim_{x \to a-0} -1 = -1$

$$\lim_{x \to a+0} f(x) = \lim_{x \to a+0} 1 = 1$$

$\Rightarrow \quad \lim_{x \to a} f(x)$ does not exist.

But $\quad |f(x)| = 1 \ \forall\ x \ \Rightarrow \ \lim_{x \to a} |f(x)| = 1$ exists.

Example 10. *If $f(x) \leq g(x) \leq h(x)$ and $\lim_{x \to a} f(x) = l = \lim_{x \to a} h(x)$, then prove that $\lim_{x \to a} g(x)$ exists and is equal to l.* **(Squeeze Principle)**

Sol. $\quad \lim_{x \to a} f(x) = l = \lim_{x \to a} h(x)$

$\Rightarrow$ given $\varepsilon > 0, \exists\ \delta_1, \delta_2 > 0$ s.t.

$$|f(x) - l| < \varepsilon \quad \text{for } 0 < |x - a| < \delta_1$$

and $\quad |h(x) - l| < \varepsilon \quad$ for $0 < |x - a| < \delta_2$

$\Rightarrow \quad l - \varepsilon < f(x) < l + \varepsilon \quad$ for $0 < |x - a| < \delta_1$

and $\quad l - \varepsilon < h(x) < l + \varepsilon \quad$ for $0 < |x - a| < \delta_2$

Let $\delta = \min.\ \{\delta_1, \delta_2\}$, then

$$l - \varepsilon < f(x) < l + \varepsilon \text{ and } l - \varepsilon < h(x) < l + \varepsilon \text{ for } 0 < |x - a| < \delta \qquad ...(i)$$

Also $\quad f(x) \leq g(x) \leq h(x)$ (given) $\qquad ...(ii)$

From (i) and (ii), $\quad l - \varepsilon < f(x) \leq g(x) \leq h(x) < l + \varepsilon$ for $0 < |x - a| < \delta$

$\Rightarrow \quad l - \varepsilon < g(x) < l + \varepsilon$ for $0 < |x - a| < \delta$

$\Rightarrow \quad |g(x) - l| < \varepsilon$ for $0 < |x - a| < \delta \ \Rightarrow \ \lim_{x \to a} g(x) = l$.

CONTINUITY

6.8. DEFINITIONS

(i) Continuity at a point

A function $f : A \to R$ is said to be continuous at the point $a \in A$ if given $\varepsilon > 0$, however small, $\exists$ a real number $\delta > 0$, such that

$$|f(x) - f(a)| < \varepsilon \text{ whenever } x \in A \text{ and } |x - a| < \delta$$

i.e., $\quad f(x) \in (f(a) - \varepsilon, f(a) + \varepsilon)$ whenever $x \in (a - \delta, a + \delta) \cap A$.

Equivalently, a function f is continuous at $x = a$, iff $\lim_{x \to a} f(x) = f(a)$

i.e., iff $\quad \lim_{x \to a-} f(x) = \lim_{x \to a+} f(x) = f(a)$.

(ii) Continuity from the left at a point

A function $f : A \to R$ is said to be continuous from the left (or left continuous) at the point $a \in A$ if given $\varepsilon > 0$, however small, $\exists$ a real number $\delta > 0$ such that

$$|f(x) - f(a)| < \varepsilon \text{ whenever } x \in A \text{ and } a - \delta < x \leq a$$

Equivalently, a function f is continuous from the left (or left continuous) at $x = a$ iff $\lim_{x \to a-} f(x) = f(a)$.

(iii) Continuity from the right at a point

A function $f : A \to R$ is said to be continuous from the right (or right continuous) at the point $a \in A$ if given $\varepsilon > 0$, however small, $\exists$ a real number $\delta > 0$ such that

$$|f(x) - f(a)| < \varepsilon \text{ whenever } x \in A \text{ and } a \leq x < a + \delta.$$

Equivalently, a function f is continuous from the right (or right continuous) at $x = a$, iff

$$\lim_{x \to a+} f(x) = f(a).$$

Note. Clearly, f is continuous at $x = a$ iff f is left as well as right continuous at $x = a$.

(iv) Continuity in an Open Interval

A function f is said to be continuous in an open interval (a, b) if f is continuous at every point of (a, b).

Thus f is continuous in the open interval (a, b) iff for every $c \in (a, b)$, $\lim_{x \to c} f(x) = f(c)$.

(v) Continuity in a Closed Interval

A function f is said to be continuous in a closed interval $[a, b]$ if it is

(i) right continuous at a $\quad$ *i.e.,* $\quad \lim_{x \to a+} f(x) = f(a)$

(ii) continuous in the open interval (a, b) $\quad$ *i.e.,* $\quad \lim_{x \to c} f(x) = f(c)$ for every $c \in (a, b)$

(*iii*) left continuous at b *i.e.*, $\lim_{x \to b-} f(x) = f(b)$

(*vi*) **Continuity in a Semi-closed Interval**

I. A function f is said to be continuous in semi-closed interval $(a, b]$ if it is

(*i*) continuous in the open interval (a, b) *i.e.*, $\lim_{x \to c} f(x) = f(c)$ for every $c \in (a, b)$

(*ii*) left continuous at b *i.e.*, $\lim_{x \to b-} f(x) = f(b)$

II. A function f is said to be continuous in semi-closed interval $[a, b)$ if it is

(*i*) right continuous at a *i.e.*, $\lim_{x \to a+} f(x) = f(a)$

(*ii*) continuous in the open interval (a, b) *i.e.*, $\lim_{x \to c} f(x) = f(c)$ for every $c \in (a, b)$

(*vii*) **Continuity on a Set**

A function f is said to be continuous on an arbitrary set S ($\subset$ R) if for each $\varepsilon < 0$ and for every $a \in S$, $\exists$ a real number $\delta > 0$ such that $| f(x) - f(a) | < \varepsilon$ whenever $x \in S$ and $| x - a | < \delta$.

Equivalently, a function f is said to be continuous on a set S if it continuous at ~~every~~ every point of S, *i.e.*, if for every $a \in S$, $\lim_{x \to a} f(x) = f(a)$.

(*viii*) **Continuous Function**

A function $f : A \to R$ is said to be continuous iff it is continuous on A.

Thus f is continuous if it is continuous at every point of its domain.

(*ix*) **Discontinuity of a Function**

A function f which is not continuous at a point 'a' is said to be *discontinuous at the point* 'a'. 'a' is called *a point of discontinuity of f* or f ~~is said to have a~~ discontinuity at 'a'.

A function which is discontinuous even at a single point of an interval is said to be *discontinuous in the interval.*

A function f can be discontinuous at a point $x = a$ because of any one of the following reasons :

(*i*) f is not defined at 'a'.

(*ii*) $\lim_{x \to a} f(x)$ does not exist *i.e.*, $\lim_{x \to a-} f(x) \neq \lim_{x \to a+} f(x)$

(*iii*) $\lim_{x \to a} f(x)$ and $f(a)$ both exist but are not equal.

(*x*) **Types of Discontinuity**

Let f be a function defined on an interval I. Let f be discontinuous at a point $a \in I$.

(1) **Removable Discontinuity**

If $\lim_{x \to a} f(x)$ exists but is not equal to $f(a)$, then f is said to have a *removable discontinuity* at 'a'.

This type of discontinuity can be removed by defining a new function g as

$$g(x) = \begin{cases} f(x) & \text{if } x \neq a \\ \lim_{x \to a} f(x) & \text{if } x = a \end{cases}$$

Then g is continuous at 'a'.

Note. If $\lim_{x \to a} f(x)$ does not exist, then the function cannot be made continuous, no matter how we define $f(a)$.

(2) **Discontinuity of First Kind (or Jump Discontinuity)**

If $\lim_{x \to a-} f(x)$ and $\lim_{x \to a+} f(x)$ both exist but are unequal, then f is said to have *a discontinuity of first kind at 'a' or jump discontinuity at 'a'.*

f is said to have a discontinuity of the first kind from the left at 'a' if $\lim_{x \to a-} f(x)$ exists but is not equal to $f(a)$.

f is said to have a discontinuity of the first kind from the right at 'a' if $\lim_{x \to a+} f(x)$ exists but is not equal to $f(a)$.

(3) **Discontinuity of Second Kind**

If neither $\lim_{x \to a-} f(x)$ nor $\lim_{x \to a+} f(x)$ exist, then f is said to have *a discontinuity of second kind at 'a'.*

f is said to have a discontinuity of the second kind from the left at 'a' if $\lim_{x \to a-} f(x)$ does not exist.

f is said to have a discontinuity of the second kind from the right at 'a' if $\lim_{x \to a+} f(x)$ does not exist.

(4) **Mixed Discontinuity**

If a function f has a discontinuity of the second kind on one side of a and on the other side, a discontinuity of the first kind or may be continuous, then f is said to have ~~a~~ mixed discontinuity at 'a'.

Thus f has a mixed discontinuity at 'a' if either

(*i*) $\lim_{x \to a-} f(x)$ does not exist and $\lim_{x \to a+} f(x)$ exists, however $\lim_{x \to a+} f(x)$ may or may not equal $f(a)$.

or (*ii*) $\lim_{x \to a+} f(x)$ does not exist and $\lim_{x \to a-} f(x)$ exists, however $\lim_{x \to a-} f(x)$ may or may not equal $f(a)$.

(*xi*) **Piecewise Continuous Function**

A function $f : A \to R$ is said to be piecewise continuous on A if A can be divided into a finite number of parts so that f is continuous on each part.

Clearly, in such a case, f has a finite number of discontinuities and the set A is divided at the points of discontinuities.

For example, consider $f : (0, 5) \to R$ defined by $f(x) = [x]$, then f is discontinuous at 1, 2, 3 and 4. If the interval (0, 5) is divided at 1, 2, 3 and 4, then f is continuous in (0, 1), (1, 2), (2, 3), (3, 4) and (4, 5).

$\therefore$ f is piecewise continuous.

ILLUSTRATIVE EXAMPLES

Example 1. *Using ε—δ definition, prove that*

(i) $f(x) = 3x + 1$ *is continuous at* $x = 2$

(ii) $f(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & \text{if } x \neq 2 \\ 4, & \text{if } x = 2 \end{cases}$ *is continuous at* $x = 2$

(iii) $f(x) = \begin{cases} \dfrac{x^3-1}{x^2-1}, & \text{if } x \neq 1 \\ 3/2, & \text{if } x = 1 \end{cases}$ *is continuous at* $x = 1$.

Sol. *(i)* Here $f(x) = 3x + 1, f(2) = 3 \times 2 + 1 = 7$

Let $\varepsilon > 0$ be given.

Now $|f(x) - f(2)| = |(3x+1) - 7| = |3(x-2)|$.

$= 3|x-2| < \varepsilon$ whenever $3|x-2| < \varepsilon$ *i.e.,* $|x-2| < \dfrac{\varepsilon}{3}$

∴ If we choose $\delta = \dfrac{\varepsilon}{3}$, then $|f(x) - f(2)| < \varepsilon$ whenever $|x-2| < \delta$

⇒ f is continuous at $x = 2$.

(ii) Here $f(x) = \dfrac{x^2-4}{x-2}, x \neq 2$

$f(2) = 4$

Let $\varepsilon > 0$ be given.

Now $|f(x) - f(2)| = \left|\dfrac{x^2-4}{x-2} - 4\right| = \left|\dfrac{(x+2)(x-2)}{x-2} - 4\right|$

$= |(x+2) - 4| = |x-2| < \varepsilon$ whenever $|x-2| < \varepsilon$

∴ If we choose $\delta = \varepsilon$, then $|f(x) - f(2)| < \varepsilon$ whenever $|x-2| < \delta$

⇒ f is continuous at $x = 2$.

(iii) Here $f(x) = \dfrac{x^3-1}{x^2-1}, x \neq 1$

$f(1) = 3/2$

Let $\varepsilon > 0$ be given.

Now $|f(x) - f(1)| = \left|\dfrac{x^3-1}{x^2-1} - \dfrac{3}{2}\right| = \left|\dfrac{(x-1)(x^2+x+1)}{(x-1)(x+1)} - \dfrac{3}{2}\right|$

$= \left|\dfrac{x^2+x+1}{x+1} - \dfrac{3}{2}\right| = \left|\dfrac{2x^2-x-1}{2(x+1)}\right|$

$= \left|\dfrac{(x-1)(2x+1)}{2x+2}\right| = |x-1|\left|\dfrac{2x+1}{2x+2}\right|$

$\leq |x-1|$ $\left[\because \left|\dfrac{2x+1}{2x+2}\right| < 1\right]$

$< \varepsilon$ whenever $|x-1| < \varepsilon$

∴ If we choose $\delta = \varepsilon$, then $|f(x) - f(1)| < \varepsilon$ whenever $|x-1| < \delta$

⇒ f is continuous at $x = 1$.

Example 2. *Using ε—δ definition, prove that*

(i) $f(x) = \begin{cases} x \sin \dfrac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ *is continuous at* $x = 0$.

(ii) $g(x) = \begin{cases} x^2 \cos \dfrac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ *is continuous at* $x = 0$.

Sol. *(i)* Here $f(x) = x \sin \dfrac{1}{x}, x \neq 0$

$f(0) = 0$

Let $\varepsilon > 0$ be given.

Now $|f(x) - f(0)| = \left|x \sin \dfrac{1}{x} - 0\right| = \left|x \sin \dfrac{1}{x}\right| = |x|\left|\sin \dfrac{1}{x}\right| \leq |x|$ $\left[\because \left|\sin \dfrac{1}{x}\right| \leq 1\right]$

$< \varepsilon$ whenever $|x| < \varepsilon$

∴ If we choose $\delta = \varepsilon$, then $|f(x) - f(0)| < \varepsilon$ whenever $|x - 0| < \delta$

⇒ f is continuous at $x = 0$.

(ii) Here $g(x) = x^2 \cos \dfrac{1}{x}, x \neq 0$

$g(0) = 0$

Let $\varepsilon > 0$ be given.

Now $|g(x) - g(0)| = \left|x^2 \cos \dfrac{1}{x} - 0\right| = \left|x^2 \cos \dfrac{1}{x}\right| = |x^2|\left|\cos \dfrac{1}{x}\right| \leq |x^2|$ $\left[\because \left|\cos \dfrac{1}{x}\right| \leq 1\right]$

$= |x|^2 < \varepsilon$ whenever $|x|^2 < \varepsilon$ *i.e.,* whenever $|x| < \sqrt{\varepsilon}$

∴ If we choose $\delta = \sqrt{\varepsilon}$, then $|g(x) - g(0)| < \varepsilon$ whenever $|x - 0| < \delta$

⇒ g is continuous at $x = 0$.

Example 3. *Using ε—δ definition, prove that the following functions are continuous :*

(i) $|x|$ *(ii)* $\cos x$ *(iii)* $\sin x$

(iv) $\cos^2 x$ *(v)* $\sin^2 x$.

Sol. ☞[A function f is said to be continuous if it is continuous at every point of its domain.]

(i) Let $f(x) = |x|$. Domain of $f = \mathbb{R}$

Let a be any real number so that $f(a) = |a|$

Let $\varepsilon > 0$ be given.

Now $|f(x) - f(a)| = ||x| - |a|| \leq |x - a|$ $[\because ||a| - |b|| \leq |a - b|]$

$< \varepsilon$ whenever $|x - a| < \varepsilon$

∴ If we choose $\delta = \varepsilon$, then $|f(x) - f(a)| < \varepsilon$ whenever $|x - a| < \delta$

⇒ f is continuous at $x = a$

⇒ f is continuous at every $a \in \mathbb{R}$

⇒ f is continuous.

(*ii*) Let $f(x) = \cos x$. Domain of $f = \mathbb{R}$

Let a be any real number so that $f(a) = \cos a$

Let $\varepsilon > 0$ be given.

Now $\quad |f(x) - f(a)| = |\cos x - \cos a|$

$$= \left| -2\sin\frac{x+a}{2}\sin\frac{x-a}{2} \right| = 2\left|\sin\frac{x+a}{2}\right|\left|\sin\frac{x-a}{2}\right|$$

$$\le 2\left|\sin\frac{x-a}{2}\right| \qquad \left[\because \left|\sin\frac{x+a}{2}\right| \le 1\right]$$

$$< 2\left|\frac{x-a}{2}\right| \qquad [\because |\sin x| \le |x|]$$

$$= 2 \cdot \frac{|x-a|}{2} = |x-a| < \varepsilon \qquad \text{whenever } |x-a| < \varepsilon$$

$\therefore$ If we choose $\delta = \varepsilon$, then

$|f(x) - f(a)| < \varepsilon \qquad$ whenever $|x - a| < \delta$

$\Rightarrow$ f is continuous at $x = a$

$\Rightarrow$ f is continuous at every $a \in \mathbb{R}$

$\Rightarrow$ f is continuous.

(*iii*) *Please try yourself.*

(*iv*) Let $\quad f(x) = \cos^2 x$. Domain of $f = \mathbb{R}$

Let a be any real number so that $\quad f(a) = \cos^2 a$.

Let $\varepsilon > 0$ be given.

Now $\quad |f(x) - f(a)| = |\cos^2 x - \cos^2 a| = |(1 - \sin^2 x) - (1 - \sin^2 a)|$

$$= |\sin^2 a - \sin^2 x| = |\sin^2 x - \sin^2 a| = |\sin(x+a)\sin(x-a)|$$

$$\le |\sin(x-a)| \qquad [\because |\sin(x+a)| \le 1]$$

$$\le |x-a| \qquad [\because |\sin x| \le |x|]$$

$$< \varepsilon \quad \text{whenever } |x-a| < \varepsilon$$

$\therefore$ If we choose $\delta = \varepsilon$, then $|f(x) - f(a)| < \varepsilon$ whenever $|x - a| < \delta$

$\Rightarrow$ f is continuous at $x = a$

$\Rightarrow$ f is continuous at every $a \in \mathbb{R}$

$\Rightarrow$ f is continuous.

(*v*) *Please try yourself.*

Example 4. *Examine the continuity of the following functions at the indicated point. Also point out the type of discontinuity, if any.*

(*i*) $f(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & \text{if } x \ne 2 \\ 4, & \text{if } x = 2 \end{cases}$ at $x = 2$ $\qquad$ (*ii*) $f(x) = \begin{cases} \dfrac{x^2-9}{x-3}, & \text{if } x \ne 3 \\ 5, & \text{if } x = 3 \end{cases}$ at $x = 3$

(*iii*) $f(x) = \dfrac{x^3-8}{x-2}$ at $x = 2$ $\qquad$ (*iv*) $f(x) = \begin{cases} \dfrac{\sin 2x}{x}, & \text{if } x \ne 0 \\ 1, & \text{if } x = 0 \end{cases}$ at $x = 0$

(*v*) $f(x) = \begin{cases} \dfrac{\sin^{-1} x}{2x}, & \text{if } x \ne 0 \\ \frac{1}{2}, & \text{if } x = 0 \end{cases}$ at $x = 0$.

Sol. (*i*) Here $f(2) = 4$

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^2-4}{x-2} = \lim_{x \to 2} \frac{(x+2)(x-2)}{x-2}$$

[Cancelling $(x - 2)$, since $x \to 2 \Rightarrow x \ne 2$]

$$= \lim_{x \to 2} (x+2) = 2 + 2 = 4$$

Since $\lim_{x \to 2} f(x) = f(2)$, f is continuous at $x = 2$.

(*ii*) Here $\qquad f(3) = 5$

$$\lim_{x \to 3} f(x) = \lim_{x \to 3} \frac{x^2-9}{x-3} = \lim_{x \to 3} \frac{(x+3)(x-3)}{x-3} = \lim_{x \to 3} (x+3) = 3 + 3 = 6$$

Thus, $\lim_{x \to 3} f(x)$ exists but $\lim_{x \to 3} f(x) \ne f(3)$.

$\therefore$ **f has a removable discontinuity at x = 3.**

f can be made continuous at $x = 3$ be redefining it as follows :

$$f(x) = \begin{cases} \dfrac{x^2-9}{x-3}, & \text{if } x \ne 3 \\ 6, & \text{if } x = 3 \end{cases}$$

(*iii*) $f(x) = \dfrac{x^3-8}{x-2}$ is not defined at $x = 2$, since $f(2)$ assumes the form $\dfrac{0}{0}$.

However, $\quad \lim_{x \to 2} f(x) = \lim_{x \to 2} \dfrac{x^3-8}{x-2} = \lim_{x \to 2} \dfrac{(x-2)(x^2+2x+4)}{x-2}$

$$= \lim_{x \to 2} (x^2 + 2x + 4) = 2^2 + 2 \times 2 + 4 = 12$$

Thus, $\lim_{x \to 2} f(x)$ exists. Therefore, *f has a removable discontinuity at $x = 2$.*

f can be made continuous at $x = 2$ by redefining it as follows :

$$f(x) = \begin{cases} \dfrac{x^3-8}{x-2}, & \text{if } x \ne 2 \\ 12, & \text{if } x = 2 \end{cases}$$

(*iv*) Here $\qquad f(0) = 1$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin 2x}{x} = \lim_{x \to 0} 2 \times \frac{\sin 2x}{2x} = 2 \times 1 = 2$$

Thus $\lim_{x \to 0} f(x)$ exists but $\lim_{x \to 0} f(x) \ne f(0)$.

$\Rightarrow$ **f has a removable discontinuity at x = 0**

f can be made continuous at $x = 0$ by redefining it as follows :

$$f(x) = \begin{cases} \dfrac{\sin 2x}{x}, & \text{if } x \ne 0 \\ 2, & \text{if } x = 0 \end{cases}$$

(*v*) Here $\qquad f(0) = \dfrac{1}{2}$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sin^{-1} x}{2x}$$

[Put $\sin^{-1} x = \theta$ so that $x = \sin\theta$ As $x \to 0,\ \theta \to 0$]

$$= \lim_{\theta\to 0} \frac{\theta}{2\sin\theta} = \frac{1}{2}\lim_{\theta\to 0}\frac{\theta}{\sin\theta} = \frac{1}{2}\times 1 = \frac{1}{2}$$

Since $\lim_{x\to 0} f(x) = f(0)$, f is continuous at $x = 0$.

Example 5. *Examine the continuity of the following functions at the indicated point. Also point out the type of discontinuity, if any.*

(*i*) $f(x) = \begin{cases} \dfrac{e^{1/x}-1}{e^{1/x}+1}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ at $x = 0$

(*ii*) $f(x) = \begin{cases} \dfrac{e^{1/x}}{1+e^{1/x}}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ at $x = 0$

(*iii*) $f(x) = \begin{cases} \dfrac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}}, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$ at $x = 0$

(*iv*) $f(x) = \begin{cases} (x-a)\dfrac{e^{1/x-a}-1}{e^{1/x-a}+1}, & \text{if } x \neq a \\ 0, & \text{if } x = a \end{cases}$ at $x = a$

(*v*) $f(x) = \begin{cases} \dfrac{x\,e^{1/x}}{1+e^{1/x}}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ at $x = 0$.

(*vi*) *Show that the function f defined on R as* $f(x) = x \cdot \dfrac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}}$ *if* $x \neq 0$ *and* $f(0) = 0$ *is continuous at* $x = 0$.

Sol. (*i*) Here $f(0) = 0$

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0-} \frac{e^{1/x}-1}{e^{1/x}+1} = \frac{0-1}{0+1} = -1 \quad \left(\because \text{ as } x\to 0-, \frac{1}{x}\to -\infty \ \therefore\ e^{1/x}\to 0\right)$$

and $$\lim_{x\to 0+} f(x) = \lim_{x\to 0+} \frac{e^{1/x}-1}{e^{1/x}+1} \qquad \text{(dividing the num. and denom. by } e^{1/x}\text{)}$$

$$= \lim_{x\to 0+} \frac{1-e^{-1/x}}{1+e^{-1/x}} = \frac{1-0}{1+0} = 1$$

$$\left(\because \text{ as } x\to 0+, \frac{1}{x}\to\infty \ \therefore\ e^{1/x}\to\infty \text{ and } e^{-1/x}\to 0\right)$$

Thus, $\lim_{x\to 0-} f(x)$ and $\lim_{x\to 0+} f(x)$ both exist but are not equal.

$\Rightarrow$ $\lim_{x\to 0} f(x)$ does not exist.

Also, none of the left and right limits is equal to $f(0)$.

$\therefore$ *f has a discontinuity of the first kind at* $x = 0$.

(*ii*) Here $f(0) = 0$

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0-} \frac{e^{1/x}}{1+e^{1/x}} = \frac{0}{1+0} = 0 \quad \left(\because \text{ as } x\to 0-, \frac{1}{x}\to -\infty \ \therefore\ e^{1/x}\to 0\right)$$

and $$\lim_{x\to 0+} f(x) = \lim_{x\to 0+} \frac{e^{1/x}}{1+e^{1/x}} \qquad \text{(dividing the num. and denom. by } e^{1/x}\text{)}$$

$$= \lim_{x\to 0+} \frac{1}{e^{-1/x}+1} = \frac{1}{0+1} = 1$$

$$\left(\because \text{ as } x\to 0+, \frac{1}{x}\to\infty \ \therefore\ e^{1/x}\to\infty \text{ and } e^{-1/x}\to 0\right)$$

Thus, $\lim_{x\to 0-} f(x)$ and $\lim_{x\to 0+} f(x)$ both exist but are not equal.

$\Rightarrow$ $\lim_{x\to 0} f(x)$ does not exist.

Since $\lim_{x\to 0-} f(x) = f(0) \neq \lim_{x\to 0+} f(x)$

Therefore, *f is continuous from the left at* $x = 0$ *and has a discontinuity of the first kind from the right at* $x = 0$.

(*iii*) Here $f(0) = 1$

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0-} \frac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}}$$

$$= \lim_{x\to 0-} \frac{e^{2/x}-1}{e^{2/x}+1} = \frac{0-1}{0+1} = -1 \quad \left(\because \text{ as } x\to 0-, \frac{2}{x}\to -\infty \ \therefore\ e^{2/x}\to 0\right)$$

and $$\lim_{x\to 0+} f(x) = \lim_{x\to 0+} \frac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}}$$

$$= \lim_{x\to 0+} \frac{1-e^{-2/x}}{1+e^{-2/x}} = \frac{1-0}{1+0} = 1$$

$$\left(\because \text{ as } x\to 0+, \frac{2}{x}\to\infty \ \therefore\ e^{2/x}\to\infty \text{ and } e^{-2/x}\to 0\right)$$

Thus, $\lim_{x\to 0-} f(x)$ and $\lim_{x\to 0+} f(x)$ both exist but are not equal.

$\Rightarrow$ $\lim_{x\to 0} f(x)$ does not exist.

Since $\lim_{x\to 0-} f(x) \neq f(0) = \lim_{x\to 0+} f(x)$

Therefore, *f is continuous from the right at* $x = 0$ *and has a discontinuity of the first kind from the left at* $x = 0$.

(*iv*) Here $f(a) = 0$

$$\lim_{x\to a-} f(x) = \lim_{x\to a-} (x-a)\frac{e^{1/x-a}-1}{e^{1/x-a}+1} \quad [\text{Put } x = a-h,\ h > 0 \text{ so that as } x\to a-,\ h\to 0+]$$

$$= \lim_{x \to 0+} -h \frac{e^{-1/h} - 1}{e^{-1/h} + 1}$$

$$= -(0) \times \frac{0-1}{0+1} = 0 \left[\because \text{ as } h \to 0+, \frac{1}{h} \to \infty \therefore e^{1/h} \to \infty \text{ and } e^{-1/h} \to 0\right]$$

and $\lim_{x \to a+} f(x) = \lim_{x \to a+} (x-a) \cdot \frac{e^{1/x-a} - 1}{e^{1/x-a} + 1}$

[Put $x = a + h, h > 0$ so that as $x \to a+, h \to 0+$]

$$= \lim_{h \to 0+} h \cdot \frac{e^{1/h} - 1}{e^{1/h} + 1} = \lim_{h \to 0+} h \cdot \frac{1 - e^{-1/h}}{1 + e^{-1/h}} = 0 \times \frac{1-0}{1+0} = 0$$

Since $\lim_{x \to a-} f(x) = 0 = \lim_{x \to a+} f(x)$

$\therefore$ $\lim_{x \to a} f(x) = 0$. Also $f(a) = 0$

Hence f is continuous at $x = a$.

(*v*) Here $f(0) = 0$

$\lim_{x \to 0-} f(x) = \lim_{x \to 0-} \frac{x e^{1/x}}{1 + e^{1/x}}$ [Put $x = 0 - h, h > 0$ so that as $x \to 0-, h \to 0+$]

$$= \lim_{x \to 0+} \frac{-h e^{-1/h}}{1 + e^{-1/h}} = -\frac{0 \times 0}{1+0} = 0$$

$\left(\because \text{ as } h \to 0+, \frac{1}{h} \to \infty \therefore e^{1/h} \to \infty \text{ and } e^{-1/h} \to 0\right)$

and $\lim_{x \to 0+} f(x) = \lim_{x \to 0+} \frac{x e^{1/x}}{1 + e^{1/x}}$ (dividing the num. and denom. by $e^{1/x}$)

$$= \lim_{x \to 0+} \frac{x}{e^{-1/x} + 1} = \frac{0}{0+1} = 0$$

Since $\lim_{x \to 0-} f(x) = 0 = \lim_{x \to 0+} f(x)$

$\therefore$ $\lim_{x \to 0} f(x) = 0$. Also $f(0) = 0$

Hence f is continuous at $x = 0$.

(*vi*) *Please try yourself.*

Example 6. *Examine the continuity of the following functions at the indicated point. Also point out the type of discontinuity, if any.*

(*i*) $f(x) = \begin{cases} e^{1/x}, & when\ x \neq 0 \\ 0, & when\ x = 0 \end{cases}$ at $x = 0$ (*ii*) $f(x) = \begin{cases} e^{-1/x}, & when\ x \neq 0 \\ 0, & when\ x = 0 \end{cases}$ at $x = 0$

(*iii*) $f(x) = \begin{cases} \dfrac{e^{-\frac{1}{x^2}}}{1 - e^{\frac{1}{x^2}}}, & when\ x \neq 0 \\ 0, & when\ x = 0 \end{cases}$ at $x = 0$ (*iv*) $f(x) = \begin{cases} \dfrac{x}{1 + e^{1/x}}, & when\ x \neq 0 \\ 0, & when\ x = 0 \end{cases}$ at $x = 0$

(*v*) $f(x) = \begin{cases} \dfrac{1}{1 - e^{1/x}}, & when\ x \neq 0 \\ 0, & when\ x = 0 \end{cases}$ at $x = 0$ (*vi*) $f(x) = \begin{cases} \dfrac{x-1}{1 + e^{1/x-1}}, & when\ x \neq 1 \\ 0, & when\ x = 1 \end{cases}$ at $x = 1$.

Sol. (*i*) Here $f(0) = 0$

$\lim_{x \to 0-} f(x) = \lim_{x \to 0-} e^{1/x} = 0$ $\left[\because \text{ as } x \to 0-, \frac{1}{x} \to -\infty \therefore e^{1/x} \to 0\right]$

and $\lim_{x \to 0+} f(x) = \lim_{x \to 0+} e^{1/x} = \infty$ *i.e.*, $\lim_{x \to 0+} f(x)$ does not exist.

$\therefore$ *f has a discontinuity of the second kind from the right at $x = 0$.*

(*ii*) *Please try yourself.* **[Ans. Discontinuity of the second kind from the left.]**

(*iii*) Here $f(0) = 0$

$\lim_{x \to 0-} f(x) = \lim_{x \to 0-} \frac{e^{\frac{1}{x^2}}}{1 - e^{\frac{1}{x^2}}}$ [Put $x = 0 - h, h > 0$, so that as $x \to 0-, h \to 0+$]

$$= \lim_{h \to 0+} \frac{e^{\frac{1}{h^2}}}{1 - e^{\frac{1}{h^2}}}$$ (dividing the num. and denom. by $e^{\frac{1}{h^2}}$)

$$= \lim_{h \to 0+} \frac{1}{e^{-\frac{1}{h^2}} - 1} = \frac{1}{0-1} = -1$$

$\left(\because \text{ as } h \to 0+, \frac{1}{h^2} \to \infty \therefore e^{\frac{1}{h^2}} \to \infty \text{ and } e^{-\frac{1}{h^2}} \to 0\right)$

and $\lim_{x \to 0+} f(x) = \lim_{x \to 0+} \frac{e^{\frac{1}{x^2}}}{1 - e^{\frac{1}{x^2}}} = \lim_{x \to 0+} \frac{1}{e^{-\frac{1}{x^2}} - 1} = \frac{1}{0-1} = -1$

Since $\lim_{x \to 0-} f(x) = -1 = \lim_{x \to 0+} f(x)$

$\therefore$ $\lim_{x \to 0} f(x) = -1$ But $f(0) = 0$ so that $\lim_{x \to 0} f(x) \neq f(0)$

Thus, f has a removable discontinuity at $x = 0$.

(*iv*) Here $f(0) = 0$

$\lim_{x \to 0-} f(x) = \lim_{x \to 0-} \frac{1}{1 + e^{1/x}}$

[Put $x = 0 - h, h > 0$, so that as $x \to 0-, h \to 0+$]

$$= \lim_{h \to 0+} \frac{-h}{1 + e^{-1/h}} = -\frac{0}{1+0} = 0$$

$\left(\because \text{ as } h \to 0+, \frac{1}{h} \to \infty \therefore e^{1/h} \to \infty \text{ and } e^{-1/h} \to 0\right)$

and $\lim_{x \to 0+} f(x) = \lim_{x \to 0+} \frac{x}{1+e^{1/x}}$ (dividing the num. and denom. by $e^{1/x}$)

$$= \lim_{x \to 0+} \frac{xe^{-1/x}}{e^{-1/x}+1} = \frac{0 \times 0}{0+1} = 0$$

Since $\lim_{x \to 0-} f(x) = 0 = \lim_{x \to 0+} f(x)$

$\therefore$ $\lim_{x \to 0} f(x) = 0$. Also $f(0) = 0$

$\therefore$ f is continuous at $x = 0$.

(*v*) Here $f(0) = 0$

$$\lim_{x \to 0-} f(x) = \lim_{x \to 0-} \frac{1}{1-e^{1/x}} = \frac{1}{1-0} = 1$$

$$\left(\because \text{ as } x \to 0-, \frac{1}{x} \to -\infty \therefore e^{1/x} \to 0\right)$$

and $\lim_{x \to 0+} f(x) = \lim_{x \to 0+} \frac{1}{1-e^{1/x}} = \lim_{x \to 0+} \frac{e^{-1/x}}{e^{-1/x}-1}$

$$= \frac{0}{0-1} = 0 \quad \left(\because \text{ as } x \to 0+, \frac{1}{x} \to \infty \therefore e^{1/x} \to \infty \text{ and } e^{-1/x} \to 0\right)$$

Thus, $\lim_{x \to 0-} f(x)$ and $\lim_{x \to 0+} f(x)$ both exist but are not equal.

$\Rightarrow$ $\lim_{x \to 0} f(x)$ does not exist.

Since $\lim_{x \to 0-} f(x) \neq f(0) = \lim_{x \to 0+} f(x)$

Therefore, f is continuous from the right at $x = 0$ and has a discontinuity of the first kind from the left at $x = 0$.

(*vi*) Here $f(1) = 0$

$$\lim_{x \to 1-} f(x) = \lim_{x \to 1-} \frac{x-1}{1+e^{1/x-1}}$$

[Put $x = 1 - h, h > 0$, so that as $x \to 1-, h \to 0+$]

$$= \lim_{h \to 0+} \frac{-h}{1+e^{-1/h}} = -\frac{0}{1+0} = 0$$

$$\left(\because \text{ as } h \to 0+, \frac{1}{h} \to \infty \therefore e^{1/h} \to \infty \text{ and } e^{-1/h} \to 0\right)$$

and $\lim_{x \to 1+} f(x) = \lim_{x \to 1+} \frac{x-1}{1+e^{1/x-1}}$

[Put $x = 1 + h, h > 0$ so that as $x \to 1+, h \to 0+$]

$$= \lim_{h \to 0+} \frac{h}{1+e^{1/h}} = \lim_{h \to 0+} \frac{h\,e^{-1/h}}{e^{-1/h}+1} = \frac{0 \times 0}{0+1} = 0$$

Since $\lim_{x \to 0-} f(x) = \lim_{x \to 0+} f(x) = f(0)$

$\therefore$ f is continuous at $x = 0$.

Example 7. *Examine the continuity of the following functions at the indicated point. Also point out the type of discontinuity, if any.*

(*i*) $f(x) = \begin{cases} e^{1/x-2}, & \text{if } x \neq 2 \\ 0, & \text{if } x = 2 \end{cases}$ *at* $x = 2$ (*ii*) $f(x) = \begin{cases} e^{-\frac{1}{(x-2)^2}}, & \text{if } x \neq 2 \\ 0, & \text{if } x = 2 \end{cases}$ *at* $x = 2$.

Sol. (*i*) Here $f(2) = 0$

$\lim_{x \to 2-} f(x) = \lim_{x \to 2-} e^{1/x-2}$ [Put $x = 2 - h, h > 0$, so that as $x \to 2-, h \to 0+$]

$$= \lim_{h \to 0+} e^{-1/h} = 0$$

and $\lim_{x \to 2+} f(x) = \lim_{x \to 2+} e^{1/x-2}$ [Put $x = 2 + h, h > 0$, so that as $x \to 2+, h \to 0+$]

$$= \lim_{h \to +0} e^{1/h} = \infty \quad i.e., \quad \lim_{x \to 2+} f(x) \text{ does not exist.}$$

$\therefore$ f has a discontinuity of the second kind from the right at $x = 0$.

(*ii*) *Please try yourself.* [**Ans.** Continuous]

Example 8. *Examine the continuity of the following functions at the indicated point. Also point out the type of discontinuity, if any.*

(*i*) $f(x) = \begin{cases} \sin \frac{1}{x}, & \text{for } x \neq 0 \\ 0, & \text{for } x = 0 \end{cases}$ *at* $x = 0$ (*ii*) $f(x) = \begin{cases} \cos \frac{1}{x}, & \text{for } x \neq 0 \\ 0, & \text{for } x = 0 \end{cases}$ *at* $x = 0$.

Sol. (*i*) *We shall show that* $\lim_{x \to 0} f(x)$ *does not exist.*

Let, if possible, $\lim_{x \to 0} f(x)$ exist and be l.

Take $\varepsilon = \frac{1}{2}$, then $\exists$ a real number $\delta > 0$, such that $\left|\sin \frac{1}{x} - l\right| < \frac{1}{2}$ whenever $0 < |x| < \delta$

Now, let $0 < |x_1| < \delta$ and $0 < |x_2| \leq \delta$

then $\left|\sin \frac{1}{x_1} - l\right| < \frac{1}{2}$ and $\left|\sin \frac{1}{x_2} - l\right| < \frac{1}{2}$

$$\therefore \left|\sin \frac{1}{x_1} - \sin \frac{1}{x_2}\right| = \left|\left(\sin \frac{1}{x_1} - l\right) - \left(\sin \frac{1}{x_2} - l\right)\right|$$

$$\leq \left|\sin \frac{1}{x_1} - l\right| + \left|\sin \frac{1}{x_2} - l\right| < \frac{1}{2} + \frac{1}{2} = 1 \quad \ldots(1)$$

Taking $x_1 = \frac{1}{2n\pi + \frac{\pi}{2}}$ and $x_2 = \frac{1}{2n\pi - \frac{\pi}{2}}$ with n so large that $|x_1|$ and $|x_2|$ are both $< \delta$, we have, from (1)

$$\left|\sin\left(2n\pi+\frac{\pi}{2}\right)-\sin\left(2n\pi-\frac{\pi}{2}\right)\right|<1$$

or $\left|\sin\frac{\pi}{2}-\sin\left(-\frac{\pi}{2}\right)\right|<1$ or $|1-(-1)|<1$ or $2<1$ which is false.

$\Rightarrow$ Our supposition that $\lim_{x\to 0}\sin\frac{1}{x}$ exists leads to a contradiction.

$\Rightarrow$ Our supposition is wrong.

$\Rightarrow$ $\lim_{x\to 0} f(x)$ does not exist.

Hence f is discontinuous at $x=0$ and the discontinuity is of the second kind.

(ii) Please try yourself. **[Ans.** Discontinuity of the second kind.]

[**Hint.** Show that $\lim_{x\to 0} f(x)$ does not exist. *Take* $x_1=\frac{1}{2n\pi}$, $x_2=\frac{1}{(2n+1)\pi}$.]

Example 9. *Examine the continuity of the following functions at the indicated point. Also point out the type of discontinuity, if any.*

(i) $f(x)=\begin{cases}\frac{1-\cos x}{x^2}, & \text{if } x\neq 0\\ 1, & \text{if } x=0\end{cases}$ at $x=0$ *(ii)* $f(x)=\begin{cases}\frac{\sin 2x}{\sin 5x}, & \text{when } x\neq 0\\ 1, & \text{when } x=0\end{cases}$ at $x=0$

(iii) $f(x)=\begin{cases}\frac{x^4+x^3+2x^2}{\tan^{-1}x}, & \text{if } x\neq 0\\ 0, & \text{if } x=0\end{cases}$ at $x=0$ *(iv)* $f(x)=\begin{cases}\frac{\sin(x-a)}{x-a}, & \text{if } x\neq 0\\ 0, & \text{if } x=0\end{cases}$ at $x=a$.

Sol. *(i)* Here $f(0)=1$

$$\lim_{x\to 0} f(x)=\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{2\sin^2 x/2}{x^2}$$

$$=\lim_{x/2\to 0}\frac{1}{2}\cdot\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2=\frac{1}{2}\times 1^2=\frac{1}{2}$$

Since $\lim_{x\to 0} f(x)$ exists but is not equal to $f(0)$.

$\therefore$ f has removable discontinuity at $x=0$.

To remove discontinuity at $x=0$, the definition of f should be modified as under :

$$f(x)=\begin{cases}\frac{1-\cos x}{x^2}, & \text{if } x\neq 0\\ \frac{1}{2}, & \text{if } x=0\end{cases}$$

(ii) Please try yourself. **[Ans.** Removable discontinuity]

(iii) Here $f(0)=0$

$$\lim_{x\to 0} f(x)=\lim_{x\to 0}\frac{x^4+x^3+2x^2}{\tan^{-1}x}$$ [Put $\tan^{-1}x=\theta$ so that $x=\tan\theta$. As $x\to 0$, $\theta\to 0$]

$$=\lim_{\theta\to 0}\frac{\tan^4\theta+\tan^3\theta+2\tan^2\theta}{\theta}$$

$$=\lim_{\theta\to 0}\frac{\tan\theta}{\theta}(\tan^3\theta+\tan^2\theta+2\tan\theta)=1\times(0+0+0)=0$$

Since $\lim_{x\to 0} f(x)=0=f(0)$.

$\therefore$ f is continuous at $x=0$.

(iv) Here $f(a)=0$

$$\lim_{x\to a} f(x)=\lim_{x\to a}\frac{\sin(x-a)}{x-a}$$ [Put $x=a+h$, so that as $x\to a$, $h\to 0$]

$$=\lim_{h\to 0}\frac{\sin h}{h}=1$$

Since $\lim_{x\to a} f(x)$ exists but is not equal to $f(a)$.

$\therefore$ f has removable discontinuity at $x=a$.

To remove discontinuity at $x=a$, the definition of f should be modified as under :

$$f(x)=\begin{cases}\frac{\sin(x-a)}{x-a}, & \text{if } x\neq a\\ 1, & \text{if } x=a.\end{cases}$$

Example 10. *Discuss the continuity of the following functions at $x=0$. Specify the type of discontinuity, if any.*

(i) $f(x)=\begin{cases}2^{1/x}, & \text{if } x\neq 0\\ 0, & \text{if } x=0\end{cases}$ *(ii)* $f(x)=\begin{cases}x\sin\frac{1}{x}, & \text{if } x\neq 0\\ 0, & \text{if } x=0\end{cases}$

(iii) $f(x)=\begin{cases}x\cos\frac{1}{x}, & \text{if } x\neq 0\\ 0, & \text{if } x=0\end{cases}$ *(iv)* $f(x)=\begin{cases}x^2\sin\frac{1}{x}, & \text{if } x\neq 0\\ 0, & \text{if } x=0\end{cases}$

(v) $f(x)=\begin{cases}x^2\cos\frac{1}{x}, & \text{if } x\neq 0\\ 0, & \text{if } x=0\end{cases}$ *(vi)* $f(x)=\begin{cases}\cos x, & \text{if } x\geq 0\\ -\cos x, & \text{if } x<0\end{cases}$

Sol. *(i)* Here $f(0)=0$

$$\lim_{x\to 0-} f(x)=\lim_{x\to 0-} 2^{1/x}=0$$ $\left[\because \text{ as } x\to 0-, \frac{1}{x}\to-\infty \therefore 2^{1/x}\to 0\right]$

$\lim_{x\to 0+} f(x)=\lim_{x\to 0+} 2^{1/x}=\infty$ *i.e.,* $\lim_{x\to 0+} f(x)$ does not exist.

$\therefore$ f has a discontinuity of the second kind from the right at $x=0$.

(ii) Here $f(0)=0$

$$\lim_{x\to 0} f(x)=\lim_{x\to 0} x\sin\frac{1}{x}$$

$$=0$$ $\left[\because \lim_{x\to 0} x=0 \text{ and } \left|\sin\frac{1}{x}\right|\leq 1\right]$

$$=f(0)$$

$\therefore$ f is continuous at $x=0$.

(*iii*) *Please try yourself.* [Ans. Continuous]

(*iv*) *Please try yourself.* [Ans. Continuous]

(*v*) *Please try yourself.* [Ans. Continuous]

(*vi*) Here $f(0) = \cos 0 = 1$

$$\lim_{x \to 0-} f(x) = \lim_{x \to 0-} -\cos x \qquad [\text{Put } x = -h,\ h > 0, \text{ so that as } x \to 0-,\ h \to 0+]$$

$$= \lim_{h \to 0+} -\cos(-h) = \lim_{h \to 0+} -\cos h = -1$$

and
$$\lim_{x \to 0+} f(x) = \lim_{x \to 0+} \cos x = 1$$

Since $\lim_{x \to 0-} f(x) \neq f(0) = \lim_{x \to 0+} f(x)$

$\therefore$ f is right continuous at $x = 0$ and has a discontinuity of the first kind from the left at $x = 0$.

Example 11. (*a*) *Discuss the continuity of the following functions at $x = 0$. Specify the type of discontinuity, if any.*

(*i*) $f(x) = \begin{cases} \dfrac{x \sin \frac{1}{x}}{x} - 1, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ (*ii*) $f(x) = \begin{cases} \sin x \cos \dfrac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$

(*iii*) $f(x) = \begin{cases} x^m \sin \dfrac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ $m > 0$.

(*b*) *Let f be a function defined by* $f(x) = \begin{cases} x \sin \dfrac{1}{x} & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ 2x \sin \dfrac{1}{x} & \text{if } x < 0 \end{cases}$

Discuss continuity of f at $x = 0$.

Sol. (*a*) (*i*) Here $f(0) = 0$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0}\left(x \sin \frac{1}{x} - 1\right) = 0 - 1 = -1 \qquad \left[\because \lim_{x \to 0} x = 0 \text{ and } \left|\sin \frac{1}{x}\right| \leq 1\right]$$

Since $\lim_{x \to 0} f(x) \neq f(0)$, the function f has a removable discontinuity at $x = 0$.

(*ii*) *Please try yourself.* [Ans. Continuous]

(*iii*) *Please try yourself.* [Ans. Continuous]

[**Hint.** Since $m > 0$, $\lim_{x \to 0} x^m = 0$]

(*b*) *Please try yourself.* [Ans. Continuous]

Example 12. *Discuss the continuity of the following functions at $x = a$. Specify the type of discontinuity, if any.*

(*i*) $f(x) = \begin{cases} (x - a) \sin \dfrac{1}{x - a}, & \text{if } x \neq a \\ 0, & \text{if } x = a \end{cases}$ (*ii*) $f(x) = \begin{cases} (x - a) \cos \dfrac{1}{x - a}, & \text{if } x \neq a \\ 0, & \text{if } x = a \end{cases}$

(*iii*) $f(x) = \begin{cases} \dfrac{1}{x - a} \operatorname{cosec}(x - a), & \text{if } x \neq a \\ 0, & \text{if } x = a \end{cases}$

Sol. (*i*) Here $f(a) = 0$

$$\lim_{x \to a} f(x) = \lim_{x \to a} (x - a) \sin \frac{1}{x - a} \qquad [\text{Put } x = a + h, \text{ so that as } x \to a,\ h \to 0]$$

$$= \lim_{h \to 0} h \sin \frac{1}{h} = 0 \qquad \left[\because \lim_{h \to 0} h = 0 \text{ and } \left|\sin \frac{1}{h}\right| \leq 1\right]$$

Since $\lim_{x \to a} f(x) = 0 = f(a)$

$\therefore$ f is continuous at $x = a$.

(*ii*) *Please try yourself.*

(*iii*) Here $f(a) = 0$

$$\lim_{x \to a-} f(x) = \lim_{x \to a-} \frac{1}{x - a} \operatorname{cosec}(x - a)$$

$$[\text{Put } x = a - h,\ h > 0, \text{ so that as } x \to a-,\ h \to 0+]$$

$$= \lim_{h \to 0+} \frac{1}{-h} \operatorname{cosec}(-h)$$

$$= \lim_{x \to a+} \frac{-\operatorname{cosec} h}{-h} = \lim_{h \to 0+} \frac{1}{h} \times \frac{1}{\sin h} = \infty$$

i.e., $\lim_{x \to a-} f(x)$ does not exist (finitely).

Also $\lim_{x \to a+} f(x) = \lim_{x \to a+} \dfrac{1}{x - a} \operatorname{cosec}(x - a)$ [Put $x = a + h$, $h > 0$, so that as $x \to a+$, $h \to 0+$]

$$= \lim_{h \to 0+} \frac{1}{h} \operatorname{cosec} h = \lim_{h \to 0+} \frac{1}{h} \times \frac{1}{\sin h} = \infty \text{ i.e., } \lim_{x \to a+} f(x) \text{ also does not exist.}$$

Thus f has a discontinuity of the second kind at $x = a$.

Example 13. *Examine the discontinuity of the following functions at the indicated point. Also point out the type of discontinuity, if any.*

(*i*) $f(x) = \begin{cases} \dfrac{|x|}{x}, & \text{when } x \neq 0 \\ 1, & \text{when } x = 0 \end{cases}$ at $x = 0$

(*ii*) $f(x) = |x| + |x - 1|$ at $x = 0$ and $x = 1$

(*iii*) $f(x) = \begin{cases} \dfrac{x - |x|}{x}, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$ at $x = 0$ (*iv*) $f(x) = \begin{cases} \dfrac{|x - 2|}{x - 2}, & \text{if } x \neq 2 \\ -1, & \text{if } x = 2 \end{cases}$ at $x = 2$

(*v*) $f(x) = \begin{cases} \dfrac{x}{|x| + x^2}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ at $x = 0$ (*vi*) $f(x) = \begin{cases} \dfrac{|x|}{x^2 + 2x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ at $x = 0$

(*vii*) $f(x) = \begin{cases} \dfrac{2|x| + x^2}{x}, & \text{if } x \neq 0 \\ 2, & \text{if } x = 0 \end{cases}$ at $x = 0$ (*viii*) $f(x) = \begin{cases} \dfrac{x}{\sqrt{x^2}}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ at $x = 0$.

Sol. (*i*) Here $f(0) = 1$

$$\lim_{x \to 0-} f(x) = \lim_{x \to 0-} \frac{|x|}{x} = \lim_{x \to 0-} \frac{-x}{x} \qquad [\because \ x \to 0- \ \Rightarrow \ x < 0 \ \therefore \ |x| = -x]$$

$$= \lim_{x \to 0-} (-1) = -1$$

and
$$\lim_{x \to 0+} f(x) = \lim_{x \to 0+} \frac{|x|}{x} = \lim_{x \to 0+} \frac{x}{x} \qquad [\because \ x \to 0+ \Rightarrow x > 0 \ \therefore \ |x| = x]$$

$$= \lim_{x \to 0+} 1 = 1$$

Since $\lim_{x \to 0-} f(x) \neq f(0) = \lim_{x \to 0+} f(x)$

$\therefore$ f has a discontinuity of the first kind from the left at $x = 0$ and is right continuous at $x = 0$.

(*ii*) **Continuity at x = 0**

Here $f(0) = |0| + |0 - 1| = 0 + 1 = 1$

$$\lim_{x \to 0-} f(x) = \lim_{x \to 0-} |x| + |x - 1| \qquad | \text{ Put } x = 0 - h, h > 0$$

$$= \lim_{h \to 0+} |-h| + |-h-1| = \lim_{h \to 0+} h + (1 + h) = 1$$

$$\lim_{x \to 0+} f(x) = \lim_{x \to 0+} |x| + |x - 1| = \lim_{x \to 0+} x + (1 - x) = 1$$

$\therefore$ $\lim_{x \to 0} f(x) = 1 = f(0) \Rightarrow f$ is continuous at $x = 0$

Continuity at x = 1

Here $f(1) = |1| + |1 - 1| = 1 + 0 = 1$

$$\lim_{x \to 1-} f(x) = \lim_{x \to 1-} |x| + |x - 1| \qquad | \text{ Put } x = 1 - h, h > 0$$

$$= \lim_{h \to 0+} |1 - h| + |-h| = \lim_{h \to 0+} (1 - h) + h = 1$$

$$\lim_{x \to 1+} f(x) = \lim_{x \to 1+} |x| + |x - 1| \qquad | \text{ Put } x = 1 + h, h > 0$$

$$= \lim_{h \to 0+} |1 + h| + |h| = \lim_{h \to 0+} (1 + h) + h = 1$$

$\therefore$ $\lim_{x \to 1} f(x) = 1 = f(1)$

$\Rightarrow$ f is continuous at $x = 1$.

(*iii*) Here $f(0) = 1$

$$\lim_{x \to 0-} f(x) = \lim_{x \to 0-} \frac{x - |x|}{x} = \lim_{x \to 0-} \frac{x - (-x)}{x} = \lim_{x \to 0-} 2 = 2$$

$$\lim_{x \to 0+} f(x) = \lim_{x \to 0+} \frac{x - |x|}{x} = \lim_{x \to 0+} \frac{x - x}{x} = \lim_{x \to 0+} 0 = 0$$

Since $\lim_{x \to 0-} f(x)$ and $\lim_{x \to 0+} f(x)$ both exist but are unequal, f has a discontinuity of the first kind at $x = 0$.

(*iv*) Here $f(2) = -1$

$$\lim_{x \to 2-} f(x) = \lim_{x \to 2-} \frac{|x - 2|}{x - 2} \qquad | \text{ Put } x = 2 - h, h > 0$$

$$= \lim_{h \to 0+} \frac{|-h|}{-h} = \lim_{h \to 0+} \frac{h}{-h} = \lim_{h \to 0+} (-1) = -1$$

$$\lim_{x \to 2+} f(x) = \lim_{x \to 2+} \frac{|x - 2|}{x - 2} \qquad | \text{ Put } x = 2 + h, h > 0$$

$$= \lim_{h \to 0+} \frac{|h|}{h} = \lim_{h \to 0+} \frac{h}{h} = \lim_{h \to 0+} 1 = 1$$

Since $\lim_{x \to 2-} f(x) = f(2) \neq \lim_{x \to 2+} f(x)$

$\therefore$ f is left continuous at $x = 2$ and has a discontinuity of the first kind from the right at $x = 2$.

(*v*) *Please try yourself.* [**Ans.** Discontinuity of the first kind]

(*vi*) *Please try yourself.* [**Ans.** Discontinuity of the first kind]

(*vii*) *Please try yourself.* [**Ans.** Right continuous at $x = 0$ and discontinuity of the first kind from the left at $x = 0$]

(*viii*) Here $f(0) = 0$

Also $\sqrt{x^2} = |x|$

$$\therefore \quad \lim_{x \to 0-} f(x) = \lim_{x \to 0-} \frac{x}{|x|} = \lim_{x \to 0-} \frac{x}{-x} = \lim_{x \to 0-} (-1) = -1$$

$$\lim_{x \to 0+} f(x) = \lim_{x \to 0+} \frac{x}{|x|} = \lim_{x \to 0+} \frac{x}{x} = \lim_{x \to 0+} 1 = 1$$

Since $\lim_{x \to 0-} f(x)$ and $\lim_{x \to 0+} f(x)$ both exist but are unequal, f has a discontinuity of the first kind at $x = 0$.

Example 14. *Discuss the continuity of the function $f(x) = [x]$ at the points $\frac{1}{2}$ and 1, where $[x]$ denotes the largest integer $\leq x$.*

Sol. Continuity at $x = \frac{1}{2}$

$$f\left(\frac{1}{2}\right) = \left[\frac{1}{2}\right] = 0$$

$$\lim_{x \to \frac{1}{2}-} f(x) = \lim_{x \to \frac{1}{2}-} [x] \qquad | \text{ Put } x = \frac{1}{2} - h, h > 0$$

$$= \lim_{h \to 0+} \left[\frac{1}{2} - h\right] = \lim_{h \to 0+} 0 = 0 \qquad | \because \ 0 < \frac{1}{2} - h < 1$$

$$\lim_{x \to \frac{1}{2}+} f(x) = \lim_{x \to \frac{1}{2}+} [x] \qquad | \text{ Put } x = \frac{1}{2} + h, h > 0$$

$$= \lim_{h \to 0+} \left[\frac{1}{2} + h\right] = \lim_{h \to 0+} 0 = 0 \qquad | \because \ 0 < \frac{1}{2} + h < 1$$

Since $\lim\limits_{x \to \frac{1}{2}} f(x) = 0 = f\left(\frac{1}{2}\right)$

$\therefore$ f is continuous at $x = \frac{1}{2}$.

Continuity at x = 1

$$f(1) = [1] = 1$$

$$\lim_{x \to 1-} f(x) = \lim_{x \to 1-} [x] \quad | \text{ Put } x = 1 - h, h > 0$$

$$= \lim_{h \to 0+} [1 - h] = \lim_{h \to 0+} 0 = 0 \quad | \because 0 < 1 - h < 1$$

$$\lim_{x \to 1+} f(x) = \lim_{x \to 1+} [x] \quad | \text{ Put } x = 1 + h, h > 0$$

$$= \lim_{h \to 0+} [1 + h] = \lim_{h \to 0+} 1 = 1 \quad | \because 1 < 1 + h < 2$$

Since $\lim\limits_{x \to 1-} f(x) \neq f(1) = \lim\limits_{x \to 1+} f(x)$

$\therefore$ f is right continuous at $x = 1$ and has a discontinuity of the first kind from the left at $x = 1$.

Example 15. *Discuss the continuity of f at $x = 1$, where $f(x) = [1 - x] + [x - 1]$.*

Sol. Here $f(1) = [1 - 1] + [1 - 1] = [0] + [0] = 0 + 0 = 0$

$$\lim_{x \to 1-} f(x) = \lim_{x \to 1-} [1 - x] + [x - 1] \quad | \text{ Put } x = 1 - h, h > 0$$

$$= \lim_{h \to 0+} [h] + [-h] = \lim_{h \to 0+} (0) + (-1) = -1 \quad \Big| \because 0 < h < 1 \text{ and } -1 < -h < 0$$

$$\lim_{x \to 1+} f(x) = \lim_{x \to 1+} [1 - x] + [x - 1] \quad | \text{ Put } x = 1 + h, h > 0$$

$$= \lim_{h \to 0+} [-h] + [h] = \lim_{h \to 0+} (-1) + (0) = -1$$

$\therefore$ $\lim\limits_{x \to 1} f(x) = -1 \neq f(1)$

$\Rightarrow$ f has a removable discontinuity at $x = 1$. Discontinuity at $x = 1$ can be removed if f is defined as under :

$$f(x) = \begin{cases} [1 - x] + [x - 1], & \text{if } x \neq 1 \\ -1, & \text{if } x = 1 \end{cases}$$

Example 16. *Show that the function f defined by*

$$f(x) = \begin{cases} [x - 1] + |x - 1|, & \text{if } x \neq 1 \\ 0, & \text{if } x = 1 \end{cases} \text{ is discontinuous at } x = 1.$$

Sol. *Please try yourself.*

Example 17. *Examine the continuity of f at $x = 2$, where* $f(x) = \begin{cases} x - [x], & \text{if } x < 2 \\ 1, & \text{if } x = 2 \\ 3x - 5, & \text{if } x > 2. \end{cases}$

Sol. Here $f(2) = 1$

$$\lim_{x \to 2-} f(x) = \lim_{x \to 2-} x - [x] \quad | \text{ Put } x = 2 - h, h > 0$$

$$= \lim_{h \to 0+} 2 - h - [2 - h] = \lim_{h \to 0+} 2 - h - 1 = 1 \quad | \because 1 < 2 - h < 2$$

$$\lim_{x \to 2+} f(x) = \lim_{x \to 2+} (3x - 5) \quad | \text{ Put } x = 2 + h, h > 0$$

$$= \lim_{h \to 0+} 3(2 + h) - 5 = \lim_{h \to 0+} (1 + 3h) = 1$$

Since $\lim\limits_{x \to 2} f(x) = 1 = f(2)$

$\therefore$ f is continuous at $x = 2$.

Example 18. *Show that the function $[x]$, where $[x]$ denotes the largest integer $\leq x$, is discontinuous at each integral value and the discontinuities are of first kind. Also show that the function is continuous at each non-integral value.*

Sol. Here $f(x) = [x]$

Let 'a' be any integer, then $f(a) = [a] = a$

$$\lim_{x \to a-} f(x) = \lim_{x \to a-} [x] \quad | \text{ Put } x = a - h, h > 0$$

$$= \lim_{h \to 0+} [a - h] = \lim_{h \to 0+} (a - 1) \quad | \because a - 1 < a - h < a$$

$$= a - 1$$

$$\lim_{x \to a+} f(x) = \lim_{x \to a+} [x] \quad | \text{ Put } x = a + h, h > 0$$

$$= \lim_{h \to 0+} [a + h] = \lim_{h \to 0+} a \quad | \because a < a + h < a + 1$$

$$= a$$

Thus $\lim\limits_{x \to a-} f(x)$ and $\lim\limits_{x \to a+} f(x)$ both exist but are unequal.

$\therefore$ f has a discontinuity of the first kind from the left at $x = a$.

Since 'a' is any integer, we conclude that f is discontinuous at each integral value and the discontinuities are of the first king from the left.

Now, let $a \in R - Z$.

If n is the greatest integer less than a, then $[a] = n$, where $n < a < n + 1$

Now $f(a) = [a] = n$

$$\lim_{x \to a-} f(x) = \lim_{x \to a-} [x] \quad | \text{ Put } x = a - h, h > 0$$

$$= \lim_{h \to 0+} [a - h] = \lim_{h \to 0+} n = n \quad | \because n < a - h < n + 1$$

$$\lim_{x \to a+} f(x) = \lim_{x \to a+} [x] \quad | \text{ Put } x = a + h, h > 0$$

$$= \lim_{h \to 0+} [a + h] = \lim_{h \to 0+} n = n \quad | \because n < a + h < n + 1$$

Thus $\lim\limits_{x \to a} f(x) = n = f(a)$

$\therefore$ f is continuous at $x = a$.

Since $a \in R - Z$ is arbitrary, we conclude that f is continuous at each non-integral value.

Example 19. *Show that the function $f(x) = x - [x]$ is discontinuous at every integral value of x and all discontinuities are of the first kind from left.*

Sol. Here $f(x) = x - [x]$

Let 'a' be any integer, then $f(a) = a - [a] = a - a = 0$

$$\lim_{x \to a-} f(x) = \lim_{x \to a-} x - [x] \qquad | \text{ Put } x = a - h, h > 0$$

$$= \lim_{h \to 0+} a - h - [a - h] = \lim_{h \to 0+} a - h - (a - 1) \qquad | \because a - 1 < a - h < a$$

$$= \lim_{h \to 0+} 1 - h = 1$$

$$\lim_{x \to a+} f(x) = \lim_{x \to a+} x - [x] \qquad | \text{ Put } x = a + h, h > 0$$

$$= \lim_{h \to 0+} a + h - [a + h] = \lim_{h \to 0+} a + h - a \qquad | \because a < a + h < a + 1$$

$$= \lim_{h \to 0+} h = 0$$

Thus $\lim_{x \to a-} f(x) \neq f(a) = \lim_{x \to a+} f(x)$

$\therefore$ f is right continuous at $x = a$ and has a discontinuity of the first kind from the left at $x = a$.

Since 'a' is any integer, we conclude that f is discontinuous at each integral value and the discontinuities are of the first kind from the left.

Example 20. (*i*) *Show that* $f(x) = \frac{1}{x - a}$ *has a discontinuity of the second kind at* $x = a$.

(*ii*) *Show that the reciprocal function* $\frac{1}{x}$ *has a discontinuity of the second kind at* $x = 0$.

Sol. (*i*) Here $f(x) = \frac{1}{x - a}$ which is not defined at $x = a$.

$$\lim_{x \to a-} f(x) = \lim_{x \to a-} \frac{1}{x - a} \qquad | \text{ Put } x = a - h, h > 0$$

$$= \lim_{h \to 0+} \frac{1}{-h} = -\infty \text{ i.e., } \lim_{x \to a-} f(x) \text{ does not exist.}$$

Also $\lim_{x \to a+} f(x) = \lim_{x \to a+} \frac{1}{x - a} \qquad | \text{ Put } x = a + h > 0$

$$= \lim_{h \to 0+} \frac{1}{h} = \infty \text{ i.e., } \lim_{x \to a+} f(x) \text{ does not exist.}$$

Since neither $\lim_{x \to a-} f(x)$ nor $\lim_{x \to a+} f(x)$ exists, therefore, f has a discontinuity of the second kind at $x = a$.

(*ii*) *Please try yourself.*

Example 21. *Prove that* $f(x) = \frac{\sin^2 ax}{x^2}$ *for* $x \neq 0$ *and* $f(0) = 1$ *is discontinuous at* $x = 0$ *unless* $a = \pm 1$.

Sol. Here $f(0) = 1$.

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin^2 ax}{x^2} = \lim_{x \to 0} a^2 \left(\frac{\sin ax}{ax}\right)^2 = a^2 \times 1^2 = a^2$$

If $a^2 = 1$, *i.e.*, $a = \pm 1$, then $\lim_{x \to 0} f(x) = 1 = f(0)$ so that f is continuous at $x = 0$.

If $a^2 \neq 1$, then $\lim_{x \to 0} f(x) = a^2 \neq f(0)$ so that f is discontinuous at $x = 0$.

Hence f is discontinuous at $x = 0$ unless $a = \pm 1$.

Example 22. *Is the function* $f(x) = \frac{3x + 4 \tan x}{x}$ *continuous at* $x = 0$. *If not, how may the function be defined to make it continuous at this point.*

Sol. Here $f(x) = \frac{3x + 4 \tan x}{x}$ which is not defined at $x = 0$ $\left[\because f(0) = \frac{0}{0} \text{ is indeterminate}\right]$

However, $\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{3x + 4 \tan x}{x} = \lim_{x \to 0} 3 + 4\left(\frac{\tan x}{x}\right) = 3 + 4 \times 1 = 7$ exists.

$\therefore$ f has a removable discontinuity at $x = 0$.

Discontinuity at $x = 0$ can be removed by defining f as under :

$$f(x) = \begin{cases} \frac{3x + 4 \tan x}{x}, & \text{if } x \neq 0 \\ 7, & \text{if } x = 0 \end{cases}$$

Example 23. *A function f is defined as follows :*

$$f(x) = \begin{cases} \frac{x^2 - x - 6}{x^2 - 2x - 3}, & \text{when } x \neq 3 \\ \frac{5}{3}, & \text{when } x = 3 \end{cases}$$

Show that f is discontinuous at $x = 3$. *Can the definition of f for* $x = 3$ *be modified so as to make it continuous there ?*

Sol. Here $f(3) = \frac{5}{3}$

$$\lim_{x \to 3} f(x) = \lim_{x \to 3} \frac{x^2 - x - 6}{x^2 - 2x - 3}$$

$$= \lim_{x \to 3} \frac{(x - 3)(x + 2)}{(x - 3)(x + 1)} = \lim_{x \to 3} \frac{x + 2}{x + 1} = \frac{3 + 2}{3 + 1} = \frac{5}{4} \neq f(3)$$

$\Rightarrow$ f is discontinuous at $x = 3$

Since $\lim_{x \to 3} f(x)$ exists, f has a removable discontinuity at $x = 3$.

Discontinuity at $x = 3$ can be removed and f can be made continuous at $x = 3$ if the definition of f for $x = 3$ is modified as under :

$$f(x) = \begin{cases} \frac{x^2 - x - 6}{x^2 - 2x - 3}, & \text{when } x \neq 3 \\ \frac{5}{4}, & \text{when } x = 3. \end{cases}$$

Example 24. *Let $f : R \to R$ be such that* $f(x) = \begin{cases} \dfrac{\sin(a+1)x + \sin x}{x} & \text{for } x < 0 \\ c & \text{for } x = 0 \\ \dfrac{(x+bx^2)^{1/2} - x^{1/2}}{bx^{3/2}} & \text{for } x > 0 \end{cases}$

Determine the values of a, b, c for which the function is continuous at $x = 0$.

Sol. Here $f(0) = c$

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0-} \frac{\sin(a+1)x + \sin x}{x} \qquad \Big|\ \text{Put } x = 0-h,\ h>0$$

$$= \lim_{h\to 0+} \frac{\sin(a+1)(-h) + \sin(-h)}{-h} = \lim_{h\to 0+} \frac{-\sin(a+1)h - \sin h}{-h}$$

$$= \lim_{h\to 0+} \frac{\sin(a+1)h + \sin h}{h} = \lim_{h\to 0+} \frac{2\sin\left(\frac{a}{2}+1\right)h\cos\frac{ah}{2}}{-h}$$

$$= \lim_{h\to 0+} 2\left(\frac{a}{2}+1\right)\cdot\frac{\sin\left(\frac{a}{2}+1\right)h}{\left(\frac{a}{2}+1\right)h}\cdot\cos\frac{ah}{2} = 2\left(\frac{a}{2}+1\right)\times 1\times 1 = a+2$$

Also $$\lim_{x\to 0+} f(x) = \lim_{x\to 0+} \frac{(x+bx^2)^{1/2} - x^{1/2}}{bx^{3/2}}$$

$$= \lim_{x\to 0+} \frac{x^{1/2}(1+bx)^{1/2} - x^{1/2}}{bx^{3/2}} = \lim_{x\to 0+} \frac{(1+bx)^{1/2} - 1}{bx}$$

$$= \lim_{x\to 0+} \frac{\sqrt{1+bx}-1}{bx} \times \frac{\sqrt{1+bx}+1}{\sqrt{1+bx}+1} = \lim_{x\to 0+} \frac{(1+bx)-1}{bx(\sqrt{1+bx}+1)}$$

$$= \lim_{x\to 0+} \frac{bx}{bx(\sqrt{1+bx}+1)} = \lim_{x\to 0+} \frac{1}{\sqrt{1+bx}+1} \qquad (\because\ b \neq 0)$$

$$= \frac{1}{\sqrt{1+0}+1} = \frac{1}{2}$$

This is independent of b so that b can have any non-zero real value.

Since f is continuous at $x = 0$, we have

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0+} f(x) = f(0)$$

$$\Rightarrow \qquad a + 2 = \frac{1}{2} = c \ \Rightarrow\ a = -\frac{3}{2}, \text{ and } c = \frac{1}{2}$$

Hence $$a = -\frac{3}{2},\ b \neq 0,\ c = \frac{1}{2}.$$

Example 25. *Prove that the function* $f(x) = \begin{cases} \dfrac{x^2}{a} - a, & \text{for } 0 < x < a \\ 0, & \text{for } x = a \\ a - \dfrac{a^3}{x^2}, & \text{for } x > a \end{cases}$ *is continuous at $x = a$.*

Sol. Here $f(a) = 0$

$$\lim_{x\to 0-} f(a) = \lim_{x\to 0-} \left(\frac{x^2}{a} - a\right) \qquad \Big|\ \text{Put } x = a-h,\ h>0$$

$$= \lim_{h\to 0+} \left[\frac{(a-h)^2}{a} - a\right] = \lim_{h\to 0+} \left[\frac{a^2 - 2ah + h^2 - a^2}{a}\right] = \lim_{h\to 0+} \frac{h^2 - 2ah}{a} = 0$$

and $$\lim_{x\to a+} f(x) = \lim_{x\to a+} \left(a - \frac{a^3}{x^2}\right) \qquad \Big|\ \text{Put } x = a+h,\ h>0$$

$$= \lim_{h\to 0+} \left[a - \frac{a^3}{(a+h)^2}\right] = \lim_{h\to 0+} \left[\frac{a(a^2+2ah+h^2) - a^3}{(a+h)^2}\right] = \lim_{h\to 0+} \frac{2a^2h + ah^2}{(a+h)^2} = 0$$

Since $\lim_{x\to a-} f(x) = \lim_{x\to a+} f(x) = f(a)$

$\therefore$ f is continuous at $x = a$.

Example 26. *A function f is defined as follows :*

$$f(x) = \begin{cases} 1+x, & \text{if } x \le 2 \\ 5-x, & \text{if } x > 2 \end{cases}$$

Is f continuous at $x = 2$?

Sol. *Please try yourself.* **[Ans. Yes]**

Example 27. *Examine the continuity of the function*

$$f(x) = \begin{cases} -x^2 & \text{for } x \le 0 \\ 5x - 4 & \text{for } 0 < x \le 1 \\ 4x^2 - 3x & \text{for } 1 < x < 2 \\ 3x + 4 & \text{for } x \ge 2 \end{cases} \text{ at } x = 0, 1 \text{ and } 2.$$

Sol. At x = 0, $\quad f(0) = -0^2 = 0$

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0-} -x^2 \qquad \Big|\ \text{Put } x = 0-h,\ h>0$$

$$= \lim_{h\to 0+} -(-h)^2 = \lim_{h\to 0+} -h^2 = 0$$

$$\lim_{x\to 0+} f(x) = \lim_{x\to 0+} (5x-4) \qquad \Big|\ \text{Put } x = 0+h,\ h>0$$

$$= \lim_{h\to 0+} (5h-4) = -4$$

Since $\lim_{x\to 0-} f(x) = f(0) \neq \lim_{x\to 0+} f(x)$

$\therefore$ f is left continuous at $x = 0$ and has a discontinuity of the first kind from the right at $x = 0$.

At x = 1, $f(1) = 5 \times 1 - 4 = 1$

$$\lim_{x \to 1-} f(x) = \lim_{x \to 1-} (5x - 4) \qquad | \text{ Put } x = 1 - h, h > 0$$

$$= \lim_{h \to 0+} 5(1-h) - 4 = \lim_{h \to 0+} (1 - 5h) = 1$$

$$\lim_{x \to 1+} f(x) = \lim_{x \to 1+} (4x^2 - 3x) \qquad | \text{ Put } x = 1 + h, h > 0$$

$$= \lim_{h \to 0+} [4(1+h)^2 - 3(1+h)] = 4 - 3 = 1$$

Since $\lim_{x \to 1-} f(x) = f(1) = \lim_{x \to 1+} f(x)$,

$\therefore$ f is continuous at $x = 1$.

At x = 2, $f(2) = 3 \times 2 + 4 = 10$

$$\lim_{x \to 2-} f(x) = \lim_{x \to 2-} (4x^2 - 3x) \qquad | \text{ Put } x = 2 - h, h > 0$$

$$= \lim_{h \to 0+} [4(2-h)^2 - 3(2-h)] = 4 \times 4 - 3 \times 2 = 10$$

$$\lim_{x \to 2+} f(x) = \lim_{x \to 2+} (3x + 4) \qquad | \text{ Put } x = 2 + h, h > 0$$

$$= \lim_{h \to 0+} [3(2+h) + 4] = 3 \times 2 + 4 = 10$$

Since $\lim_{x \to 2-} f(x) = f(2) = \lim_{x \to 2+} f(x)$

$\therefore$ f is continuous at $x = 2$.

Example 28. *Discuss the continuity at $x = 1, 2, 3$ of the following function :*

$$f(x) = \begin{cases} x + 4 & \text{for } x \le 1 \\ x^2 + 3x + 1 & \text{for } 1 < x \le 2 \\ 2x^2 + x + 1 & \text{for } 2 < x < 3 \\ 8x - 2 & \text{for } x \ge 3. \end{cases}$$

Sol. *Please try yourself.* [**Ans.** Continuous at each]

Example 29. *Obtain the points of discontinuity of the function defined on [0, 1] as follows :*

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{2} - x & \text{if } 0 < x < \frac{1}{2} \\ \frac{1}{2} & \text{if } x = \frac{1}{2} \\ \frac{3}{2} - x & \text{if } \frac{1}{2} < x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

Also, examine the kinds of discontinuities.

Sol. Since f is a polynomial on $\left(0, \frac{1}{2}\right)$ and $\left(\frac{1}{2}, 1\right)$, therefore, f is continuous on $\left(0, \frac{1}{2}\right)$ and $\left(\frac{1}{2}, 1\right)$.

Right continuity at x = 0.

$f(0) = 0$

$$\lim_{x \to 0+} f(x) = \lim_{x \to 0+} \left(\frac{1}{2} - x\right) \qquad | \text{ Put } x = 0 + h, h > 0$$

$$= \lim_{h \to 0+} \left[\frac{1}{2} - (0 + h)\right] = \lim_{h \to 0+} \left(\frac{1}{2} - h\right) = \frac{1}{2}$$

Since $\lim_{x \to 0+} f(x)$ exists but $\lim_{x \to 0+} f(x) \ne f(0)$, therefore, f has removable discontinuity at $x = 0$.

Continuity at $x = \frac{1}{2}$

$$f\left(\frac{1}{2}\right) = \frac{1}{2}$$

$$\lim_{x \to \frac{1}{2}-} f(x) = \lim_{x \to \frac{1}{2}-} \left(\frac{1}{2} - x\right) \qquad | \text{ Put } x = \frac{1}{2} - h, h > 0$$

$$= \lim_{h \to 0+} \left[\frac{1}{2} - \left(\frac{1}{2} - h\right)\right] = \lim_{h \to 0+} h = 0$$

$$\lim_{x \to \frac{1}{2}+} f(x) = \lim_{x \to \frac{1}{2}+} \left(\frac{3}{2} - x\right) \qquad | \text{ Put } x = \frac{1}{2} + h, h > 0$$

$$= \lim_{h \to 0+} \left[\frac{3}{2} - \left(\frac{1}{2} + h\right)\right] = \lim_{h \to 0+} (1 - h) = 1$$

Since $\lim_{x \to \frac{1}{2}-} f(x)$ and $\lim_{x \to \frac{1}{2}+} f(x)$ both exist but none is equal to $f\left(\frac{1}{2}\right)$, therefore, f has a discontinuity of the first kind at $x = \frac{1}{2}$ on both sides.

Left continuity at x = 1

$f(x) = 1$

$$\lim_{x \to 1-} f(x) = \lim_{x \to 1-} \left(\frac{3}{2} - x\right) \qquad | \text{ Put } x = 1 - h, h > 0$$

$$= \lim_{h \to 0+} \left[\frac{3}{2} - (1 - h)\right] = \lim_{h \to 0+} \left(\frac{1}{2} + h\right) = \frac{1}{2}$$

Since $\lim_{x \to 1-} f(x)$ exists but $\lim_{x \to 1-} f(x) \ne f(1)$, therefore, f has removable discontinuity at $x = 1$.

Example 30. *Let f be a function on [0, 1] defined by*

$$f(x) = \begin{cases} (-1)^r & \text{if } \frac{1}{r+1} \le x < \frac{1}{r}, r = 1, 2, 3, \ldots \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x = 1 \end{cases}$$

Examine the continuity of f at $1, \frac{1}{2}, \frac{1}{3}, \ldots$

Sol. Continuity at x = 1

$$f(1) = 1 \quad \text{and} \quad f(x) = -1 \text{ if } \frac{1}{2} \le x < 1$$

$$\lim_{x \to 1-} f(x) = \lim_{x \to 1-} (-1) = -1 \ne f(1)$$

$\Rightarrow$ f is not left continuous at $x = 1$.

$\Rightarrow$ f is not continuous at $x = 1$.

Continuity at $\mathbf{x = \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots}$

When $x = \frac{1}{2n}$, $n \in \mathbb{N}$

$$f(x) = \begin{cases} 1 & \text{if } \frac{1}{2n+1} \le x < \frac{1}{2n} \\ -1 & \text{if } \frac{1}{2n} \le x < \frac{1}{2n-1} \end{cases}$$

$\therefore$ $$f\left(\frac{1}{2n}\right) = -1$$

$$\lim_{x \to \frac{1}{2n}-} f(x) = \lim_{x \to \frac{1}{2n}-} (1) = 1$$

$$\lim_{x \to \frac{1}{2n}+} f(x) = \lim_{x \to \frac{1}{2n}+} (-1) = -1$$

Since $$\lim_{x \to \frac{1}{2n}-} f(x) \ne \lim_{x \to \frac{1}{2n}+} f(x)$$

$\therefore$ $\lim_{x \to \frac{1}{2n}} f(x)$ does not exist. $\Rightarrow$ f is discontinuous at $x = \frac{1}{2n}$.

Continuity at $\mathbf{x = \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots}$

When $x = \frac{1}{2n+1}$, $n \in \mathbb{N}$

$$f(x) = \begin{cases} -1 & \text{if } \frac{1}{2n+2} \le x < \frac{1}{2n+1} \\ 1 & \text{if } \frac{1}{2n+1} \le x < \frac{1}{2n} \end{cases}$$

$\therefore$ $$f\left(\frac{1}{2n+1}\right) = 1$$

$$\lim_{x \to \frac{1}{2n+1}-} f(x) = \lim_{x \to \frac{1}{2n+1}-} (-1) = -1$$

$$\lim_{x \to \frac{1}{2n+1}+} f(x) = \lim_{x \to \frac{1}{2n+1}+} (1) = 1$$

Since $$\lim_{x \to \frac{1}{2n+1}-} f(x) \ne \lim_{x \to \frac{1}{2n+1}+} f(x)$$

$\therefore$ $\lim_{x \to \frac{1}{2n+1}} f(x)$ does not exist. $\Rightarrow$ f is discontinuous at $x = \frac{1}{2n+1}$.

Hence f is discontinuous at $1, \frac{1}{2}, \frac{1}{3}, \ldots$

Example 31. *Discuss the continuity of the function* $f(x) = \frac{x^3 - 7x^2 + 3x - 1}{x^2 - 3x}$ *for* $0 < x < 3$.

Sol. Let c be any point of the open interval $(0, 3)$.

$$f(c) = \frac{c^3 - 7c^2 + 3c - 1}{c^2 - 3c}$$

$$\lim_{x \to c-} f(x) = \lim_{x \to c-} \frac{x^3 - 7x^2 + 3x - 1}{x^2 - 3x} \qquad | \text{ Put } x = c - h, h > 0$$

$$= \lim_{h \to 0+} \frac{(c-h)^3 - 7(c-h)^2 + 3(c-h) - 1}{(c-h)^2 - 3(c-h)} = \frac{c^3 - 7c^2 + 3c - 1}{c^2 - 3c}$$

$$\lim_{x \to c+} f(x) = \lim_{x \to c+} \frac{x^3 - 7x^2 + 3x - 1}{x^2 - 3x} \qquad | \text{ Put } x = c + h, h > 0$$

$$= \lim_{h \to 0+} \frac{(c+h)^3 - 7(c+h)^2 + 3(c+h) - 1}{(c+h)^2 - 3(c+h)} = \frac{c^3 - 7c^2 + 3c - 1}{c^2 - 3c}$$

Since $\lim_{x \to c-} f(x) = f(c) = \lim_{x \to c+} f(x)$, therefore, f is continuous at $x = c$.

But c is any point of $(0, 3)$.

$\therefore$ f is continuous at every point of $(0, 3)$, *i.e.*, f is continuous on $(0, 3)$.

Note. A rational function is a function of the form $R(x) = \dfrac{P(x)}{Q(x)}$, where P(x) and Q(x) are polynomials and $Q(x) \neq 0$.

The set of points of discontinuity of $R(x) = \{x : Q(x) = 0, x \text{ is real}\}$

In example 31, $f(x)$ is a rational function with points of discontinuity at $x = 0, 3$. None of the points of discontinuity is in (0, 3). Thus f is continuous everywhere in (0, 3).

Example 32. *Determine the constants a and b so that the function f defined below is continuous everywhere :*

$$(i)\ f(x) = \begin{cases} 2x+1 & \text{for } x \le 1 \\ ax^2+b & \text{for } 1 < x < 3 \\ 5x+2a & \text{for } x \ge 3 \end{cases} \qquad (ii)\ f(x) = \begin{cases} 1 & \text{when } x \le 3 \\ ax+b & \text{when } 3 < x < 5 \\ 7 & \text{when } x \ge 5 \end{cases}$$

$$(iii)\ f(x) = \begin{cases} 3 & \text{if } x \le 2 \\ ax^2+bx+1 & \text{if } 2 < x < 3 \\ 7-ax & \text{if } x \ge 3 \end{cases}$$

Sol. (*i*) Since f is continuous everywhere, it is also continuous at $x = 1$ and $x = 3$, the breaking points of the domain.

Continuity at x = 1

$$f(1) = 2 \times 1 + 1 = 3$$

$$\lim_{x \to 1-} f(x) = \lim_{x \to 1-} (2x+1) \qquad | \text{ Put } x = 1-h, h > 0$$

$$= \lim_{h \to 0+} [2(1-h)+1] = 3$$

$$\lim_{x \to 1+} f(x) = \lim_{x \to 1+} (ax^2+b) \qquad | \text{ Put } x = 1+h, h > 0$$

$$= \lim_{h \to 0+} [a(1+h)^2+b] = a+b$$

Since f is continuous at $x = 1$

$$\therefore \quad \lim_{x \to 1-} f(x) = \lim_{x \to 1+} f(x) = f(1)$$

$$\Rightarrow \quad 3 = a+b = 3 \quad \therefore \quad a+b = 3 \qquad \ldots(1)$$

Continuity at x = 3

$$f(3) = 5 \times 3 + 2a = 2a + 15$$

$$\lim_{x \to 3-} f(x) = \lim_{x \to 3-} (ax^2+b) \qquad | \text{ Put } x = 3-h, h > 0$$

$$= \lim_{h \to 0+} [a(3-h)^2+b] = 9a+b$$

$$\lim_{x \to 3+} f(x) = \lim_{x \to 3+} (5x+2a) \qquad | \text{ Put } x = 3+h, h > 0$$

$$= \lim_{h \to 0+} [5(3+h)+2a] = 2a+15$$

Since f is continuous at $x = 3$

$$\therefore \quad \lim_{x \to 3-} f(x) = \lim_{x \to 3+} f(x) = f(3)$$

$$\Rightarrow \quad 9a+b = 2a+15 = 2a+15$$

$$\therefore \quad 7a+b = 15 \qquad \ldots(2)$$

From (1) and (2), $\quad a = 2, b = 1$.

(*ii*) *Please try yourself.* **[Ans.** $a = 3, b = -8$**]**

(*iii*) *Please try yourself.* **[Ans.** $a = \frac{1}{2}, b = 0$**]**

6.9. THEOREM

A function $f : S \to R$ is continuous at a point 'a' $\in$ S if and only if for every sequence $< x_n >$ in S which converges to 'a', the sequence $< f(x_n) >$ converges to f(a). **(Heine's Definition)**

Proof. (*i*) Let f be continuous at 'a' $\in$ S and let $< x_n >$ be a sequence in S such that $\lim_{n \to \infty} x_n = a$

We shall prove that $\lim_{n \to \infty} f(x_n) = f(a)$.

Since f is continuous at 'a', therefore, given $\varepsilon > 0$, $\exists\ \delta > 0$ such that

$$| f(x) - f(a) | < \varepsilon \text{ whenever } | x - a | < \delta \qquad \ldots(1)$$

Also, since $\lim_{n \to \infty} x_n = a$, therefore, $\exists$ a positive integer m such that

$$| x_n - a | < \delta \quad \forall\ n \ge m \qquad \ldots(2)$$

Putting $x = x_n$ in (1), we have

$$| f(x_n) - f(a) | < \varepsilon \text{ whenever } | x_n - a | < \delta \qquad \ldots(3)$$

From (2) and (3), we have $| f(x_n) - f(a) | < \varepsilon \quad \forall\ n \ge m$

$$\Rightarrow \quad \lim_{n \to \infty} f(x_n) = f(a) \quad i.e., \quad < f(x_n) > \text{ converges to } f(a).$$

(*ii*) Let the sequence $< f(x_n) >$ converge to $f(a)$ whenever the sequence $< x_n >$ converges to 'a'. ...(4)

We shall prove that f is continuous at 'a'.

Suppose f is not continuous at 'a'. Then for some $\varepsilon > 0$ and for every $\delta > 0$, $\exists\ x \in$ S such that

$$| x - a | < \delta \quad \text{and} \quad | f(x) - f(a) | \ge \varepsilon$$

By taking $\delta = \dfrac{1}{n}$, we find that for each positive integer n, $\exists\ x_n \in$ S such that

$$| x_n - a | < \frac{1}{n} \quad \text{and} \quad | f(x_n) - f(a) | \ge \varepsilon \quad \forall\ n$$

$\Rightarrow$ $< x_n >$ converges to a but $< f(x_n) >$ does not converge to $f(a)$.

But this contradicts (4).

$\therefore$ Our supposition that f is not continuous at 'a' is wrong.

Hence f is continuous at 'a'.

6.10. ALGEBRA OF CONTINUOUS FUNCTIONS

Theorem I. *Let f and g be defined on an interval I. If f and g are continuous at a* ~~$\in$ I~~, *then f + g is continuous at a.*

Proof. f is continuous at a $\quad \Rightarrow \quad \lim_{x \to a} f(x) = f(a)$

g is continuous at a $\Rightarrow \lim_{x \to a} g(x) = g(a)$

Now $\lim_{x \to a} (f+g)(x) = \lim_{x \to a} [f(x) + g(x)]$

$= \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = f(a) + g(a) = (f+g)(a)$

$\Rightarrow$ $f+g$ is continuous at a.

(Second Method)

Let $< a_n >$ be any sequence converging to a.

f is continuous at a $\Rightarrow \lim_{n \to \infty} f(a_n) = f(a)$

g is continuous at a $\Rightarrow \lim_{n \to \infty} g(a_n) = g(a)$

Now $\lim_{n \to \infty} (f+g)(a_n) = \lim_{n \to \infty} (f(a_n) + g(a_n))$

$= \lim_{n \to \infty} f(a_n) + \lim_{n \to \infty} g(a_n) = f(a) + g(a) = (f+g)(a)$

$\Rightarrow$ The sequence $< (f+g)(a_n) >$ converges to $(f+g)(a)$

$\Rightarrow$ $f+g$ is continuous at a.

Theorem II. *Let f and g be defined on an interval I. If f and g are continuous at $a \in I$, then $f-g$ is continuous at a.*

Proof. *Please try yourself by both methods.*

Theorem III. *Let f and g be defined on an interval I. If f and g are continuous at $a \in I$, then fg is continuous at a.*

Proof. f is continuous at a $\Rightarrow \lim_{x \to a} f(x) = f(a)$

g is continuous at a $\Rightarrow \lim_{x \to a} g(x) = g(a)$

Now $\lim_{x \to a} (fg)(x) = \lim_{x \to a} f(x)g(x)$

$= \left(\lim_{x \to a} f(x)\right)\left(\lim_{x \to a} g(x)\right) = f(a)g(a) = (fg)(a)$

$\Rightarrow$ fg is continuous at a.

(Second Method)

Let $< a_n >$ be any sequence converging to a.

f is continuous at a $\Rightarrow \lim_{n \to \infty} f(a_n) = f(a)$

g is continuous at a $\Rightarrow \lim_{n \to \infty} g(a_n) = g(a)$

Now $\lim_{n \to \infty} (fg)(a_n) = \lim_{n \to \infty} f(a_n)g(a_n) = \left(\lim_{n \to \infty} f(a_n)\right)\left(\lim_{n \to \infty} g(a_n)\right) = f(a)g(a) = (fg)(a)$

$\Rightarrow$ The sequence $< (fg)(a_n) >$ converges to $(fg)(a)$

$\Rightarrow$ fg is continuous at a.

Theorem IV. *Let f and g be defined on an interval I and let $g(a) \neq 0$. If f and g are continuous at $a \in I$, then $\frac{f}{g}$ is continuous at a.*

Proof. f is continuous at a $\Rightarrow \lim_{x \to a} f(x) = f(a)$

g is continuous at a $\Rightarrow \lim_{x \to a} g(x) = g(a) \neq 0$

Now $\lim_{x \to a} \left(\frac{f}{g}\right)(x) = \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{f(a)}{g(a)} = \left(\frac{f}{g}\right)(a)$

$\Rightarrow$ $\frac{f}{g}$ is continuous at a.

(Second Method)

Let $< a_n >$ be any sequence converging to a.

f is continuous at a $\Rightarrow \lim_{n \to \infty} f(a_n) = f(a)$

g is continuous at a $\Rightarrow \lim_{n \to \infty} g(a_n) = g(a)$

Since $g(a) \neq 0$, $\exists$ a positive integer m such that

$g(a_n) \neq 0 \quad \forall\, n \geq m.$

Now $\lim_{n \to \infty} \left(\frac{f}{g}\right)(a_n) = \lim_{n \to \infty} \frac{f(a_n)}{g(a_n)} = \frac{\lim_{n \to \infty} f(a_n)}{\lim_{n \to \infty} g(a_n)} = \frac{f(a)}{g(a)} = \left(\frac{f}{g}\right)(a)$

$\Rightarrow$ The sequence $< \left(\frac{f}{g}\right)(a_n) >$ converges to $\left(\frac{f}{g}\right)(a)$

$\Rightarrow$ $\frac{f}{g}$ is continuous at a.

Theorem V. *If f is continuous at a point 'a' and $c \in R$, then cf is continuous at 'a'.*

Proof. f is continuous at a $\Rightarrow \lim_{x \to a} f(x) = f(a)$

Now $\lim_{x \to a} (cf)(x) = \lim_{x \to a} cf(x) = c \lim_{x \to a} f(x) = cf(a) = (cf)(a)$

$\Rightarrow$ cf is continuous at a.

Note. Prove it by the second method yourself.

Theorem VI. *If a function f is continuous at a, then $|f|$ is also continuous at a, but not conversely.*

Proof. f is continuous at a

$\Rightarrow$ given $\varepsilon > 0$, $\exists$ a real number $\delta > 0$ such that

$| f(x) - f(a) | < \varepsilon$ whenever $| x - a | < \delta$ ...(1)

But we know that $||f(x)| - |f(a)|| \leq |f(x) - f(a)|$ ...(2)

$[\because \ |a - b| \geq ||a| - |b||]$

From (1) and (2), we have $||f(x)| - |f(a)|| < \varepsilon$ whenever $|x - a| < \delta$

$\Rightarrow$ $|f|$ is continuous at a.

The converse of the above theorem may or may not be true.

Thus $|f|$ is continuous at a does not necessarily imply that f is continuous at a.

For example, consider the function

$$f(x) = \begin{cases} +1 \text{ if } x \geq 0 \\ -1 \text{ if } x < 0 \end{cases} \quad \text{then} \quad |f|(x) = |f(x)| = 1 \quad \forall x \in R.$$

$$\lim_{x \to 0} |f|(x) = \lim_{x \to 0} 1 = 1 = |f|(0)$$

$\Rightarrow$ $|f|$ is continuous at 0.

However, $\lim_{x \to 0-} f(x) = \lim_{x \to 0-} -1 = -1$ and $\lim_{x \to 0+} f(x) = \lim_{x \to 0+} 1 = 1$

so that $\lim_{x \to 0} f(x)$ does not exist.

$\therefore$ f is not continuous at 0.

Theorem VII. *If f and g are two continuous functions at a, then the functions max. $\{f, g\}$ and min. $\{f, g\}$ are both continuous at a.*

Proof. max. $\{f, g\} = \frac{1}{2}(f+g) + \frac{1}{2}|f-g|$; min. $\{f, g\} = \frac{1}{2}(f+g) - \frac{1}{2}|f-g|$

Now f and g are continuous at a

$\Rightarrow$ $f+g$ is continuous at a [Th. 1]

$\Rightarrow$ $\frac{1}{2}(f+g)$ is continuous at a ...(1) [Th. 5]

Also $f-g$ is continuous at a [Th. 2]

$\Rightarrow$ $|f-g|$ is continuous at a [Th. 6]

$\Rightarrow$ $\frac{1}{2}|f-g|$ is continuous at a ...(2) [Th. 5]

From (1) and (2), we have $\frac{1}{2}(f+g) + \frac{1}{2}|f-g|$ and $\frac{1}{2}(f+g) - \frac{1}{2}(f-g)$ are continuous at a.

$\Rightarrow$ max. $\{f, g\}$ and min. $\{f, g\}$ are continuous at a.

Theorem VIII. [Composition of Continuous Functions]

Let f and g be defined on intervals I and J respectively and let $f(I) \subset J$. If f is continuous at $a \in I$ and g is continuous at $f(a) \in J$, then gof is continuous at a.

Proof. Since g is continuous at $f(a)$.

$\therefore$ Given $\varepsilon > 0$, $\exists$ a real number $\delta > 0$ such that

$$|g(f(x)) - g(f(a))| < \varepsilon \text{ whenever } |f(x) - f(a)| < \delta \qquad ...(1)$$

Also f is continuous at a

$\therefore$ (For $\varepsilon = \delta$), $\exists\, \delta_1 > 0$ such that

$$|f(x) - f(a)| < \delta \text{ whenever } |x - a| < \delta_1 \qquad ...(2)$$

From (1) and (2), we have, for any $\varepsilon > 0$, $\exists\, \delta_1 > 0$ such that

$$|g(f(x)) - g(f(a))| < \varepsilon \text{ whenever } |x - a| < \delta_1$$

$\Rightarrow$ $|(gof)(x) - (gof)(a)| < \varepsilon$ whenever $|x - a| < \delta_1$

$\Rightarrow$ gof is continuous at $x = a$.

Note 1. We know that $f(x) = x^n$, where n is a whole number, is continuous on R.

For $c \in$ R, cx^n is continuous on R.

$\therefore$ Using algebra of continuous functions, the polynomial function $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots\ldots + a_{n-1}x + a_n$ is continuous on R.

Note 2. $f(x) = \sin x$ is continuous on R.

$g(x) = x^3$ is continuous on R.

$\therefore$ By the theorem on composite functions,

$$(gof)(x) = g(f(x)) = g(\sin x) = \sin^3 x \text{ is continuous on R.}$$

Note 3. $f(x) = x^3$ is continuous on R.

$g(x) = \cos x$ is continuous on R.

$\therefore$ By the theorem on composite functions,

$(gof)(x) = g(f(x)) = g(x^3) = \cos x^3$ is continuous on R.

6.11. THEOREM

A function f is continuous at a if and only if for each $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x_1) - f(x_2)| < \varepsilon$, whenever $x_1, x_2 \in (a - \delta, a + \delta)$.

Proof. (*i*) Let f be continuous at a.

Then, given $\varepsilon > 0$, $\exists\, \delta > 0$ such that

$$|f(x) - f(a)| < \frac{\varepsilon}{2} \text{ whenever } |x - a| < \delta$$

i.e., $|f(x) - f(a)| < \frac{\varepsilon}{2}$ whenever $a - \delta < x < a + \delta$

i.e., $|f(x) - f(a)| < \frac{\varepsilon}{2}$ whenever $x \in (a - \delta, a + \delta)$

For $x_1, x_2 \in (a - \delta, a + \delta)$, we have

$$|f(x_1) - f(a)| < \frac{\varepsilon}{2} \quad \text{and} \quad |f(x_2) - f(a)| < \frac{\varepsilon}{2} \qquad ...(1)$$

$\therefore$
$$|f(x_1) - f(x_2)| = |f(x_1) - f(a) + f(a) - f(x_2)| = |(f(x_1) - f(a)) + (f(a) - f(x_2))|$$
$$\leq |f(x_1) - f(a)| + |f(a) - f(x_2)| = |f(x_1) - f(a)| + |f(x_2) - f(a)|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \qquad |\text{ using (1)}$$

$\Rightarrow$ $|f(x_1) - f(x_2)| < \varepsilon$.

(*ii*) Conversely, suppose for each $\varepsilon > 0$, $\exists\, \delta > 0$ such that

$$|f(x_1) - f(x_2)| < \varepsilon \text{ whenever } x_1, x_2 \in (a - \delta, a + \delta)$$

Taking $x_1 = x$ and $x_2 = a$, we have

$$|f(x) - f(a)| < \varepsilon \text{ whenever } x \in (a - \delta, a + \delta)$$

$\Rightarrow$ f is continuous at a.

6.12. BOUNDEDNESS OF CONTINUOUS FUNCTIONS

Theorem I. *If a function f is continuous at a, then it is bounded in some neighbourhood of a.*

Proof. Since f is continuous at a.

$\therefore$ Given $\varepsilon > 0$, $\exists$ a real number $\delta > 0$ such that

$$|f(x) - f(a)| < \varepsilon \text{ whenever } |x - a| < \delta, x \in D_f$$

$\Rightarrow$ $f(a) - \varepsilon < f(x) < f(a) + \varepsilon$ whenever $a - \delta < x < a + \delta$, $x \in D_f$

Let $M = \max. \{|f(a) - \varepsilon|, |f(a) + \varepsilon|\}$, then

$$-M \leq f(x) \leq M \text{ whenever } x \in (a - \delta, a + \delta) \cap D_f$$

$\Rightarrow$ $|f(x)| \leq M$ whenever $x \in (a - \delta, a + \delta) \cap D_f$

$\Rightarrow$ f is bounded in some nbd. of a.

Theorem II. *If f is continuous on $[a, b]$, then given $\varepsilon > 0$, however small, the interval $[a, b]$ can be divided into a finite number of sub-intervals in each of which the oscillation of f is less than ε*

i.e., $\quad | f(x_1) - f(x_2) | < \varepsilon$

for any two points x_1 and x_2 in the same sub-interval.

Proof. We shall prove the theorem by contradiction. Let us assume that the theorem is false.

Divide $[a, b] = I$ into two halves $[a, c]$ and $[c, b]$ at the point c where $c = \dfrac{a+b}{2}$.

Since the theorem is false on $[a, b]$, it must be false either on both $[a, c]$ and $[c, b]$ or on at least one of them. Let us denote that sub-interval by $I_1 = [a_1, b_1]$ whose length is $\frac{1}{2}(b-a)$ *i.e.*, $l(I_1) = \frac{1}{2}(b-a)$

Again divide $I_1 = [a_1, b_1]$ into two halves $[a_1, c_1]$ and $[c_1, b_1]$ at the point c_1 where $c_1 = \dfrac{a_1+b_1}{2}$. The theorem must be false on at least one of these sub-intervals. Let us denote that sub-interval by $I_2 = [a_2, b_2]$ where

$$l(I_2) = \frac{1}{2}\left\{\frac{1}{2}(b-a)\right\} = \frac{1}{2^2}(b-a)$$

Continuing this process of repeated bisection indefinitely, we get a sequence of closed intervals

$$[a_1, b_1], [a_2, b_2], \ldots\ldots, [a_n, b_n], \ldots\ldots$$

i.e., $\quad < I_n > = < [a_n, b_n] >$ such that

(*i*) $I_1 \supset I_2 \supset I_3 \supset \ldots\ldots \supset I_n \supset \ldots\ldots \qquad$ (*ii*) $l(I_n) = \dfrac{b-a}{2^n} \to 0$ as $n \to \infty$

(*iii*) the theorem is false on each I_n.

By Nested Interval Property, $\bigcap\limits_{n=1}^{\infty} I_n$ is a singelton $= \{\alpha\}$, say. Since f is continuous on I, f is continuous at α also.

$\therefore$ Given any $\varepsilon > 0$, $\exists$ a real number $\delta > 0$ such that

$$| f(x) - f(\alpha) | < \frac{\varepsilon}{2} \text{ whenever } | x - \alpha | < \delta \qquad \ldots(1)$$

Since $\lim\limits_{n\to\infty} l(I_n) = 0$, we can choose a positive integer m so large that $I_m \subset (\alpha - \delta, \alpha + \delta)$.

$\therefore$ If x_1 and x_2 are any two points in I_m, then from (1), we have

$$| f(x_1) - f(\alpha) | < \frac{\varepsilon}{2} \quad \text{and} \quad | f(x_2) - f(\alpha) | < \frac{\varepsilon}{2}$$

Now $\quad | f(x_1) - f(x_2) | = | (f(x_1) - f(\alpha)) + (f(\alpha) - f(x_2)) |$

$\qquad \leq | f(x_1) - f(\alpha) | + | f(\alpha) - f(x_2) | = | f(x_1) - f(\alpha) | + | f(x_2) - f(\alpha) |$

$\qquad < \dfrac{\varepsilon}{2} + \dfrac{\varepsilon}{2} = \varepsilon$

This contradicts the fact that the theorem is false on each I_n.

Hence the theorem must be true.

Theorem III. *Every function defined and continuous on a closed interval is bounded in that interval.*

Proof. Let f be continuous in the closed interval $[a, b]$, then given $\varepsilon > 0$, we can divide $[a, b]$ into a finite number of sub-intervals, say

$$[a = a_0, a_1], [a_1, a_2], [a_2, a_3], \ldots\ldots, [a_{n-1}, a_n = b]$$

such that $\quad | f(x_1) - f(x_2) | < \varepsilon \qquad \ldots(1)$

for any two points x_1, x_2 belonging to the same sub-interval.

Let x be any point in the first sub-interval $[a, a_1]$, then by (1), we have

$$| f(x) - f(a) | < \varepsilon, \quad \forall x \in [a, a_1]$$

$\therefore \quad | f(x) | = | (f(x) - f(a)) + f(a) |$

$\qquad \leq | f(x) - f(a) | + | f(a) | < \varepsilon + | f(a) |$

In particular, when $x = a_1$, we have $\quad | f(a_1) | < | f(a) | + \varepsilon \qquad \ldots(2)$

Now, let x be any point in the second sub-interval $[a_1, a_2]$, then by (1), we have

$$| f(x) - f(a_1) | < \varepsilon, \quad \forall x \in [a_1, a_2]$$

$\therefore \quad | f(x) | = | (f(x) - f(a_1)) + f(a_1) | \leq | f(x) - f(a_1) | + | f(a_1) |$

$\qquad < \varepsilon + | f(a_1) | < \varepsilon + (| f(a) | + \varepsilon) = | f(a) | + 2\varepsilon \qquad$ [using (2)]

In particular, when $x = a_2$, we have $| f(a_2) | < | f(a) | + 2\varepsilon$

Proceeding similarly, we have

$$| f(x) | < | f(a) | + n\varepsilon, \quad \forall x \in [a_{n-1}, a_n = b]$$

This inequality is satisfied over the whole interval $[a, b]$.

$\therefore \quad f$ is bounded on $[a, b]$.

(Second Method)

Let f be a continuous function on closed interval $[a, b] = I$. Suppose the theorem is false and f is not bounded on $[a, b]$.

Let f be not bounded above on $[a, b]$.

Divide $I = [a, b]$ into two halves $[a, c]$ and $[c, b]$ at the point c where $c = \dfrac{a+b}{2}$.

Since f is unbounded above on $[a, b]$, it must be unbounded above in it at least one of the sub-intervals $[a, c]$ and $[c, b]$. Let us denote that sub-interval by $I_1 = [a_1, b_1]$ whose length is $\frac{1}{2}(b-a)$. *i.e.*, $l(I_1) = \frac{1}{2}(b-a)$.

Again, divide $I_1 = [a_1, b_1]$ into two halves $[a_1, c_1]$ and $[c_1, b_1]$ at the point c_1, where $c_1 = \dfrac{a_1+b_1}{2}$. Then f must be unbounded above in at least one of the sub-intervals.

Let us denote that sub-interval by $I_2 = [a_2, b_2]$ where $l(I_2) = \frac{1}{2}\{\frac{1}{2}(b-a)\} = \dfrac{1}{2^2}(b-a)$.

Continuing this process of repeated bisection indefinitely, we get a sequence of closed intervals

$$[a_1, b_1], [a_2, b_2], \ldots\ldots, [a_n, b_n], \ldots\ldots$$

i.e., $\quad < I_n > = < [a_n, b_n] >$ such that

(*i*) $I_1 \supset I_2 \supset I_3 \supset \ldots\ldots \supset I_n \supset \ldots\ldots \qquad$ (*ii*) $l(I_n) = \dfrac{b-a}{2^n} \to 0$ as $n \to \infty$

(*iii*) the theorem is false on each I_n, *i.e.*, f is not bounded above on $I_n \quad \forall n \in N$.

By Cantor's intersection theorem, $\bigcap_{n=1}^{\infty} I_n$ is a singleton $= \{\alpha\}$, say.

Since f is continuous on I, f is continuous at α also.

$\therefore$ Given any $\varepsilon > 0$, $\exists$ a real number $\delta > 0$ such that

$$|f(x) - f(\alpha)| < \varepsilon \text{ whenever } |x - \alpha| < \delta \quad \ldots(1)$$

Since $\lim_{n \to \infty} l(I_n) = 0$, we can choose n so large that $I_n \subset (\alpha - \delta, \alpha + \delta)$.

If x is any point of I_n, then

$$|f(x)| = |f(x) - f(\alpha) + f(\alpha)|$$
$$\leq |f(x) - f(\alpha)| + |f(\alpha)| < \varepsilon + |f(\alpha)| \quad \text{[using (1)]}$$

showing that f is bounded on I_m. This contradicts the fact that f is unbounded above on each sub-interval I_n.

Hence f must be bounded above on I.

Similarly, f is bounded below on I. Hence f is bounded on $I = [a, b]$.

(Third Method)

Let f be a continuous function on closed interval $[a, b]$. Suppose the theorem is false and f is not bounded on $[a, b]$. Let f be not bounded above on $[a, b]$.

Then for every $n \in N$, $\exists\, x_n \in [a, b]$ such that $f(x_n) > n$. Now $< x_n >$ is a sequence in $[a, b]$. Therefore, by Bolzano-Weierstrass theorem, the sequence $< x_n >$ has a limit point c in $[a, b]$. Hence there must exist a subsequence $< x_{n_k} >$ of $< x_n >$ such that

$$\lim_{n \to \infty} x_{n_k} = c$$

Since $f(x_n) > n \quad \forall\, n \in N$, therefore, $f(x_{n_k}) > n_k$ for all k.

$\Rightarrow$ The sequence $< f(x_{n_k}) >$ diverges to $+\infty$.

Now c is a point of I such that $< x_{n_k} >$ is a sequence in I that converges to c but $< f(x_{n_k}) >$ does not converge to $f(c)$. This means that f is not continuous at c which is a contradiction.

$\therefore$ Our supposition that f is not bounded above is wrong.

$\therefore$ f is bounded above on $[a, b]$.

Similarly, we can show that f is bounded below on $[a, b]$.

Hence f is bounded on $[a, b]$.

Note. The above theorem need not be true if the interval is not closed. For example, let $f(x) = \dfrac{1}{x}$ for all x in $(0, 1]$. Then f is continuous on $(0, 1]$ but is not bounded above therein.

Theorem IV. *Every function defined and continuous on a closed interval attains its bounds.*

Or

If a function f(x) is continuous on [a, b], then it attains its supermum and infimum at least once in [a, b].

(Mostest Theorem)

Proof. Let f be continuous on $[a, b]$, then f is bounded on $[a, b]$.

$\Rightarrow$ Sup. f and Inf. f on $[a, b]$ exist.

Let $u = \text{Sup. } f$ and $l = \text{Inf. } f$ on $[a, b]$.

We have to show that f attains its bounds, i.e., $\exists\, \alpha, \beta \in [a, b]$ such that $f(\alpha) = u$ and $f(\beta) = l$.

Now $\quad u = \text{Sup. } f$ on $[a, b] \quad \Rightarrow \quad f(x) \leq u \quad \forall\, x \in [a, b]$

Suppose f does not attain u on $[a, b]$, then

$$f(x) \neq u \text{ for any } x \in [a, b] \quad \Rightarrow \quad u - f(x) \neq 0 \text{ for any } x \in [a, b].$$

Since u is a constant, it is continuous for all x. Also f is continuous on $[a, b]$.

$\therefore$ $u - f(x)$ is continuous on $[a, b]$

$\Rightarrow$ $\dfrac{1}{u - f(x)}$ is continuous on $[a, b] \quad [\because\; u - f(x) \neq 0]$

$\Rightarrow$ $\dfrac{1}{u - f(x)}$ is bounded on $[a, b]$

$\Rightarrow$ $\exists$ a real number $k > 0$ such that

$$\frac{1}{u - f(x)} \leq k \quad \forall\, x \in [a, b] \quad \Rightarrow \quad u - f(x) \geq \frac{1}{k} \quad \forall\, x \in [a, b]$$

$$\Rightarrow \quad f(x) \leq u - \frac{1}{k} < u \quad \forall\, x \in [a, b] \qquad (\because\; k > 0)$$

$\Rightarrow$ $u - \dfrac{1}{k}$ is an upper bound of f on $[a, b]$ and this upper bound is less than sup. f on $[a, b]$. This contradicts the hypothesis that u is sup. f on $[a, b]$. Hence there must exist some $\alpha \in [a, b]$ such that $f(\alpha) = u$.

$\Rightarrow$ f attains its supermum at least once on $[a, b]$.

Similarly, there must exist at least one value β of x in $[a, b]$ such that $f(\beta) = l = \text{Inf. } f$ on $[a, b]$.

Note. The above theorem need not be true if the interval is not closed.

For example, consider a function f on $(0, 1]$ defined by $f(x) = x$.

f is continuous and bounded on $(0, 1]$ with Sup. $f = 1$ and Inf. $f = 0$. Clearly, f attains the supermum but not the infimum.

Similarly, f on $[0, 1)$ attains the infimum but not the supermum while f on $(0, 1)$ attains none.

Theorem V. *If a function f is continuous on a closed interval [a, b], then f is bounded and attains its bounds at least once in [a, b].*

It is Theorems 3 and 4 combined.

6.13. INTERMEDIATE VALUE THEOREM

Here we shall prove a number of theorems ending with the intermediate value theorem.

Theorem I. *Let f be a continuous function on [a, b] and let $a < c < b$. Then*

(i) $f(a) > 0$ implies that there exists a $\delta > 0$ such that $f(x) > 0 \quad \forall\, x \in [a, a + \delta)$

(ii) $f(a) < 0$ implies that there exists a $\delta > 0$ such that $f(x) < 0 \quad \forall\, x \in [a, a + \delta)$

(iii) $f(c) > 0$ implies that there exists a $\delta > 0$ such that $f(x) > 0 \quad \forall\, x \in (c - \delta, c + \delta)$

(iv) $f(c) < 0$ implies that there exists a $\delta > 0$ such that $f(x) < 0 \quad \forall\, x \in (c - \delta, c + \delta)$

(v) $f(b) > 0$ implies that there exists a $\delta > 0$ such that $f(x) > 0 \quad \forall\, x \in (b - \delta, b]$

(vi) $f(b) < 0$ implies that there exists a $\delta > 0$ such that $f(x) < 0 \quad \forall\, x \in (b - \delta, b]$.

Proof. (*i*) f is continuous on $[a, b] \Rightarrow f$ is right continuous at a

$\therefore$ Given any $\varepsilon > 0$, we can find a $\delta > 0$ such that

$$|f(x) - f(a)| < \varepsilon \qquad \forall\, x \in [a, a + \delta)$$

$$\Rightarrow \quad f(a) - \varepsilon < f(x) < f(a) + \varepsilon \qquad \forall\, x \in [a, a + \delta)$$

Taking $\varepsilon = f(a) > 0$, we have

$$0 < f(x) < 2f(a) \quad \forall\, x \in [a, a+\delta)$$
$$\Rightarrow \quad f(x) > 0 \quad \forall\, x \in [a, a+\delta).$$

(*ii*) *Please try yourself.* **[Hint.** Take $\varepsilon = -f(a) > 0$]

(*iii*) f is continuous on $[a, b]$ and $a < c < b$

$\Rightarrow$ f is continuous at c

$\therefore$ Given any $\varepsilon > 0$, we can find a $\delta > 0$ such that

$$|f(x) - f(c)| < \varepsilon \quad \forall\, x \in (c-\delta, c+\delta)$$
$$\Rightarrow \quad f(c) - \varepsilon < f(x) < f(c) + \varepsilon \quad \forall\, x \in (c-\delta, c+\delta)$$

Taking $\varepsilon = f(c) > 0$, we have $0 < f(x) < 2f(c)$. $\quad \forall\, x \in (c-\delta, c+\delta)$

$$\Rightarrow \quad f(x) > 0 \quad \forall\, x \in (c-\delta, c+\delta).$$

(*iv*) *Please try yourself.* **[Hint.** Take $\varepsilon = -f(c) > 0$]

(*v*) f is continuous on $[a, b]$ $\Rightarrow$ f is left continuous at b.

$\therefore$ Given any $\varepsilon > 0$, we can find a $\delta > 0$ such that

$$|f(x) - f(b)| < \varepsilon \quad \forall\, x \in (b-\delta, b]$$
$$\Rightarrow \quad f(b) - \varepsilon < f(x) < f(b) + \varepsilon \quad \forall\, x \in (b-\delta, b]$$

Taking $\varepsilon = f(b) > 0$, we have

$$0 < f(x) < 2f(b) \quad \forall\, x \in (b-\delta, b]$$
$$\Rightarrow \quad f(x) > 0 \quad \forall\, x \in (b-\delta, b].$$

(*vi*) *Please try yourself.* **[Hint.** Take $\varepsilon = -f(b) > 0$.]

Note. The above theorem asserts that if a function is continuous at a point, then its sign is invariable near the point *i.e.*, the function retains its sign near the point.

Theorem II. *If $f(x)$ is continuous on $[a, b]$ and $a < c < b$ such that $f(c) \neq 0$, then there exists a $\delta > 0$ such that $f(x)$ has the same sign as $f(c)$ for all x in $(c - \delta, c + \delta)$.*

Proof. Since $f(c) \neq 0$, either $f(c) > 0$ or $f(c) < 0$.

Case 1. When $f(c) > 0$

Reproduce Theorem 1 (*iii*).

Case 2. When $f(c) < 0$.

Reproduce Theorem 1 (*iv*).

Theorem III. *If a function f is continuous on a closed interval $[a, b]$ and $f(a)$ and $f(b)$ are of opposite signs, then there exists at least one point $c \in (a, b)$ such that $f(c) = 0$.*

Proof. Without loss of generality, we suppose that $f(a) < 0$ and $f(b) > 0$.

Let $\quad S = \{x : x \in [a, b] \text{ and } f(x) < 0\}$

Since $a \in [a, b]$ and $f(a) < 0$, therefore, $a \in S$

$\Rightarrow \quad S \neq \phi$

Also S is bounded above $\quad$ ($\because$ b is an upper bound of S)

$\therefore$ By completeness property of reals, **S has the l.u.b.** c (say).

We shall show that $f(c) = 0$ where $a < c < b$.

To prove that a < c < b.

Since f is continuous on $[a, b]$

$$\therefore \quad f(a) < 0 \Rightarrow \exists\, a\ \delta_1 > 0 \text{ such that}$$
$$f(x) < 0 \quad \forall\, x \in [a, a + \delta_1)$$

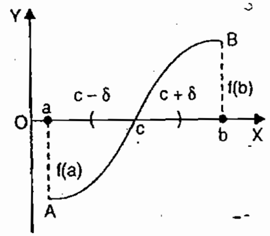
This means that $[a, a + \delta_1) \subset S$ and consequently the l.u.b. of S must be $\geq a + \delta_1$.

Thus $\quad a < a + \delta_1 \leq c \quad i.e., \quad a < c \quad \ldots(1)$

Also $f(b) > 0 \Rightarrow \exists\, a\ \delta_2 > 0$ such that $f(x) > 0 \ \forall\, x \in (b - \delta_2, b]$

This means that $b - \delta_2$ is an upper bound of S and consequently

$$c \leq b - \delta_2 < b \quad i.e., \quad c < b \quad \ldots(2)$$



From (1) and (2), $\quad a < c < b$.

To prove that f(c) = 0.

Suppose $f(c) \neq 0$, then either $f(c) < 0$ or $f(c) > 0$.

Case 1. f(c) < 0.

$$f(c) < 0 \Rightarrow \exists\, a\ \delta_3 > 0 \text{ such that } f(x) < 0 \quad \forall\, x \in (c - \delta_3, c + \delta_3)$$

Let d be any point such that $c < d < c + \delta_3$, then $f(d) < 0$.

$\Rightarrow$ $d \in S$, but this contradicts the fact that c = l.u.b. S.

$$\therefore \quad f(c) \nless 0.$$

Case 2. f(c) > 0.

$$f(c) > 0 \Rightarrow \exists\, a\ \delta_4 > 0 \text{ such that } f(x) > 0 \quad \forall\, x \in (c - \delta_4, c + \delta_4).$$

Since c = l.u.b. S, $\exists\, d' \in S$ such that $c - \delta_4 < d' < c$

($\because$ $c - \delta_4 < c$ is not an upper bound of S)

Now $\quad d' \in S \Rightarrow f(d') < 0$

But $\quad d' \in (c - \delta_4, c) \Rightarrow f(d') > 0$

This contradiction shows that $f(c) \ngtr 0$.

Since $f(c)$ is neither negative nor positive, therefore, $f(c) = 0$.

Theorem IV. (Intermediate Value Theorem)

If a function f is continuous on a closed interval $[a, b]$ and $f(a) \neq f(b)$, then f assumes every value between $f(a)$ and $f(b)$ at least once.

Proof. Since $f(a) \neq f(b)$, let us assume that $f(a) < (b)$.

Let k be any number such that $f(a) < k < f(b)$.

Consider a function g defined on $[a, b]$ such that $g(x) = f(x) - k$

Since f is continuous on $[a, b]$ and k is a constant, therefore, g is also continuous on $[a, b]$.

Also $\quad g(a) = f(a) - k < 0$ and $g(b) = f(b) - k > 0 \quad | \because f(a) < k < f(b)$

Now g is continuous on $[a, b]$ and $g(a)$ and $g(b)$ have opposite signs.

$\Rightarrow$ $\exists$ at least one point $c \in (a, b)$ such that

$$g(c) = 0 \quad i.e., \quad f(c) - k = 0 \quad i.e., \quad f(c) = k$$

Since k is any value between $f(a)$ and $f(b)$, it follows that f assumes every value between $f(a)$ and $f(b)$ at least once.

Note. Converse of the Intermediate Value Theorem need not be true.

For example, consider the function $f(x) = \begin{cases} x & \text{if } 0 \leq x < 1 \\ x - 1 & \text{if } 1 \leq x \leq 3 \end{cases}$ so that $f(0) = 0$ and $f(3) = 3 - 1 = 2$.

Here f assumes every value between 0 and 2 as x moves between 0 and 3, but f is discontinuous at $x = 1$ and hence discontinuous on $[0, 3]$.

Theorem V. *If a function f is continuous on a closed interval [a, b], then f assumes every value between its bounds.*

Proof. f is continuous on $[a, b]$

$\Rightarrow$ f is bounded on $[a, b]$ and attains its bounds on $[a, b]$.

$\Rightarrow$ $\exists\, \alpha, \beta \in [a, b]$ such that $f(\alpha) = M$ and $f(\beta) = m$ where M = l.u.b. of f and m = g.l.b. of f.

Case 1. If $m = M$, then f is constant on $[a, b]$ so that $m = M = c$ and $f(x) = c\ \forall\, x \in [a, b]$. The result follows.

Case 2. If $m \neq M$, then $\alpha \neq \beta$

$\Rightarrow$ either $\alpha < \beta$ or $\alpha > \beta$.

(*i*) If $\alpha < \beta$, then $a \le \alpha < \beta \le b$, so that f is continuous on $[\alpha, \beta]$ and $M = f(\alpha) \neq f(\beta) = m$.

$\therefore$ By Intermediate Value Theorem, f assumes every value between m and M.

(*ii*) If $\alpha > \beta$, then $a \le \beta < \alpha \le b$, so that f is continuous on $[\beta, \alpha]$ and $M = f(\alpha) \neq f(\beta) = m$.

$\therefore$ By Intermediate Value Theorem, f assumes every value between m and M.

Hence f assumes every value between its bounds.

Theorem VI. *The image of a closed interval under a continuous function is a closed interval.*

Proof. Let f be a continuous function on a closed interval $I = [a, b]$.

Let M = l.u.b. of f on I and m = g.l.b. of f on I.

Then $\quad m \le f(x) \le M \quad \forall\, x \in I$

$\Rightarrow \quad f(I) \subset [m, M]$ ...(1)

Since f is continuous on I, f assumes every value between m and M.

$\therefore$ If $m \le c \le M$, then for some $x \in I$, $f(x) = c$ so that $[m, M] \subset f(I)$ ...(2)

From (1) and (2), we have $\quad f(I) = [m, M]$.

6.14. MONOTONIC FUNCTIONS

Let f be a function defined on $[a, b]$. Then

(*i*) f is said to be **monotonically increasing** on $[a, b]$ if

$$x_1, x_2 \in [a, b] \text{ and } x_1 < x_2 \Rightarrow f(x_1) \le f(x_2).$$

(*ii*) f is said to be **strictly monotonically increasing** on $[a, b]$ if

$$x_1, x_2 \in [a, b] \text{ and } x_1 < x_2 \Rightarrow f(x_1) < f(x_2).$$

(*iii*) f is said to be **monotonically decreasing** on $[a, b]$ if

$$x_1, x_2 \in [a, b] \text{ and } x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2).$$

(*iv*) f is said to be **strictly monotonically decreasing** on $[a, b]$ if

$$x_1, x_2 \in [a, b] \text{ and } x_1 < x_2 \Rightarrow f(x_1) > f(x_2).$$

(*v*) f is said to be **monotonic** if it is either monotonically increasing or monotonically decreasing.

6.15. THEOREM

(*i*) *A function $f : R \to R$ is continuous on R if and only if for every open set G in R, $f^{-1}(G)$ is an open set in R.*

(*ii*) *A function $f : R \to R$ is continuous on R if and only if for every closed set F in R, $f^{-1}(F)$ is a closed set in R.*

Proof. (*i*) **The condition is necessary.**

Let f be continuous on R and let G be any open subset of R.

We shall show that $f^{-1}(G)$ is open.

If $f^{-1}(G) = \phi$, then it is open.

If $f^{-1}(G) \neq \phi$, let a be any point of $f^{-1}(G)$.

$$a \in f^{-1}(G) \Rightarrow f(a) \in G$$

Since G is an open set, G is a nbd. of $f(a)$.

$\therefore \quad \exists\, \varepsilon > 0$ such that $\quad (f(a) - \varepsilon, f(a) + \varepsilon) \subset G$.

Now, f being continuous at a, for the above choice of ε, $\exists$ a $\delta > 0$, such that

$$| f(x) - f(a) | < \varepsilon \text{ whenever } | x - a | < \delta$$

$\Rightarrow \quad f(x) \in (f(a) - \varepsilon, f(a) + \varepsilon)$ whenever $x \in (a - \delta, a + \delta)$

$\Rightarrow \quad f(x) \in (f(a) - \varepsilon, f(a) + \varepsilon) \subset G$ whenever $x \in (a - \delta, a + \delta)$

$\Rightarrow \quad x \in f^{-1}(G)$ whenever $x \in (a - \delta, a + \delta)$

$\Rightarrow \quad a \in (a - \delta, a + \delta) \subset f^{-1}(G)$

$\Rightarrow \quad f^{-1}(G)$ is a *nbd.* of a

Since a is any point of $f^{-1}(G)$, therefore, $f^{-1}(G)$ is open.

The condition is sufficient.

Let $f^{-1}(G)$ be open whenever G is open.

We shall show that f is continuous on R.

Consider any point $a \in R$. Let $\varepsilon > 0$ be any positive number.

Now $\quad I = (f(a) - \varepsilon, f(a) + \varepsilon)$ is an open set containing $f(a)$.

$\therefore \quad f^{-1}(I)$ is an open set containing a. $\quad [\because$ G is open $\Rightarrow f^{-1}(G)$ is open$]$

Consequently, there exists a $\delta > 0$ such that

$$(a - \delta, a + \delta) \subset f^{-1}(I) \quad \Rightarrow \quad f((a - \delta, a + \delta)) \subset I$$

Thus we have found a $\delta > 0$ such that

$$x \in (a - \delta, a + \delta) \quad \Rightarrow \quad f(x) \in I = (f(a) - \varepsilon, f(a) + \varepsilon)$$

or $$| x - a | < \delta \quad \Rightarrow \quad | f(x) - f(a) | < \varepsilon$$

$\therefore \quad f$ is continuous at a.

Since a is any point of R, therefore, f is continuous on R.

(*ii*) **The condition is necessary.**

Let f be continuous on R and let F be any closed sub-set of R.

We shall show that $f^{-1}(F)$ is closed.

F is a closed subset of R.

$\Rightarrow \quad F^c$ is an open subset of R.

$\Rightarrow \quad f^{-1}(F^c)$ is an open subset of R $\quad$ [Using part (*i*)]

Since $\quad f^{-1}(F^c) = (f^{-1}(F))^c$

$\therefore \quad (f^{-1}(F))^c$ is an open subset of R

$\Rightarrow \quad f^{-1}(F)$ is a closed subset of R.

The condition is sufficient.

Let $f^{-1}(F)$ be closed whenever F is closed.

We shall show that f is continuous on R.

Let G be any open subset of R, then G^c is a closed subset of R.

$\therefore$ $f^{-1}(G^c)$ is a closed subset of R. [$\because$ F is closed $\Rightarrow$ $f^{-1}(F)$ is closed]

Since $f^{-1}(G^c) = (f^{-1}(G))^c$

$\therefore$ $(f^{-1}(G))^c$ is a closed subset of R

$\Rightarrow$ $f^{-1}(G)$ is an open subset of R.

Thus $f^{-1}(G)$ is open whenever G is open.

$\therefore$ f is continuous on R. [by part (*i*)]

6.16. THEOREM (CONTINUITY OF INVERSE FUNCTION)

If a one-to-one function f is continuous and strictly monotonic on a closed interval [a, b], then f^{-1} is also continuous on [f(a), f(b)] or [f(b), f(a)] according as f is increasing or decreasing.

Proof. f is continuous and strictly monotonic on $[a, b]$.

$\Rightarrow$ f is bounded and attains its bounds at a and b.

$\Rightarrow$ $\exists$ real numbers m and M such that

$$f(a) = m < M = f(b) \text{ or } f(b) = m < M = f(a)$$

according as f is strictly monotonically increasing or decreasing.

$\Rightarrow$ f is a one-to-one function from $[a, b]$ onto $[f(a), f(b)]$ or $[f(b), f(a)]$ according as f is strictly monotonically increasing or decreasing.

$\Rightarrow$ f^{-1} is a one-to-one function from $[f(a), f(b)]$

or $[f(b), f(a)]$ onto $[a, b]$ according as f is strictly monotonically increasing or decreasing.

Case 1. Let f be strictly monotonically increasing.

Then $f : [a, b] \to [f(a), f(b)]$ and $f^{-1} : [f(a), f(b)] \to [a, b]$

We shall show that f^{-1} is continuous on $[f(a), f(b)]$.

Consider any point $y_0 \in [f(a), f(b)]$. Then $\exists$ a unique $x_0 \in [a, b]$ such that

$$f^{-1}(y_0) = x_0 \quad i.e., \quad f(x_0) = y_0$$

Let $\varepsilon > 0$ be given.

Let $f(x_0 - \varepsilon) = y_0 - \delta_1$ and $f(x_0 + \varepsilon) = y_0 + \delta_2$ where δ_1 and δ_2 are positive reals.

Since f is strictly monotonically increasing.

$$\therefore \quad x \in (x_0 - \varepsilon, x_0 + \varepsilon) \Rightarrow f(x) \in (f(x_0 - \varepsilon), f(x_0 + \varepsilon))$$

$$\text{or} \quad | x - x_0 | < \varepsilon \Rightarrow y = f(x) \in (y_0 - \delta_1, y_0 + \delta_2)$$

If $\delta = \min. \{\delta_1, \delta_2\}$, then $(y_0 - \delta, y_0 + \delta) \subset (y_0 - \delta_1, y_0 + \delta_2)$

$\therefore$ $| x - x_0 | < \varepsilon$ for $y \in (y_0 - \delta, y_0 + \delta)$

$\Rightarrow$ $| x - x_0 | < \varepsilon$ for $| y - y_0 | < \delta \Rightarrow | f^{-1}(y) - f^{-1}(y_0) | < \varepsilon$ for $| y - y_0 | < \delta$

$\Rightarrow$ f^{-1} is continuous at y_0.

Since y_0 is any point of $[f(a), f(b)]$; therefore, f^{-1} is continuous on $[f(a), f(b)]$.

Case 2. Let f be strictly monotonically decreasing.

Let us define a function $g(x) = -f(x)$ so that g is a one-to-one continuous function on $[a, b]$.

Since f is strictly monotonically decreasing on $[a, b]$.

$\therefore$ g is strictly monotonically increasing on $[a, b]$.

By case I, g^{-1} is continuous on $[g(a), g(b)]$.

$\Rightarrow$ $-g^{-1}$ is continuous on $[-g(b), -g(a)]$ and consequently, $f^{-1} = (-g)^{-1} = -g^{-1}$ is continuous on

$$[-g(b), -g(a)] = [f(b), f(a)].$$

6.17. UNIFORM CONTINUITY

We know that a function f is continuous at a point x_0 of an interval I if given $\varepsilon > 0$, $\exists$ a $\delta > 0$ such that

$$| f(x) - f(x_0) | < \varepsilon \text{ whenever } | x - x_0 | < \delta$$

Now, this δ depends in general, not only on ε but also on the point x_0 at which the continuity of f is being considered *i.e.*, in general, $\delta = \delta(\varepsilon, x_0)$

For example, consider the function f defined by $f(x) = x^2 \ \forall x \in$ R.

Let us take $\varepsilon = \frac{1}{4}$ and $x_0 = 0$, then $| f(x) - f(x_0) | = | x^2 - 0 | = | x^2 | = | x |^2 < \frac{1}{4}$ whenever $| x |^2 < \frac{1}{4}$ *i.e.*, whenever $| x | < \frac{1}{2}$

Taking $\delta = \frac{1}{2}$, we have $| f(x) - f(x_0) | < \varepsilon$ whenever $| x - x_0 | < \delta$

Thus $\delta = \frac{1}{2}$ works at $x_0 = 0$ corresponding to $\varepsilon = \frac{1}{4}$.

Now let us take $\varepsilon = \frac{1}{4}$ and $x_0 = 1$, then $\delta = \frac{1}{2}$ does not work. For, if we take $x = 1.4$, then

$$| x - x_0 | = | 1.4 - 1 | = .4 < \tfrac{1}{2}$$

$$\text{But} \quad | f(x) - f(x_0) | = | 1.96 - 1 | = .96 > \tfrac{1}{4}.$$

Thus, given $\varepsilon > 0$, the same value of δ does not work for different points of the interval.

If a continuous function f is such that given $\varepsilon > 0$, we can find a uniform $\delta > 0$ which depends only on ε and not on the point x_0 at which the continuity is considered, then we say that f is *uniformly continuous*.

Definition. A function f defined on an interval I is said to be uniformly continuous on I if given $\varepsilon > 0$, $\exists$ a $\delta = \delta(\varepsilon) > 0$ such that $| f(x_1) - f(x_2) | < \varepsilon$ whenever $| x_1 - x_2 | < \delta$ where $x_1, x_2 \in$ I.

Note 1. Uniform continuity of a function is a *global property*. We talk of uniform continuity *on a set and not at a point*. Continuity on the other hand is a *local property*.

Note 2. A function f is not uniformly continuous on I if $\exists$ some $\varepsilon > 0$ for which no δ works, *i.e.*, for any $\delta > 0$, $\exists\, x_1, x_2 \in$ I such that $| f(x_1) - f(x_2) | \geq \varepsilon$ and $| x_1 - x_2 | < \delta$.

6.18. THEOREM (*Uniform Continuity* $\Rightarrow$ *Continuity*)

If a function f is uniformly continuous on an interval I, then it is continuous on I.

Proof. f is uniformly continuous on I

$\Rightarrow$ Given $\varepsilon > 0$, $\exists$ $\delta = \delta(\varepsilon) > 0$ such that

$$| f(x_1) - f(x_2) | < \varepsilon \text{ whenever } | x_1 - x_2 | < \delta, \text{ where } x_1, x_2 \in \text{I}.$$

Let c be any point of I.

Taking $x_1 = x$ and $x_2 = c$, we have $| f(x) - f(c) | < \varepsilon$ whenever $| x - c | < \delta$

$\Rightarrow$ f is continuous at c.

But c is any point of I.

$\therefore$ f is continuous at every point of I.

Hence f is continuous on I.

Note 1. I is any interval, open or closed or semi-open.

Note 2. Converse of the above theorem need not be true *i.e., every continuous function need not be uniformly continuous.*

Consider the function $f : \text{I} = (0, 1] \to \text{R}$ defined by $f(x) = \dfrac{1}{x}$

Since x is continuous on I and $x \neq 0$ in I.

$\therefore \quad \frac{1}{x}$ is continuous on I.

Now, for any $\delta > 0$, $\exists\, m \in N$ such that $\frac{1}{n} < \delta \;\forall\; n \geq m$

Let $\quad x_1 = \frac{1}{2m}$ and $x_2 = \frac{1}{m}$ so that $x_1, x_2 \in I$

and $\quad | x_1 - x_2 | = \left| \frac{1}{2m} - \frac{1}{m} \right| = \frac{1}{2m} < \frac{1}{2}\delta < \delta$

But $\quad | f(x_1) - f(x_2) | = | 2m - m | = m$ which cannot be less than every $\varepsilon > 0$.

$\Rightarrow$ f is not uniformly continuous on I.

Remark. If $f(x) = \frac{1}{x}$ is considered on a closed interval $I = [a, b]$ where $a > 0$, then f is uniformly continuous because of the following theorem.

6.19. THEOREM

If a function f is continuous on a closed interval $[a, b]$, then it is uniformly continuous on $[a, b]$.

Proof. Since f is continuous on $[a, b]$.

$\therefore$ Given $\varepsilon > 0$, we can divide $[a, b]$ into a finite number, say n, of sub-intervals.

$$[a = t_0, t_1], [t_1, t_2], \ldots\ldots, [t_{r-1}, t_r], [t_r, t_{r+1}], \ldots\ldots, [t_{n-1}, t_n = b]$$

such that $\quad | f(x_1) - f(x_2) | < \frac{\varepsilon}{2}$ for x_1, x_2 belonging to the same sub-interval.

Let $\quad \delta = \frac{1}{2} \min \{ | t_r - t_{r-1} | \} > 0, 1 \leq r \leq n.$

Let x_1, x_2 be any two points of $[a, b]$ such that $| x_1 - x_2 | < \delta$. Then x_1, x_2 either belong to the same sub-interval or they belong to two consecutive sub-intervals with a common end point.

Case (*i*) When x_1, x_2 belong to the same sub-interval, we have

$$| f(x_1) - f(x_2) | < \frac{\varepsilon}{2} < \varepsilon \text{ for } | x_1 - x_2 | < \delta.$$

Case (*ii*) When x_1, x_2 belong to two consecutive sub-intervals with a common end point, say t_r, we have

$$| f(x_1) - f(t_r) | < \frac{\varepsilon}{2} \quad \text{and} \quad | f(t_r) - f(x_2) | < \frac{\varepsilon}{2}$$

$$\therefore \quad | f(x_1) - f(x_2) | = | f(x_1) - f(t_r) + f(t_r) - f(x_2) |$$

$$\leq | f(x_1) - f(t_r) | + | f(t_r) - f(x_2) | < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \text{ for } | x_1 - x_2 | < \delta$$

Thus in either case, we have for any $\varepsilon > 0, \exists$ a $\delta > 0$ such that

$$| f(x_1) - f(x_2) | < \varepsilon \text{ for } x_1, x_2 \in [a, b] \quad \text{and} \quad | x_1 - x_2 | < \delta.$$

$\Rightarrow$ f is uniformly continuous on $[a, b]$.

Hence continuity on a closed interval $\Rightarrow$ uniform continuity.

6.20. THEOREM

Let K be a compact set. If a function $f : K \to R$ is continuous on K then f is uniformly continuous on K.

Proof. Let $\varepsilon > 0$ be given and let k be any point of K.

Since f is continuous on K, therefore, f is continuous at $x = k$.

$\therefore$ We can find a $\delta_k > 0$ such that

$$x \in K \text{ and } | x - k | < \delta_k \Rightarrow | f(x) - f(k) | < \frac{\varepsilon}{2} \qquad \ldots(1)$$

The family $F = \left\{ \left(k - \frac{\delta_k}{2}, k + \frac{\delta_k}{2} \right); k \in K \right\}$ is an open cover of K, because each $k \in K$ is covered by the corresponding interval.

$$\therefore \quad K \subset \bigcup_{k \in K} \left(k - \frac{\delta_k}{2}, k + \frac{\delta_k}{2} \right)$$

Since K is compact, therefore, by Heine-Borel Theorem, there exists a finite sub-cover for K.

$\Rightarrow$ $\exists$ finitely many points $k_1, k_2, \ldots\ldots, k_n$ such that $\bigcup_{i=1}^{n} \left(k_i - \frac{\delta_{k_i}}{2}, k_i + \frac{\delta_{k_i}}{2} \right) \supset K.$

Let $\quad \delta = \frac{1}{2} \min. \{ \delta_{k_i} : i = 1, 2, \ldots\ldots, n \}$

Now, let x, y be any two arbitrary points of K such that $| x - y | < \delta$

Since $x \in K$, $\exists$ some j $(1 \leq j \leq n)$ such that $x \in \left(k_j - \frac{\delta_{k_j}}{2}, k_j + \frac{\delta_{k_j}}{2} \right)$

whence $\quad | x - k_j | < \frac{\delta_{k_j}}{2} < \delta_{k_j} \qquad \ldots(2)$

Also $\quad | y - k_j | = | y - x + x - k_j | \leq | y - x | + | x - k_j | < \delta + \frac{\delta_{k_j}}{2}$

$$\leq \frac{\delta_{k_j}}{2} + \frac{\delta_{k_j}}{2} = \delta_{k_j}, \text{ since } \delta \leq \frac{\delta_{k_j}}{2} \qquad \ldots(3)$$

From (1), (2) and (3), we have

$$| f(x) - f(k_j) | < \frac{\varepsilon}{2} \quad \text{and} \quad | f(y) - f(k_j) | < \frac{\varepsilon}{2}$$

$$\therefore \quad | f(x) - f(y) | = | f(x) - f(k_j) + f(k_j) - f(y) | \leq | f(x) - f(k_j) | + | f(k_j) - f(y) |$$

$$= | f(x) - f(k_j) | + | f(y) - f(k_j) | < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Thus, given $\varepsilon > 0$, we have found a $\delta > 0$ such that x,

$$y \in K \text{ and } | x - y | < \delta \Rightarrow | f(x) - f(y) | < \varepsilon.$$

Hence f is uniformly continuous on K.

ILLUSTRATIVE EXAMPLES

Example 1. *Let f be continuous on R. Then show that*

(i) $A = \{x \in R \mid f(x) > 0\}$ *is an open set.*

(ii) $B = \{x \in R \mid f(x) < 0\}$ *is an open set.*

(iii) $C = \{x \in R \mid f(x) \geq 0\}$ *is a closed set.* *(iv)* $D = \{x \in R \mid f(x) \leq 0\}$ *is a closed set.*

(v) $E = \{x \in R \mid f(x) = 0\}$ *is a closed set.*

Sol. *(i)* Let $a \in A$, then $f(a) > 0$ | by def. of A

$\therefore$ $\exists$ a $\delta > 0$ such that $f(x) > 0 \quad \forall x \in (a - \delta, a + \delta)$ | See Theorem 1 with 6.13

$\Rightarrow$ $x \in (a - \delta, a + \delta) \subset A$

$\Rightarrow$ A is a nbd. of a

Since a is any element of A, therefore, A is a nbd. of each of its elements.

Hence A is an open set.

(ii) Please try yourself.

(iii) $C = R - B = B^c$

Since B is an open set, therefore, C is a closed set.

(iv) $D = R - A = A^c$.

Since A is an open set, therefore, D is a closed set.

(v) $E = R - (A \cup B) = (A \cup B)^c$

Since A and B are open sets, therefore, $A \cup B$ is an open set and hence E is a closed set.

Example 2. *Let f be a continuous function on R and let c be any real number. Then show that*

(i) $A = \{x \in R \mid f(x) > c\}$ *is an open set.* *(ii)* $B = \{x \in R \mid f(x) < c\}$ *is an open set.*

(iii) $C = \{x \in R \mid f(x) \geq c\}$ *is a closed set.* *(iv)* $D = \{x \in R \mid f(x) \leq c\}$ *is a closed set.*

(v) $E = \{x \in R \mid f(x) = c\}$ *is a closed set.*

Sol. *(i)* Let $g(x) = f(x) - c$

Then f is continuous $\Rightarrow$ g is continuous and $f(x) > c \Rightarrow g(x) > 0$

$\therefore$ $A = \{x \in R \mid g(x) > 0\}$

Let $a \in A$, then $g(a) > 0$ | by def. of A

$\therefore \exists\, a\ \delta > 0$ such that $g(x) > 0 \quad \forall x \in (a - \delta, a + \delta)$

$\Rightarrow$ $x \in (a - \delta, a + \delta) \subset A$

$\Rightarrow$ A is a nbd. of a

Since a is any element of A, therefore, A is a nbd. of each of its elements.

Hence A is an open set.

(ii) Please try yourself. *(iii) Please try yourself.*
(iv) Please try yourself. *(v) Please try yourself.* } (As in Example 1)

Example 3. *Show that the function* $f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ *is continuous for every real x.*

Sol. Let x be any real number, then either $x = 0$ or $x \neq 0$.

Case I. When $x = 0$.

Let $\varepsilon > 0$ be given.

$$| f(x) - f(0) | = \left| x \sin \frac{1}{x} - 0 \right| = \left| x \sin \frac{1}{x} \right| = | x | \left| \sin \frac{1}{x} \right| \leq | x | \qquad \left[\because \left| \sin \frac{1}{x} \right| \leq 1 \right]$$

$< \varepsilon$ whenever $| x | < \varepsilon$.

Taking $\delta = \varepsilon$, we have :

Given $\varepsilon > 0, \exists\, a\ \delta > 0$ such that $| f(x) - f(0) | < \varepsilon$ whenever $| x | < \delta$

$\Rightarrow$ f is continuous at $x = 0$.

Case II. When $x \neq 0$

Since $x \neq 0, g(x) = \frac{1}{x}$ is continuous.

Also $h(x) = \sin x$ is continuous.

$\therefore$ $(hog)(x) = h(g(x)) = h\left(\frac{1}{x}\right) = \sin \frac{1}{x}$ is continuous.

Now x and $\sin \frac{1}{x}$ are continuous. $\Rightarrow$ $x \sin \frac{1}{x}$ is continuous.

$\therefore$ f is continuous when $x \neq 0$.

Combining the two cases, f is continuous for every real x.

Example 4. *Examine the continuity of the function*

$$f(x) = \begin{cases} x \cos \frac{1}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases} \quad at\ x \in R.$$

Sol. Let x be any real number, then either $x = 0$ or $x \neq 0$.

Case I. When $x = 0$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} x \cos \frac{1}{x} = 0 \qquad \left(\because \lim_{x \to 0} x = 0 \text{ and } \left| \cos \frac{1}{x} \right| \leq 1 \right)$$

But $f(0) = 1$.

Since $\lim_{x \to 0} f(x)$ exists but is not equal to $f(0)$, therefore, f has removable discontinuity at $x = 0$.

Case II. When $x \neq 0$.

Since $x \neq 0$, $g(x) = \frac{1}{x}$ is continuous.

Also $h(x) = \cos x$ is continuous.

$\because$ $(hog)(x) = h(g(x)) = h\left(\frac{1}{x}\right) = \cos \frac{1}{x}$ is continuous.

Now x and $\cos \frac{1}{x}$ are continuous. $\Rightarrow$ $x \cos \frac{1}{x}$ is continuous.

$\therefore$ f is continuous when $x \neq 0$

Combining the two cases, f has removable discontinuity at $x = 0$ and is continuous elsewhere.

Example 5. *Show that the Dirichlet's function f defined by*

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ -1, & \text{if } x \text{ is irrational} \end{cases}$$ *is discontinuous for every real x.*

Sol. Let x be any real number, then either x is rational or x is irrational.

Case I. *When x is a rational number.*

Since in any interval there lie infinitely many rationals as well as infinitely many irrationals, therefore, for each $n \in N$, $\exists$ an irrational number x_n such that

$$x - \frac{1}{n} < x_n < x + \frac{1}{n}, \Rightarrow \ | x_n - x | < \frac{1}{n} \ \forall n$$

$\Rightarrow$ The sequence $< x_n >$ converges to x.

But $f(x_n) = -1$ for all n and $f(x) = 1$, so that $\lim\limits_{n \to \infty} f(x_n) = -1 \neq f(x)$

$\therefore$ f is discontinuous at x, any rational number.

Case II. *When x is an irrational number.*

Since in any interval there lie infinitely many rationals as well as infinitely many irrationals, therefore, for each $n \in N$, $\exists$ a rational number x_n such that

$$x - \frac{1}{n} < x_n < x + \frac{1}{n} \ \Rightarrow \ | x_n - x | < \frac{1}{n} \ \forall n$$

$\Rightarrow$ The sequence $< x_n >$ converges to x.

But $f(x_n) = 1$ for all n and $f(x) = -1$, so that $\lim\limits_{n \to \infty} f(x_n) = 1 \neq f(x)$

$\therefore$ f is discontinuous at x, any irrational number.

Hence f is discontinuous for every real x.

Example 6. *Prove that the function f defined by*

$$f(x) = \begin{cases} \frac{1}{2}, & \text{if } x \text{ is rational} \\ \frac{1}{3}, & \text{if } x \text{ is irrational} \end{cases}$$ *is discontinuous everywhere.*

Sol. *Please try yourself.*

Example 7. *Show that the function f defined by*

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

is discontinuous at every point and discontinuity is of second kind.

Sol. Let a be any real number, rational or irrational.

Let $(a - \delta, a)$ be a deleted left nbd. of a. Between $a - \delta$ and a, there lie infinitely many rational numbers as well as infinitely many irrational numbers. Since f takes value 1 at each rational number and value 0 at each irrational number, it follows that f oscillates and does not approach any unique real number.

Thus $\lim\limits_{x \to a-} f(x)$ does not exist.

Similarly $\lim\limits_{x \to a+} f(x)$ does not exist.

$\therefore$ f is discontinuous at every point and discontinuity is of second kind.

Example 8. *Show that the function f defined by*

$$f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ -x, & \text{if } x \text{ is irrational} \end{cases}$$ *is continuous only at x = 0.*

Sol. Let x be any real number. For each $n \in N$, $\exists$ a rational number a_n and an irrational number b_n such that

$$x - \frac{1}{n} < a_n < x + \frac{1}{n} \quad \text{and} \quad x - \frac{1}{n} < b_n < x + \frac{1}{n}$$

$\Rightarrow$ $| a_n - x | < \frac{1}{n}$ and $| b_n - x | < \frac{1}{n} \ \forall n$

$\Rightarrow$ $\lim\limits_{n \to \infty} a_n = x = \lim\limits_{n \to \infty} b_n$ ...(1)

If f is continuous at x, we must have

$$\lim_{n \to \infty} f(a_n) = f(x) = \lim_{n \to \infty} f(b_n)$$

But $f(a_n) = a_n$ and $f(b_n) = -b_n$

$\therefore$ $\lim\limits_{n \to \infty} a_n = f(x) = \lim\limits_{n \to \infty} -b_n$

$\Rightarrow$ $x = f(x) = -x$ [from (1)]

$\Rightarrow$ $2x = 0 \ \therefore \ x = 0$

Thus 0 is the only possible point of continuity.

Now, we shall show that f is actually continuous at 0.

Let $\varepsilon > 0$ be given. Also $f(0) = 0$.

For a rational number x, we have $| f(x) - f(0) | = | x - 0 | = | x |$

For an irrational number x, we have $| f(x) - f(0) | = | -x - 0 | = | -x | = | x |$

In either case, $| f(x) - f(0) | = | x | < \varepsilon$ whenever $| x | < \varepsilon$

Choose $\delta = \varepsilon$, then $| f(x) - f(0) | < \varepsilon$ whenever $| x | < \delta$

$\Rightarrow$ f is continuous at 0.

Example 9. (*a*) *Show that the function f defined by*

$$f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$ *is continuous only at x = 0.*

(*b*) *Show that the function f defined by*

$$f(x) = \begin{cases} x, & \text{when } x \text{ is irrational} \\ -x, & \text{when } x \text{ is rational} \end{cases}$$ *is continuous only at x = 0.*

Sol. *Please try both parts yourself.*

Example 10. *Show that the function f defined by $f(x) = x^2$ is uniformly continuous on $[-2, 2]$.*

Sol. Let $\varepsilon > 0$ be given.

Let x_1, x_2 be any two points of $[-2, 2]$ so that $| x_1 | \leq 2$, $| x_2 | \leq 2$.

Now $| f(x_1) - f(x_2) | = | x_1^2 - x_2^2 |$

$$= | (x_1 - x_2)(x_1 + x_2) | = | x_1 - x_2 | \cdot | x_1 + x_2 | \leq | x_1 - x_2 | \cdot (| x_1 | + | x_2 |)$$

$$\leq | x_1 - x_2 | \cdot (2 + 2) = 4 | x_1 - x_2 | < \varepsilon \text{ whenever } | x_1 - x_2 | < \varepsilon/4$$

Choose $\delta = \frac{\varepsilon}{4}$, then $| f(x_1) - f(x_2) | < \varepsilon$ whenever $| x_1 - x_2 | < \delta$

$\Rightarrow$ f is uniformly continuous on $[-2, 2]$.

Example 11. *Show that the function f defined by $f(x) = x^2$ is uniformly continuous on $[-1, 1]$.*

Sol. *Please try yourself.*

Example 12. *Show that the function f defined by $f(x) = x^3$ is uniformly continuous on $[-1, 1]$.*

Sol. Let $\varepsilon > 0$ be given.

Let x_1, x_2 be any two points of $[-1, 1]$ so that $|x_1| \le 1$, $|x_2| \le 1$.

Now
$$\begin{aligned} |f(x_1) - f(x_2)| &= |x_1^3 - x_2^3| = |(x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2)| \\ &= |x_1 - x_2| \cdot |x_1^2 + x_1x_2 + x_2^2| \\ &\le |x_1 - x_2| \, |x_1^2| + |x_1x_2| + |x_2^2|) \\ &= |x_1 - x_2| \, (|x_1|^2 + |x_1| \, |x_2| + |x_2|^2) \\ &\le |x_1 - x_2| \, (1 + 1 \times 1 + 1) \\ &= 3 \, |x_1 - x_2| < \varepsilon \text{ whenever } |x_1 - x_2| < \frac{\varepsilon}{3} \end{aligned}$$

Choose $\delta = \frac{\varepsilon}{3}$, then $|f(x_1) - f(x_2)| < \varepsilon$ whenever $|x_1 - x_2| < \delta$

$\Rightarrow$ f is uniformly continuous on $[-1, 1]$.

Example 13. *Show that the function f defined by $f(x) = x^3$ is uniformly continuous on $[-2, 2]$.*

Sol. *Please try yourself.*

Example 14. *Show that $f(x) = x^3$ is uniformly continuous on $(0, 1)$.*

Sol. *Please try yourself.*

Example 15. *Show that the function f defined by $f(x) = 2x^2 - 3x + 5$ is uniformly continuous on $[-2, 2]$.*

Sol. Let $\varepsilon > 0$ be given.

Let x_1, x_2 be any two points of $[-2, 2]$ so that $|x_1| \le 2$, $|x_2| \le 2$.

Now
$$\begin{aligned} |f(x_1) - f(x_2)| &= |(2x_1^2 - 3x_1 + 5) - (2x_2^2 - 3x_2 + 5)| \\ &= |2(x_1^2 - x_2^2) - 3(x_1 - x_2)| = |(x_1 - x_2)(2(x_1 + x_2) - 3)| \\ &= |x_1 - x_2| \, |2(x_1 + x_2) - 3| \le |x_1 - x_2| \, [\, |2(x_1 + x_2)| + |3| \,] \\ &= |x_1 - x_2| \, (2 \, |x_1 + x_2| + 3) \le |x_1 - x_2| \, [2(|x_1| + |x_2|) + 3] \\ &\le |x_1 - x_2| \, [2(2 + 2) + 3] \\ &= 11 . |x_1 - x_2| < \varepsilon \text{ whenever } |x_1 - x_2| < \frac{\varepsilon}{11} \end{aligned}$$

Choose $\delta = \frac{\varepsilon}{11}$, then $|f(x_1) - f(x_2)| < \varepsilon$ whenever $|x_1 - x_2| < \delta$

$\Rightarrow$ f is uniformly continuous on $[-2, 2]$.

Example 16. (*a*) *Prove that the function f defined by $f(x) = x^2 + 2x + 2$ is uniformly continuous on $[1, 2]$.*

(*b*) *Prove that the function f defined by $f(x) = 2x^2 + 3x - 4$ is uniformly continuous on $[0, 2]$.*

Sol. *Please try yourself.*

Example 17. *Show that the function f defined by*

$$f(x) = \frac{x}{x+1} \text{ is uniformly continuous on } [0, 2].$$

Sol. Let $\varepsilon > 0$ be given.

Let x_1, x_2 by any two points of $[0, 2]$, so that

$$\begin{aligned} & 0 \le x_1 \le 2 && \text{and} \quad 0 \le x_2 \le 2 \\ \Rightarrow \quad & 1 \le x_1 + 1 \le 3 && \text{and} \quad 1 \le x_2 + 1 \le 3. \\ \Rightarrow \quad & |x_1 + 1| \ge 1 && \text{and} \quad |x_2 + 1| \ge 1 \\ \Rightarrow \quad & \frac{1}{|x_1 + 1|} \le 1 && \text{and} \quad \frac{1}{|x_2 + 1|} \le 1 \end{aligned}$$

Now
$$\begin{aligned} |f(x_1) - f(x_2)| &= \left| \frac{x_1}{x_1 + 1} - \frac{x_2}{x_2 + 1} \right| = \left| \frac{x_1 - x_2}{(x_1 + 1)(x_2 + 1)} \right| = \frac{|x_1 - x_2|}{|x_1 + 1| \, |x_2 + 1|} \\ &\le |x_1 - x_2| < \varepsilon \text{ whenever } |x_1 - x_2| < \varepsilon \end{aligned}$$

Choose $\delta = \varepsilon$, then $|f(x_1) - f(x_2)| < \varepsilon$ whenever $|x_1 - x_2| < \delta$

$\Rightarrow$ f is uniformly continuous on $[0, 2]$.

Example 18. *Show that the function f defined by $f(x) = \frac{2x}{2x - 1}$ is uniformly continuous on $[1, \infty)$.*

Sol. Let $\varepsilon > 0$ be given.

Let x_1, x_2 be any two points of $[1, \infty)$ so that

$$\begin{aligned} & x_1 \ge 1 && \text{and} \quad x_2 \ge 1 \\ \Rightarrow \quad & 2x_1 - 1 \ge 1 && \text{and} \quad 2x_2 - 1 \ge 1 \\ \Rightarrow \quad & |2x_1 - 1| \ge 1 && \text{and} \quad |2x_2 - 1| \ge 1 \\ \Rightarrow \quad & \frac{1}{|2x_1 - 1|} \le 1 && \text{and} \quad \frac{1}{|2x_2 - 1|} \le 1 \end{aligned}$$

Now
$$\begin{aligned} |f(x_1) - f(x_2)| &= \left| \frac{2x_1}{2x_1 - 1} - \frac{2x_2}{2x_2 - 1} \right| = \left| \frac{2(x_2 - x_1)}{(2x_1 - 1)(2x_2 - 1)} \right| = \frac{2 \, |x_1 - x_2|}{|2x_1 - 1| \, |2x_2 - 1|} \\ &\le 2 \, |x_1 - x_2| < \varepsilon \text{ whenever } |x_1 - x_2| < \frac{\varepsilon}{2} \end{aligned}$$

Choose $\delta = \frac{\varepsilon}{2}$, then $|f(x_1) - f(x_2)| < \varepsilon$ whenever $|x_1 - x_2| < \delta$

$\Rightarrow$ f is uniformly continuous on $[1, \infty)$.

Example 19. *Prove that the function f defined by $f(x) = x^2$, $x \in R$, is uniformly continuous on every closed and finite interval, but is not uniformly continuous on R.*

Sol. Let $a, b \in R$ with $a < b$. If $c = \max \{ |a|, |b| \}$, then $c > 0$.

Let $\varepsilon > 0$ be given.

Let x_1, x_2 be any two points of $[a, b]$ so that $|x_1| \le c$ and $|x_2| \le c$.

Now
$$\begin{aligned} |f(x_1) - f(x_2)| &= |x_1^2 - x_2^2| = |(x_1 + x_2)(x_1 - x_2)| \\ &= |x_1 + x_2| \, . \, |x_1 - x_2| \le (|x_1| + |x_2|) \, . \, |x_1 - x_2| \\ &\le (c + c) \, |x_1 - x_2| = 2c \, |x_1 - x_2| < \varepsilon \text{ whenever } |x_1 - x_2| < \frac{\varepsilon}{2c} \end{aligned}$$

Choose $\delta = \frac{\varepsilon}{2c} > 0$, then $|f(x_1) - f(x_2)| < \varepsilon$ whenever $|x_1 - x_2| < \delta$

$\Rightarrow$ f is uniformly continuous on $[a, b]$.

Now we shall show that f is not uniformly continuous on R.

Let $\varepsilon > 0$ be given. We shall show that for each $\delta > 0, \exists\, x_1, x_2 \in$ R such that

$$| x_1 - x_2 | < \delta \Rightarrow | f(x_1) - f(x_2) | \geq \varepsilon$$

The sequence $< a_n >$ defined by

$$a_n = \sqrt{n+2\varepsilon} - \sqrt{n} \text{ converges to } 0, \text{ since } a_n = \frac{2\varepsilon}{\sqrt{n+2\varepsilon}+\sqrt{n}} \to 0 \text{ as } n \to \infty.$$

$\therefore$ Given $\delta > 0, \exists$ a positive integer m such that

$$| a_n - 0 | < \delta \quad \forall\, n \geq m \Rightarrow | \sqrt{n+2\varepsilon} - \sqrt{n} | < \delta \quad \forall\, n \geq m$$

Let us take $x_1 = \sqrt{m+2\varepsilon}$ and $x_2 = \sqrt{m}$, then

$$| x_1 - x_2 | < \delta \text{ whereas } | f(x_1) - f(x_2) | = | (m+2\varepsilon) - m | = 2\varepsilon > \varepsilon$$

Hence f is not uniformly continuous on R.

Example 20. (a) *Show that the function f defined by $f(x) = \frac{1}{x^2}, x \neq 0$, is uniformly continuous on $[a, \infty)$ where $a > 0$, but not uniformly continuous on $(0, \infty)$.*

(b) *If $f : (0, \infty) \to R$ is a function defined by $f(x) = \frac{1}{x}$, prove that f is uniformly continuous on $[a, \infty)$ where $a > 0$. Show that f is continuous but not uniformly on $(0, \infty)$.*

Sol. (a) Let $\varepsilon > 0$ be given.

Let x_1, x_2 be any two points of $[a, \infty)$ so that

$$x_1 \geq a \quad \text{and} \quad x_2 \geq a$$

$$\Rightarrow \quad | x_1 | \geq a \quad \text{and} \quad | x_2 | \geq a$$

$$\Rightarrow \quad \frac{1}{|x_1|} \leq \frac{1}{a} \quad \text{and} \quad \frac{1}{|x_2|} \leq \frac{1}{a}$$

Now
$$| f(x_1) - f(x_2) | = \left| \frac{1}{x_1^2} - \frac{1}{x_2^2} \right| = \left| \left(\frac{1}{x_1} + \frac{1}{x_2} \right)\left(\frac{1}{x_1} - \frac{1}{x_2} \right) \right| = \left| \frac{1}{x_1} + \frac{1}{x_2} \right| \cdot \left| \frac{1}{x_1} - \frac{1}{x_2} \right|$$

$$\leq \left(\left| \frac{1}{x_1} \right| + \left| \frac{1}{x_2} \right| \right) \left| \frac{x_2 - x_1}{x_1 x_2} \right| = \left(\frac{1}{|x_1|} + \frac{1}{|x_2|} \right) \frac{|x_1 - x_2|}{|x_1||x_2|}$$

$$\leq \left(\frac{1}{a} + \frac{1}{a} \right) \frac{|x_1 - x_2|}{a.a} = \frac{2}{a^3} |x_1 - x_2| < \varepsilon \text{ whenever } |x_1 - x_2| < \frac{a^3}{2}\varepsilon$$

Choose $\delta = \frac{a^3}{2}\varepsilon > 0$, then $| f(x_1) - f(x_2) | < \varepsilon$ whenever $| x_1 - x_2 | < \delta$

$\Rightarrow$ f is uniformly continuous on $[a, \infty)$.

Now we shall show that f is not uniformly continuous on $(0, \infty)$.

Let $\varepsilon > 0$ be given. We shall show that for each $\delta > 0, \exists\, x_1, x_2 \in (0, \infty)$ such that

$$| x_1 - x_2 | < \delta \Rightarrow | f(x_1) - f(x_2) | \geq \varepsilon$$

The sequence $< a_n >$ defined by

$$a_n = \frac{1}{\sqrt{n+2\varepsilon}} - \frac{1}{\sqrt{n}} \text{ converges to } 0, \text{ since } a_n \to 0 \text{ as } n \to \infty.$$

$\therefore$ Given $\delta > 0, \exists$ a positive integer m such that

$$| a_n - 0 | < \delta \quad \forall \quad n \geq m \quad \Rightarrow \left| \frac{1}{\sqrt{n+2\varepsilon}} - \frac{1}{\sqrt{n}} \right| < \delta \quad \forall \quad n \geq m$$

Let us take $x_1 = \frac{1}{\sqrt{m+2\varepsilon}}$ and $x_2 = \frac{1}{\sqrt{m}}$, then $| x_1 - x_2 | < \delta$

whereas $| f(x_1) - f(x_2) | = | (m + 2\varepsilon) - m | = 2\varepsilon > \varepsilon$.

Hence f is not uniformly continuous on $(0, \infty)$.

(b) *Please try yourself.*

Example 21. *Prove that the function f defined by $f(x) = \sin \frac{1}{x} \quad \forall \quad x > 0$ is continuous but not uniformly continuous on R^+.*

Sol. Let a be any positive real number, then $f(a) = \sin \frac{1}{a}$

Now
$$\lim_{x \to a-} f(x) = \lim_{x \to a-} \sin \frac{1}{x} \qquad | \text{ Put } x = a - h, h > 0$$
$$= \lim_{h \to 0+} \sin \frac{1}{a-h} = \sin \frac{1}{a}$$

Also
$$\lim_{x \to a+} f(x) = \lim_{x \to a+} \sin \frac{1}{x} \qquad | \text{ Put } x = a + h, h > 0$$
$$= \lim_{h \to 0+} \sin \frac{1}{a+h} = \sin \frac{1}{a}$$

$$\therefore \quad \lim_{x \to a-} f(x) = f(a) = \lim_{x \to a+} f(x)$$

$\Rightarrow$ f is continuous at 'a'.

Since 'a' is an arbitrary point of R^+, f is continuous on R^+.

Now we shall show that f is not uniformly continuous on R^+.

Let $\varepsilon > 0$ be given. We shall show that for each $\delta > 0, \exists\, x_1, x_2 \in R^+$ such that

$$| x_1 - x_2 | < \delta \Rightarrow | f(x_1) - f(x_2) | \geq \varepsilon.$$

Now for any $\delta > 0, \exists\, m \in$ N such that

$$\frac{1}{2n\left(n\pi + \frac{\pi}{2}\right)} < \delta \quad \forall\, n \geq m$$

Take
$$x_1 = \frac{1}{m\pi + \frac{\pi}{2}} \text{ and } x_2 = -\frac{1}{m\pi}$$

then $x_1, x_2 \in R^+$ and
$$| x_1 - x_2 | = \left| \frac{1}{m\pi + \frac{\pi}{2}} - \frac{1}{m\pi} \right| = \left| \frac{-\pi/2}{m\pi\left(m\pi + \frac{\pi}{2}\right)} \right| = \frac{1}{2m\left(m\pi + \frac{\pi}{2}\right)} < \delta$$

But
$$| f(x_1) - f(x_2) | = \left| \sin\left(m\pi + \frac{\pi}{2}\right) - \sin m\pi \right| = | \cos m\pi - 0 | = | (-1)^m | = 1$$

which is not less than each $\varepsilon > 0$.

Hence f is not uniformly continuous on R^+.

Example 22. *Prove that the function f defined by*

$$f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 1-x, & \text{if } x \text{ is irrational} \end{cases} \quad \textit{is continuous only at } x = \frac{1}{2}.$$

Sol. Let x be any real number. For each $n \in N$, $\exists$ a rational number a_n and an irrational number b_n such that

$$x - \frac{1}{n} < a_n < x + \frac{1}{n} \quad \text{and} \quad x - \frac{1}{n} < b_n < x + \frac{1}{n}$$

$$\Rightarrow \quad |a_n - x| < \frac{1}{n} \quad \text{and} \quad |b_n - x| < \frac{1}{n} \quad \forall \; n$$

$$\Rightarrow \quad \lim_{n\to\infty} a_n = x = \lim_{n\to\infty} b_n \qquad \ldots(1)$$

If f is continuous at x, we must have $\lim_{n\to\infty} f(a_n) = f(x) = \lim_{n\to\infty} f(b_n)$

But $\quad f(a_n) = a_n \quad \text{and} \quad f(b_n) = 1 - b_n$

$\therefore \quad \lim_{n\to\infty} a_n = f(x) = \lim_{n\to\infty} (1 - b_n)$

$\Rightarrow \quad x = f(x) = 1 - x \qquad$ [from (1)]

$\Rightarrow \quad 2x = 1 \quad \therefore \quad x = \frac{1}{2}$.

Thus $\frac{1}{2}$ is the only possible point of continuity.

Now we shall show that f is actually continuous at $\frac{1}{2}$.

Let $\varepsilon > 0$ be given. Also $f\left(\frac{1}{2}\right) = \frac{1}{2}$.

For a rational number x, we have $\left|f(x) - f\left(\frac{1}{2}\right)\right| = \left|x - \frac{1}{2}\right|$

For an irrational number x, we have $\left|f(x) - f\left(\frac{1}{2}\right)\right| = \left|1 - x - \frac{1}{2}\right| = \left|\frac{1}{2} - x\right| = \left|x - \frac{1}{2}\right|$

In either case, $\left|f(x) - f\left(\frac{1}{2}\right)\right| = \left|x - \frac{1}{2}\right| < \varepsilon$ whenever $\left|x - \frac{1}{2}\right| < \varepsilon$.

Choose $\delta = \varepsilon$, then $\left|f(x) - f\left(\frac{1}{2}\right)\right| < \varepsilon$ whenever $\left|x - \frac{1}{2}\right| < \delta$

$\Rightarrow$ f is continuous at $\frac{1}{2}$.

Example 23. *Examine for continuity the function f defined by*

$$f(x) = \lim_{n\to\infty} \frac{e^x - x^n \sin x}{1 + x^n}, \; 0 \le x \le \frac{\pi}{2}$$

at $x = 1$. Explain why the function f does not vanish anywhere on $\left[0, \frac{\pi}{2}\right]$ although $f(0)$ and $f\left(\frac{\pi}{2}\right)$ have opposite signs.

Sol. First of all we obtain expressions for f in $\left[0, \frac{\pi}{2}\right]$ in a form free from limits.

Since $\quad \lim_{n\to\infty} x^n = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x = 1 \\ \infty & \text{if } 1 < x \le \frac{\pi}{2} \end{cases}$

$$\therefore \quad f(x) = \lim_{n\to\infty} \frac{e^x - x^n \sin x}{1 + x^n} \begin{cases} e^x & \text{if } 0 \le x < 1 \\ \dfrac{e - \sin 1}{2} & \text{if } x = 1 \\ \lim_{n\to\infty} \dfrac{x^{-n} e^x - \sin x}{x^{-n} + 1} = -\sin x & \text{if } 1 < x \le \frac{\pi}{2} \end{cases}$$

$$(\because \; x^n \to \infty \;\Rightarrow\; x^{-n} \to 0)$$

Now $\quad f(1) = \frac{1}{2}(e - \sin 1)$

$$\lim_{x\to 1-} f(x) = \lim_{x\to 1-} e^x = \lim_{h\to 0+} e^{1-h} = e$$

$$\lim_{x\to 1+} f(x) = \lim_{x\to 1+} (-\sin x) = \lim_{h\to 0+} -\sin(1+h) = -\sin 1$$

Since $\lim_{x\to 1-} f(x)$ and $\lim_{x\to 1+} f(x)$ both exist but are unequal, also neither of them is equal to $f(1)$, therefore, f has a discontinuity of the first kind at $x = 1$ on both sides.

Now $\quad f(0) = e^0 = 1 > 0 \quad \text{and} \quad f\left(\frac{\pi}{2}\right) = -\sin\frac{\pi}{2} = -1 < 0$

so that $f(0)$ and $f\left(\frac{\pi}{2}\right)$ have opposite signs. Also, it is clear that f does not vanish anywhere in $\left[0, \frac{\pi}{2}\right]$.

The function f is not continuous on $\left[0, \frac{\pi}{2}\right]$, the point $x = 1$ being a point of discontinuity.

This explains the reason why f does not vanish anywhere in $\left[0, \frac{\pi}{2}\right]$ even though $f(0)$ and $f\left(\frac{\pi}{2}\right)$ are of opposite signs.

Note. If a function f is continuous on a closed interval $[a, b]$ and $f(a) f(b) < 0$, then $\exists$ at least one point $c \in (a, b)$ such that $f(c) = 0$.

Example 24. *Discuss the nature of discontinuity of the function f defined by*

$$f(x) = \lim_{n\to\infty} \frac{\log(2+x) - x^{2n} \sin x}{1 + x^{2n}}, \; 0 \le x \le \frac{\pi}{2} \text{ at } x = 1.$$

Show that $f(0)$ and $f\left(\frac{\pi}{2}\right)$ differ in sign, and explain why f still does not vanish in $\left[0, \frac{\pi}{2}\right]$.

Sol. First of all we obtain expressions for f in $\left[0, \frac{\pi}{2}\right]$ in a form free from limits.

Since $$\lim_{n \to \infty} x^{2n} = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x = 1 \\ \infty & \text{if } 1 < x \le \frac{\pi}{2} \end{cases}$$

$$\therefore \quad f(x) = \lim_{n \to \infty} \frac{\log(2+x) - x^{2n} \sin x}{1 + x^{2n}} = \begin{cases} \log(2+x) \text{ if } 0 \le x < 1 \\ \dfrac{\log 3 - \sin 1}{2} \text{ if } x = 1 \\ \lim\limits_{n \to \infty} \dfrac{x^{-2n} \log(2+x) - \sin x}{x^{-2n} + 1} = -\sin x \text{ if } 1 < x \le \frac{\pi}{2} \end{cases}$$

$$(\because \; x^{2n} \to \infty \;\Rightarrow\; x^{-2n} \to 0)$$

Now $$f(1) = \frac{1}{2}(\log 3 - \sin 1)$$

$$\lim_{x \to 1-} (x) = \lim_{x \to 1-} \log(2+x) = \lim_{h \to 0+} \log(2+1-h) = \log 3$$

$$\lim_{x \to 1+} f(x) = \lim_{x \to 1+} (-\sin x) = \lim_{h \to 0+} -\sin(1+h) = -\sin 1$$

Since $\lim\limits_{x \to 1-} f(x)$ and $\lim\limits_{x \to 1+} f(x)$ both exist, but are unequal, also neither of them is equal to $f(1)$, therefore, f has a discontinuity of the first kind at $x = 1$ on both sides.

Now $$f(0) = \log 2 > 0 \quad \text{and} \quad f\left(\frac{\pi}{2}\right) = -\sin\frac{\pi}{2} = -1 < 0$$

so that $f(0)$ and $f\left(\frac{\pi}{2}\right)$ have opposite signs. Also, it is clear that f does not vanish anywhere in $\left[0, \frac{\pi}{2}\right]$.

The function f is not continuous on $\left[0, \frac{\pi}{2}\right]$, the point $x = 1$ being a point of discontinuity. This explains the reason why f does not vanish anywhere in $\left[0, \frac{\pi}{2}\right]$ even though $f(0)$ and $f\left(\frac{\pi}{2}\right)$ are of opposite signs.

Example 25. *Show that the function* $\phi(x) = \lim\limits_{n \to \infty} \dfrac{x^{2n} - \cos x}{x^{2n} + 1}$ *does not vanish anywhere in the interval* [0, 2] *though* $\phi(0)$ *and* $\phi(2)$ *differ in sign.*

Sol. *Please try yourself.*

Example 26. *Prove that the function f defined on (0, 1) as*

$$f(x) = \begin{cases} 0 & \text{when } x \text{ is irrational} \\ \dfrac{1}{q} & \text{when } x \text{ is a rational number of the form} \\ \dfrac{p}{q} & \text{where } p \text{ and } q \text{ are co-prime positive integers} \end{cases}$$

is continuous at every irrational point and discontinuous at each rational point.

Sol. First suppose that $a = \frac{p}{q}$ is any rational number in (0, 1). Then $f(a) = f\left(\frac{p}{q}\right) = \frac{1}{q} > 0$.

Since every nbd. of a rational number contains infinitely many irrational numbers, therefore, for each $n \in \mathrm{N}$, we can find an irrational number a_n such that

$$a - \frac{1}{n} < a_n < a + \frac{1}{n} \quad \Rightarrow \quad |a_n - a| < \frac{1}{n}$$

$\Rightarrow$ The sequence $< a_n >$ converges to a.

Since $$f(a_n) = 0 \quad \forall \quad n \in \mathrm{N} \qquad \text{(given)}$$

$$\therefore \quad \lim_{n \to \infty} f(a_n) = 0 \ne f(a)$$

$\Rightarrow$ $< f(a_n) >$ does not converge to $f(a)$.

$\Rightarrow$ f is not continuous at 'a'.

$\Rightarrow$ f is discontinuous at each rational point in (0, 1).

Next suppose 'b' is any irrational number in (0, 1). Then $f(b) = 0$.

Let $\varepsilon > 0$ be given. By Archemedian property of real numbers, we can choose a positive integer n such that $\frac{1}{n} < \varepsilon$. Clearly, there can be only a finite number of rational numbers $\frac{p}{q}$ in (0, 1) such that $q < n$. We can, therefore, find a $\delta > 0$ such that $(b - \delta, b + \delta)$ has no rational number with denominator less than n, *i.e.*, $(b - \delta, b + \delta)$ contains only those rational numbers $\frac{p}{q}$ for which $q \ge n$.

Thus, when x is irrational $|x - b| < \delta \;\Rightarrow\; |f(x) - f(b)| = |f(x)| = 0 < \varepsilon$

and when x is rational $|x - b| < \delta \;\Rightarrow\; |f(x) - f(b)| = |f(x)| = \frac{1}{q} \le \frac{1}{n} < \varepsilon$

Combining, we have $x \in \mathrm{R}$ and $|x - b| < \delta \;\Rightarrow\; |f(x) - f(b)| < \varepsilon$

$\Rightarrow$ f is continuous at 'b'.

$\Rightarrow$ f is continuous at every irrational point.

Example 27. *Let a function f satisfy the equation*

$$f(x + y) = f(x) + f(y) \quad \forall\, x, y \in R.$$

Show that if f is continuous at the point a, then it is continuous for all $x \in R$.

Sol. *Continuity at the point 'a'.*

$$\lim_{x \to a-} f(x) = \lim_{h \to 0+} f(a - h) = \lim_{h \to 0+} [f(a) + f(-h)] \quad [\because \; f(x + y) = f(x) + f(y)]$$

$$= \lim_{h \to 0+} f(a) + \lim_{h \to 0+} f(-h) = f(a) + \lim_{h \to 0+} f(-h)$$

Similarly $$\lim_{x \to a+} f(x) = \lim_{h \to 0+} f(a + h) = \lim_{h \to 0+} [f(a) + f(h)]$$

$$= \lim_{h \to 0+} f(a) + \lim_{h \to 0+} f(h) = f(a) + \lim_{h \to 0+} f(h).$$

Since f is continuous at 'a'

$$\therefore \quad \lim_{x \to a-} f(x) = \lim_{x \to a+} f(x) = f(a)$$

$$\Rightarrow \quad f(a) + \lim_{h \to 0+} f(-h) = f(a) + \lim_{h \to 0+} f(h) = f(a)$$

$$\Rightarrow \quad \lim_{h \to 0+} f(-h) = \lim_{h \to 0+} f(h) = 0 \quad \ldots(1)$$

Now, let α be any real number. Then

$$\lim_{x \to \alpha-} f(x) = \lim_{h \to 0+} f(\alpha - h) = \lim_{h \to 0+} [f(\alpha) + f(-h)]$$

$$= \lim_{h \to 0+} f(\alpha) + \lim_{h \to 0+} f(-h) = f(\alpha) + 0 = f(\alpha) \quad \text{[by (1)]}$$

Also $$\lim_{x \to \alpha+} f(x) = \lim_{h \to 0+} f(\alpha + h) = \lim_{h \to 0+} [f(\alpha) + f(h)]$$

$$= \lim_{h \to 0+} f(\alpha) + \lim_{h \to 0+} f(h) = f(\alpha) + 0 = f(\alpha) \quad \text{[by (1)]}$$

Since $$\lim_{x \to \alpha-} f(x) = f(\alpha) = \lim_{x \to \alpha+} f(x)$$

$\therefore$ f is continuous at α.

But α is any real number. Hence f is continuous for all $x \in R$.

Example 28. *Give one example of each of the following :*

(i) A function which is discontinuous everywhere.

(ii) A function which is continuous but not uniformly continuous.

(iii) A function which is continuous everywhere but is not differentiable anywhere.

(iv) A function on [0, 1] which is continuous everywhere except the end points.

(v) A function on [1, 2] which is continuous everywhere except at 3/2.

(vi) A function which is continuous on (0, 1) but is not bounded.

(vii) A function which is bounded on (0, 1) but is not continuous.

(viii) A function f which is continuous no where but $|f|$ is continuous everywhere.

Sol. (*i*) The function f defined by $f(x) = \begin{cases} 1 \text{ if } x \text{ is rational} \\ -1 \text{ if } x \text{ is irrational} \end{cases}$ is discontinuous for every real x.

(*ii*) The function f defined by $f(x) = \sin \dfrac{1}{x}, x \in R^+$ is continuous but not uniformly continuous on R^+.

(*iii*) The function f defined by $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n} \cos (3^n x) \quad \forall \quad x \in R$ is continuous everywhere but is not differentiable anywhere.

[*The proof is beyond the scope of the present book.*]

(*iv*) The function f defined by $f(x) = \begin{cases} 1, \text{when } x = 0 \\ x, \text{when } 0 < x < 1 \\ 2, \text{when } x = 1 \end{cases}$

is continuous on (0, 1) but discontinuous at $x = 0$ and $x = 1$.

(*v*) The function f defined by $f(x) = \begin{cases} 1 + x, \text{when } 1 \le x \le \dfrac{3}{2} \\ 0 \quad , \text{when } x = \dfrac{3}{2} \\ 1 - x, \text{when } \dfrac{3}{2} < x \le 2 \end{cases}$

Or

The function g defined by $g(x) = \begin{cases} x, \text{when } 1 \le x \le 2, x \ne \dfrac{3}{2} \\ 0, \text{when } x = \dfrac{3}{2} \end{cases}$

are continuous everywhere on [1, 2] except at 3/2.

(*vi*) The function $f(x) = \dfrac{1}{x}$ is continuous on (0, 1) but is not bounded.

(*vii*) The function f defined on (0, 1) by $f(x) = \begin{cases} 1 \text{ if } x \text{ is rational} \\ -1 \text{ if } x \text{ is irrational} \end{cases}$

is bounded but not continuous.

(*viii*) The function f defined on R by $f(x) = \begin{cases} 1 \text{ if } x \text{ is rational} \\ -1 \text{ if } x \text{ is irrational} \end{cases}$

is continuous no where but $|f(x)| = 1 \quad \forall x \in R$ is a constant function and hence continuous everywhere.

Example 29. *Give an example of each of the following :*

(i) A continuous and bounded function on R which attains the l.u.b. but not the g.l.b.

(ii) A continuous and bounded function on R which attains the g.l.b. but not the l.u.b.

(iii) A function continuous on an open interval but may fail to be uniformly continuous.

Sol. (*i*) The function $f(x) = \dfrac{1}{x^2 + 1}$ is continuous on R, since $x^2 + 1$ is continuous on R and $x^2 + 1 \ne 0$ for any $x \in R$.

Also $\quad x^2 + 1 \ge 1 \quad \forall x \in R \quad \Rightarrow \quad 0 < f(x) \le 1 \quad \forall x \in R$

$\Rightarrow$ f is bounded on R and attains the l.u.b. 1 but not the g.l.b. 0.

(*ii*) The function $f(x) = \dfrac{-1}{x^2 + 1}$ is continuous on R and $-1 \le f(x) < 0 \quad \forall x \in R$

$\Rightarrow$ f is bounded on R and attains the g.l.b. -1 but not the l.u.b. 0.

(*iii*) The function $f(x) = \dfrac{1}{x}$ is continuous on the open interval (0, 1) but not uniformly continuous on (0, 1).

Example 30. *If a function f is such that $|f(x)| \le |x| \quad \forall \quad x \in R$, then show that $|f|$ is continuous at 0.*

Sol. Since $\quad |f(x)| \ge 0 \quad \forall x$

$$\therefore \quad \lim_{x \to 0} |f(x)| \ge 0 \quad \ldots(1)$$

Also $\quad |f(x)| \le |x|$

$$\Rightarrow \quad \lim_{x \to 0} |f(x)| \le \lim_{x \to 0} |x| = 0 \quad \ldots(2)$$

From (1) and (2), $\displaystyle\lim_{x \to 0} |f(x)| = 0$

Also $\quad |f(0)| \le |0| = 0$

But $\quad | f(0) | \geq 0 \quad \therefore \quad | f(0) | = 0$

Thus $\quad \lim_{x \to 0} | f(x) | = | f(0) |$

Hence f is continuous at $x = 0$.

Example 31. *Let f be a continuous function at $x = 0$ and $f(0) = 0$. If g be a function such that $| g(x) | \leq | f(x) |$ for all x, then show that g is also continuous at $x = 0$.*

Sol. We have $\quad 0 \leq | g(x) | \leq | f(x) |$

Since $\quad f(0) = 0 \quad \therefore \quad g(0) = 0$

Also f is continuous at $x = 0$.

$\Rightarrow \quad \lim_{x \to 0} f(x) = f(0) = 0 \quad \ldots(1)$

Now $\quad | g(x) | \leq f(x) \quad \Rightarrow \quad -f(x) \leq g(x) \leq f(x)$

$\Rightarrow \quad \lim_{x \to 0} g(x) = 0$, by Squeeze Principle and using (1)

$\Rightarrow \quad \lim_{x \to 0} g(x) = g(0) \quad \Rightarrow \quad g$ is also continuous at $x = 0$.

Example 32. *If f and g are two continuous functions on $[a, b]$ such that $f(a) < g(a)$ and $f(b) > g(b)$, then show that there exists a real number $c \in (a, b)$ such that $f(c) = g(c)$.*

Sol. Define a new function h on $[a, b]$ by $h(x) = f(x) - g(x)$

Then $\quad h(a) = f(a) - g(a) < 0 \quad [\because f(a) < g(a)]$

and $\quad h(b) = f(b) - g(b) > 0 \quad [\because f(b) > g(b)]$

Since f and g are continuous on $[a, b]$

$\therefore \quad h = f - g$ is also continuous on $[a, b]$

Also $h(a)$ and $h(b)$ have opposite signs.

$\therefore$ By Intermediate Value Theorem, $\exists$ at least one $c \in (a, b)$ such that $h(c) = 0$.

$\Rightarrow \quad f(c) - g(c) = 0 \quad$ or $\quad f(c) = g(c)$.

Example 33. *Using Intermediate value theorem, show that $x^{41} + x + 1 = 0$ has a real root.*

Sol. $f(x) = x^{41} + x + 1$ being a polynomial is continuous for all x.

Also $\quad f(-1) = (-1)^{41} + (-1) + 1 = -1 - 1 + 1 = -1$

and $\quad f(0) = 0 + 0 + 1 = 1$

so that $f(-1)$ and $f(0)$ have opposite signs.

$\therefore$ By Intermediate value theorem, $\exists$ a real number $c \in (-1, 0)$ such that $f(c) = 0$.

$\Rightarrow \quad c$ is a real root of $f(x) = x^{41} + x + 1 = 0$.

7

The Derivative and Mean Value Theorems

7.1. DERIVATIVE OF A FUNCTION AT A POINT

Let $f : [a, b] \to \mathbf{R}$ be a function and $c \in (a, b)$, then f is said to be derivable (or differentiable) at c, if

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} \quad \text{or} \quad \lim_{h \to 0} \frac{f(c + h) - f(c)}{h}$$

exists. The limit, in case it exists, is called the derivative or the differential ~~co-efficient of the~~ function f at $x = c$ and is denoted by $f'(c)$.

Left-hand Derivative

Let $f : [a, b] \to \mathbf{R}$ be a function and $c \in (a, b)$.

If $\quad \lim_{x \to c-} \frac{f(x) - f(c)}{x - c} \quad$ or $\quad \lim_{h \to 0+} \frac{f(c - h) - f(c)}{-h}$ exists, then this limit is called the left-hand derivative of f at c and is denoted by $f'(c - 0)$ or $f'(c -)$ or $Lf'(c)$.

Right-hand Derivative

Let $f : [a, b] \to \mathbf{R}$ be a function and $c \in (a, b)$.

If $\lim_{x \to c+} \frac{f(x) - f(c)}{x - c}$ or $\lim_{h \to 0} \frac{f(c + h) - f(c)}{h}$ exists, then this limit is called the right-hand derivative of f at c and is denoted by $f'(c + 0)$ or $f'(c +)$ or $Rf'(c)$.

Clearly, the derivative $f'(c)$ exists iff $Lf'(c) = Rf'(c)$.

Note. When we say that $\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ exists, we mean that it is a real number *i.e.*, finite.

7.2. DERIVABILITY IN AN INTERVAL

A function $f : [a, b] \to \mathbf{R}$ is said to be derivable in the *open interval* (a, b) iff $f'(c)$ exists for each c such that $a < c < b$.

A function $f : [a, b] \to \mathbf{R}$ is said to be derivable in the *closed interval* $[a, b]$ if

(*i*) $f'(c)$ exists for each $c \in (a, b)$, $\quad$ (*ii*) $Rf'(a)$ exists, and

(*iii*) $Lf'(b)$ exists.

A function $f : \mathbf{I} \to \mathbf{R}$ is said to be derivable on I if f is derivable at every point of I.

7.3. DERIVATIVE OF A FUNCTION

Let f be a function whose domain is an interval I. Let $\mathbf{I}_1$ be the set of all those points x of I at which $f'(x)$ exists. Clearly $\mathbf{I}_1 \subset \mathbf{I}$. If $\mathbf{I}_1 \neq \phi$, then the function f' with domain $\mathbf{I}_1$ is called the derivative of f.

7.4. DERIVABILITY AND CONTINUITY

Theorem. *If a function is derivable at a point, then it is continuous at that point.*

Proof. Let f be derivable at $x = a$, then $\lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$ exists and is $f'(a)$.

Now $$f(x) - f(a) = \frac{f(x) - f(a)}{x - a} \times (x - a),\ x \neq a$$

$$\therefore \quad \lim_{x \to a} [f(x) - f(a)] = \lim_{x \to a} \left[\frac{f(x) - f(a)}{x - a} \times (x - a)\right]$$

$$= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \times \lim_{x \to a} (x - a) = f'(a) \times 0 = 0$$

$$\Rightarrow \quad \lim_{x \to a} f(x) - \lim_{x \to a} f(a) = 0$$

$$\Rightarrow \quad \lim_{x \to a} f(x) = \lim_{x \to a} f(a) = f(a) \qquad [\because\ f(a) \text{ is a constant}]$$

Since $$\lim_{x \to a} f(x) = f(a)$$

$\therefore$ f is continuous at $x = a$.

Hence *derivability at a point* $\Rightarrow$ *continuity at that point.*

Note 1. *The converse of this theorem need not be true.* That is, if f is continuous at a point, then it is not necessary that f must be derivable at that point.

For example, consider the modulus function $f(x) = |x|$ at $x = 0$.

We have $$f(0) = |0| = 0$$

$$\lim_{x \to 0-} f(x) = \lim_{x \to 0-} |x| = \lim_{h \to 0+} |-h| = \lim_{h \to 0+} h = 0$$

$$\lim_{x \to 0+} f(x) = \lim_{x \to 0+} |x| = \lim_{h \to 0+} |h| = \lim_{h \to 0+} h = 0$$

Since $$\lim_{x \to 0-} f(x) = \lim_{x \to 0+} f(x) = f(0)$$

$\therefore$ *$f(x)$ is continuous at $x = 0$.*

Now $$Lf'(0) = \lim_{x \to 0-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0-} \frac{|x|}{x} = \lim_{x \to 0-} \frac{-x}{x} = \lim_{x \to 0-} (-1) = -1$$

and $$Rf'(0) = \lim_{x \to 0+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0+} \frac{|x|}{x} = \lim_{x \to 0+} \frac{x}{x} = \lim_{x \to 0+} 1 = 1$$

Since $$Lf'(0) \neq Rf'(0)$$

$\therefore$ *$f(x)$ is not derivable at $x = 0$.*

☞**Note 2.** If f is not continuous at a, it cannot be derivable at a.

ILLUSTRATIVE EXAMPLES

Example 1. (*i*) *Determine if $f(x)$ has a derivative at $x = 0$ when*

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & \text{when } x \neq 0 \\ 0, & \text{when } x = 0 \end{cases}$$

(*ii*) *Examine the function* $f(x) = \begin{cases} x^2 \cos \dfrac{1}{x}, & \text{when } x \neq 0 \\ 0, & \text{when } x = 0 \end{cases}$ *for the existence of derivative at $x = 0$.*

Sol. (*i*) $$Lf'(0) = \lim_{x \to 0-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0-} \frac{x^2 \sin \frac{1}{x} - 0}{x} \qquad |\ \textbf{Put } x = 0 - h, h > 0$$

$$= \lim_{h \to 0+} \frac{(0 - h)^2 \sin\left(\frac{1}{0 - h}\right)}{-h} = \lim_{h \to 0+} -h \sin\left(\frac{1}{-h}\right) = \lim_{h \to 0+} h \sin \frac{1}{h}$$

$= 0 \times$ a finite quantity between -1 and $1 = 0$

and $$Rf'(0) = \lim_{x \to 0+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0+} \frac{x^2 \sin \frac{1}{x} - 0}{x - 0} \qquad |\ \textbf{Put } x = 0 + h, h > 0$$

$$= \lim_{h \to 0+} \frac{(0 + h)^2 \sin\left(\frac{1}{0 + h}\right)}{0 + h} = \lim_{h \to 0+} h \sin \frac{1}{h}$$

$= 0 \times$ a finite quantity between -1 and $1 = 0$.

$\because$ L.H.D. and R.H.D. both exist and are equal.

Thus $f'(0)$ also exists and is equal to zero, *i.e.*, $f'(0) = 0$.

(*ii*) *Please try yourself.* [**Ans.** $f'(0) = 0$]

Example 2. *Show that the function* $f(x) = \begin{cases} x^2 - 1, & \text{when } x \geq 1 \\ 1 - x, & \text{when } x < 1 \end{cases}$ *has no derivative at $x = 1$.*

Sol. Here $f(1) = (1)^2 - 1 = 0$.

$$Lf'(1) = \lim_{x \to 1-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1-} \frac{(1 - x) - 0}{x - 1} \qquad |\ \textbf{Put } x = 1 - h, h > 0$$

$$= \lim_{h \to 0+} \frac{1 - (1 - h)}{(1 - h) - 1} = \lim_{h \to 0+} \frac{h}{-h} = \lim_{h \to 0+} (-1) = -1$$

$$Rf'(1) = \lim_{x \to 1+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1+} \frac{(x^2 - 1) - 0}{x - 1} \qquad |\ \textbf{Put } x = 1 + h, h > 0$$

$$= \lim_{h \to 0+} \frac{(1 + h)^2 - 1}{1 + h - 1} = \lim_{h \to 0+} \frac{h^2 + 2h}{h} = \lim_{h \to 0+} (h + 2) = 2.$$

Since L.H.D. $\neq$ R.H.D., $\therefore$ derivative at $x = 1$ does not exist *i.e.*, $f(x)$ is not derivable at $x = 1$.

Example 3. *Show that the function $f(x)$, where* $f(x) = \begin{cases} x, & 0 < x \leq 1 \\ x - 1, & 1 < x \leq 2 \end{cases}$ *has no derivative at $x = 1$.*

Sol. *Please try yourself.*

Example 4. *Show that the following function is continuous at $x = 1$, for all values of p.*

$$f(x) = \begin{cases} px + 1, & \text{if } x \ge 1 \\ x^2 + p, & \text{if } x < 1 \end{cases}$$

Find the left-hand and right-hand derivatives of $f(x)$ at $x = 1$. Hence find the condition for the existence of the derivative at that point.

Sol. When $x = 1$, we have $f(x) = px + 1$, so that $f(1) = p.1 + 1 = p + 1$

Now $\lim_{x \to 1-} f(x) = \lim_{x \to 1-} (x^2 + p)$ | **Put x = 1 – h, h > 0**

$= \lim_{h \to 0+} \{(1-h)^2 + p\} = 1 + p$

and $\lim_{x \to 1+} f(x) = \lim_{x \to 1+} (px + 1)$ | **Put x = 1 + h, h > 0**

$= \lim_{h \to 0+} \{p(1+h) + 1\} = p + 1$

Since L.H.L. = R.H.L. $= p + 1$, $\therefore$ $\lim_{x \to 1} f(x)$ exists and $= p + 1$.

Now $\because$ $\lim_{x \to 1} f(x) = p + 1 = f(1)$, $\therefore$ $f(x)$ is continuous at $x = 1$, for all values of p.

Further, $$Lf'(1) = \lim_{x \to 1-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1-} \frac{(x^2 + p) - (p + 1)}{x - 1}$$

$= \lim_{x \to 1-} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1-} (x + 1)$ | **Put x = 1 – h, h > 0**

$= \lim_{h \to 0+} \{1 - h + 1\} = 2$

and $$Rf'(1) = \lim_{x \to 1+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1+} \frac{(px + 1) - (p + 1)}{x - 1}$$

$= \lim_{x \to 1+} \frac{px - p}{x - 1} = \lim_{x \to 1+} \frac{p(x-1)}{x - 1} = \lim_{x \to 1+} (p) = p$

For the existence of derivative at $x = 1$, we must have

L.H.D. = R.H.D. or $2 = p$

Thus the condition for the existence of $f'(1)$ is $p = 2$.

Example 5. *Prove that the function $f(x) = | x - 1 |$ is continuous at $x = 1$ but not derivable at $x = 1$.*

Sol. Continuity at x = 1.

L.H.L. $= \lim_{x \to 1-} f(x) = \lim_{x \to 1-} | x - 1 |$ | **Put x = 1 – h, h > 0**

$= \lim_{h \to 0+} | 1 - h - 1 | = \lim_{h \to 0+} | -h | = \lim_{h \to 0+} h = 0$

R.H.L. $= \lim_{x \to 1+} f(x) = \lim_{x \to 1+} | x - 1 |$ | **Put x = 1 + h, h > 0**

$= \lim_{h \to 0+} | 1 + h - 1 | = \lim_{h \to 0+} | h | = \lim_{h \to 0+} h = 0$

Since L.H.L. = R.H.L. ; $\therefore$ $\lim_{x \to 1} f(x)$ exists and is = 0

Also $f(1) = | 1 - 1 | = | 0 | = 0$.

$\therefore$ $\lim_{x \to 1} f(x) = 0 = f(1)$,

$\therefore$ $f(x)$ is **continuous** at $x = 1$.

Derivability at x = 1.

We have $f(1) = | 1 - 1 | = | 0 | = 0$.

Now L.H.D. $= f'(1 - 0) = \lim_{x \to 1-0} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1-0} \frac{|x - 1| - 0}{x - 1}$ | **Put x = 1 – h**

$= \lim_{h \to 0} \frac{|1 - h - 1| - 0}{1 - h - 1} = \lim_{h \to 0} \frac{|-h|}{-h} = \lim_{h \to 0} \frac{h}{-h} = \lim_{h \to 0} (-1) = -1$

R.H.D. $= f'(1 + 0) = \lim_{h \to 1+0} \frac{f(x) - f(1)}{x - 1} = \lim_{h \to 1+0} \frac{|x - 1| - 0}{x - 1}$ | **Put x = 1 + h**

$= \lim_{h \to 0} \frac{|1 + h - 1| - 0}{1 + h - 1} = \lim_{h \to 0} \frac{|h|}{h} = \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} (1) = 1$.

Since L.H.D. $\ne$ R.H.D. $\therefore$ $f(x)$ is not derivable at $x = 1$.

Example 6. *Show that $f(x) = x \sin \frac{1}{x}$ for $x \ne 0$, and $f(0) = 0$ is continuous but not derivable at $x = 0$.*

Sol. Continuity at x = 0.

L.H.L. $= \lim_{x \to 0-0} f(x) = \lim_{x \to 0-0} x \sin \frac{1}{x}$ | **Put x = 0 – h**

$= \lim_{h \to 0} (0 - h) \sin \left(\frac{1}{0 - h} \right) = \lim_{h \to 0} \left(h \sin \frac{1}{h} \right)$

= 0 × a finite quantity between – 1 and 1 = 0.

R.H.L. $= \lim_{x \to 0+0} f(x) = \lim_{x \to 0+0} x \sin \frac{1}{x}$ | **Put x = 0 + h**

$= \lim_{h \to 0} (0 + h) \sin \left(\frac{1}{0 + h} \right) = \lim_{h \to 0} \left(h \sin \frac{1}{h} \right)$

= 0 × a finite quantity between – 1 and 1 = 0.

Since L.H.L. = R.H.L. = 0, $\therefore$ $\lim_{x \to 0} f(x) = 0$.

Now $\lim_{x \to 0} f(x) = 0 = f(0)$; $\therefore$ $f(x)$ is **continuous** at $x = 0$.

Derivability at x = 0.

L.H.D. $= \lim_{x \to 0-0} \frac{f(x) - f(1)}{x - 0} = \lim_{x \to 0-0} \frac{x \sin \frac{1}{x} - 0}{x}$ | **Put x = 0 – h**

$= \lim_{h \to 0} \frac{(0 - h) \sin \left(\frac{1}{0 - h} \right)}{0 - h} = \lim_{h \to 0} - \sin \frac{1}{h}$ which does not exist

$\left[\because \text{ as } h \to 0, \sin \frac{1}{h} \text{ oscillates between } -1 \text{ and } 1 \text{ and} \right]$

$$\text{R.H.D.} = \lim_{x \to 0+0} \frac{f(x)-f(0)}{x-0} = \lim_{x \to 0+0} \frac{x \sin \frac{1}{x} - 0}{x} \qquad | \textbf{ Put } x = 0 + h$$

$$= \lim_{h \to 0} \frac{(0+h)\sin \frac{1}{h}}{0+h} = \lim_{h \to 0} \sin \frac{1}{h} \text{ which does not exist.}$$

Since neither the left hand derivative nor the right hand derivative exists at $x = 0$.

$\therefore$ $f(x)$ has no derivative at $x = 0$.

Example 7. *Test the following function for continuity and derivability at $x = 0$.*

$$f(x) = x \cos \frac{1}{x} \text{ when } x \neq 0$$
$$= 0 \text{ when } x = 0.$$

Sol. *Please try yourself.* **[Ans. Continuous but not derivable]**

Example 8. *Prove that $f(x) = x \tan^{-1} \frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$ is continuous but not derivable at $x = 0$.*

Sol.
$$\text{L.H.L.} = \lim_{x \to 0-0} f(x) = \lim_{x \to 0-0} x \tan^{-1} \frac{1}{x} \qquad | \textbf{ Put } x = 0 - h$$

$$= \lim_{h \to 0} (0-h) \tan^{-1}\left(\frac{1}{0-h}\right) = \lim_{h \to 0} h \tan^{-1} \frac{1}{h} = 0 \times \frac{\pi}{2} = 0.$$

$$\text{R.H.L.} = \lim_{x \to 0+0} f(x) = \lim_{x \to 0+0} x \tan^{-1} \frac{1}{x} \qquad | \textbf{ Put } x = 0 + h$$

$$= \lim_{h \to 0} (0+h) \tan^{-1}\left(\frac{1}{0+h}\right) = \lim_{h \to 0} h \tan^{-1} \frac{1}{h} = 0 \times \frac{\pi}{2} = 0.$$

Since L.H.L. = R.H.L. $\therefore$ $\lim_{x \to 0} f(x)$ exists and is = 0.

Now $\lim_{x \to 0} f(x) = 0 = f(0)$

$\therefore$ $f(x)$ is **continuous** at $x = 0$.

Now
$$\text{L.H.D.} = \lim_{x \to 0-0} \frac{f(x)-f(0)}{x-0} = \lim_{x \to 0-0} \frac{x \tan^{-1} \frac{1}{x} - 0}{x-0} \qquad | \textbf{ Put } x = 0 - h$$

$$= \lim_{h \to 0} \frac{(0-h)\tan^{-1}\left(\frac{1}{0-h}\right)}{0-h} = \lim_{h \to 0} \frac{h \tan^{-1} \frac{1}{h}}{-h}$$

$$= \lim_{h \to 0} \left(-\tan^{-1} \frac{1}{h}\right) = -\tan^{-1}(\infty) = -\frac{\pi}{2}.$$

$$\text{R.H.D.} = \lim_{x \to 0+0} \frac{f(x)-f(0)}{x-0} = \lim_{x \to 0+0} \frac{x \tan^{-1} \frac{1}{x} - 0}{x-0} \qquad | \textbf{ Put } x = 0 + h$$

$$= \lim_{h \to 0} \frac{(0+h)\tan^{-1}\left(\frac{1}{0+h}\right)}{(0+h)-0} = \lim_{h \to 0} \frac{h \tan^{-1} \frac{1}{h}}{h} = \lim_{h \to 0} \tan^{-1} \frac{1}{h} = \frac{\pi}{2}$$

Since L.H.D. $\neq$ R.H.D.

$\therefore$ $f(x)$ is **not derivable** at $x = 0$.

Example 9. *Discuss the continuity and differentiability of the function*

$$f(x) = x[1 + \tfrac{1}{3} \sin (\log x^2)] \text{ when } x \neq 0$$
$$= 0 \text{ when } x = 0 \text{ at the origin.}$$

Sol. [Continuity]

$$\text{L.H.S.} = \lim_{x \to 0-0} f(x) = \lim_{x \to 0-0} x\left[1 + \frac{1}{3}\sin(\log x^2)\right] \qquad | \textbf{ Put } x = 0 - h$$

$$= \lim_{h \to 0} (0-h)[1 + \tfrac{1}{3} \sin \{\log (0-h)^2\}] = \lim_{h \to 0} (-h)[1 + \tfrac{1}{3} \sin (\log h^2)]$$

$$= 0 \times [1 + \tfrac{1}{3} \text{ (a finite quantity between } -1 \text{ and } 1)] = 0.$$

[$\because$ as $h \to 0$, $\log h^2 \to -\infty$ and so $\sin (\log h^2)$ does not tend to any unique and definite limit as $h \to 0$ but oscillates between -1 and 1.]

$$\text{R.H.L.} = \lim_{x \to 0+0} f(x) = \lim_{x \to 0+0} x [1 + \tfrac{1}{3} \sin (\log h^2)] \qquad | \textbf{ Put } x = 0 + h$$

$$= \lim_{h \to 0} (0+h)[1 + \tfrac{1}{3} \sin \{\log (0+h)^2\}] = \lim_{h \to 0} h[1 + \tfrac{1}{3} \sin (\log h^2)]$$

$$= 0 \times [1 + \tfrac{1}{3} \times \text{(a finite quantity between } -1 \text{ and } 1)] = 0.$$

$\because$ L.H.L. = R.H.L., $\therefore$ $\lim_{x \to 0} f(x)$ exists and is = 0.

Now $\lim_{x \to 0} f(x) = 0 = f(0)$.

$\therefore$ **$f(x)$ is continuous at $x = 0$.**

Derivability

$$\text{L.H.D.} = \lim_{x \to 0-0} \frac{f(x)-f(0)}{x-0} = \lim_{x \to 0-0} \frac{x[1 + \frac{1}{3} \sin (\log x^2)] - 0}{x} \qquad | \textbf{ Put } x = 0 - h$$

$$= \lim_{h \to 0} \frac{(0-h)[1 + \frac{1}{3} \sin \{\log (0-h)^2\}]}{0-h} = \lim_{h \to 0} [1 + \tfrac{1}{3} \sin (\log h^2)]$$

which does not exist.

[$\because$ as $h \to 0$, $\log h^2 \to -\infty$ and so $\sin (\log h^2)$ oscillates between -1 and 1 and does not tend to any unique and definite value].

Similarly, the R.H.D. also does not exist.

Hence $f(x)$ is **not derivable** at $x = 0$.

Thus $f(x)$ is **continuous but not derivable** at $x = 0$.

Example 10. *Prove that $f(x) = x^2 \sin \frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$ is continuous and differentiable at $x = 0$. Is $f'(x)$ continuous at $x = 0$?*

Sol. Continuity of f(x) at x = 0.

$$\text{L.H.L.} = \lim_{x \to 0-0} f(x) = \lim_{x \to 0-0} x^2 \sin\frac{1}{x} \qquad | \textbf{ Put } \mathbf{x = 0 - h}$$

$$= \lim_{h \to 0} (0-h)^2 \sin\left(\frac{1}{0-h}\right) = \lim_{h \to 0}\left(-h^2 \sin\frac{1}{h}\right)$$

$= 0 \times$ a finite quantity between -1 and $1 = 0$.

$$\text{R.H.L.} = \lim_{x \to 0+0} f(x) = \lim_{x \to 0+0} x^2 \sin\frac{1}{x} \qquad | \textbf{ Put } \mathbf{x = 0 + h}$$

$$= \lim_{h \to 0} (0+h)^2 \sin\left(\frac{1}{0+h}\right) = \lim_{h \to 0} h^2 \sin\frac{1}{h}$$

$= 0 \times$ a finite quantity between -1 and $1 = 0$.

$\because$ L.H.L. = R.H.L. = 0, $\therefore \lim_{x \to 0} f(x) = 0$.

Now $\lim_{x \to 0} f(x) = 0 = f(0)$.

$\therefore$ $f(x)$ **is continuous at** $x = 0$.

[Derivability of f(x) at x = 0]

$$\text{L.H.D.} = \lim_{x \to 0-0} \frac{f(x)-f(0)}{x-0} = \lim_{x \to 0-0} \frac{x^2 \sin\frac{1}{x} - 0}{x} \qquad | \textbf{ Put } \mathbf{x = 0 - h}$$

$$= \lim_{h \to 0} \frac{(0-h)^2 \sin\left(\frac{1}{0-h}\right)}{0-h} = \lim_{h \to 0} \frac{-h^2 \sin\frac{1}{h}}{-h} = \lim_{h \to 0} h \sin\frac{1}{h}$$

$= 0 \times$ a finite quantity between -1 and $1 = 0$.

$$\text{R.H.D.} = \lim_{x \to 0+0} \frac{f(x)-f(0)}{x-0} = \lim_{x \to 0+0} \frac{x^2 \sin\frac{1}{x} - 0}{x} \qquad | \textbf{ Put } \mathbf{x = 0 + h}$$

$$= \lim_{h \to 0} \frac{(0+h)^2 \sin\left(\frac{1}{0+h}\right)}{0+h} = \lim_{h \to 0} h \sin\frac{1}{h}$$

$= 0 \times$ a finite quantity between -1 and $1 = 0$.

$\because$ L.H.D. = R.H.D. = 0.

$\therefore$ $f(x)$ **is derivable at** $x = 0$ and $f'(0) = 0$.

[Continuity of f ′(x) at x = 0.]

Now $f'(x) = 2x \sin\frac{1}{x} + x^2 \cos\frac{1}{x}\left(-\frac{1}{x^2}\right) = 2x \sin\frac{1}{x} - \cos\frac{1}{x}$

$$\text{L.H.L.} = \lim_{x \to 0-0} f'(x) = \lim_{x \to 0-0}\left(2x \sin\frac{1}{x} - \cos\frac{1}{x}\right) \qquad | \textbf{ Put } \mathbf{x = 0 - h}$$

$$= \lim_{h \to 0}\left[2(0-h)\sin\left(\frac{1}{0-h}\right) - \cos\left(\frac{1}{0-h}\right)\right] = \lim_{h \to 0}\left[2h \sin\frac{1}{h} - \cos\frac{1}{h}\right]$$

$$= 2 \lim_{h \to 0} h \,.\, \lim_{h \to 0} \sin\frac{1}{h} - \lim_{h \to 0} \cos\frac{1}{h} = 0 - \lim_{h \to 0} \cos\frac{1}{h}$$

which does not exist.

[$\because$ *as* $h \to 0$, $\cos\frac{1}{h}$ can have any value between -1 and 1 and does not tend to any unique and definite limit.]

Similarly, R.H.L. does not exist.

Now since neither the L.H.L. nor the R.H.L. exists.

$\therefore$ $\lim_{x \to 0} f'(x)$ **does not exist and hence f ′ (x) is not continuous at x = 0.**

Example 11. *Are the following functions continuous and derivable ?*

(a) $f(x) = 1 + x$ *if* $x < 2$ *and* $f(x) = 5 - x$ *if* $x \geq 2$ *at the point* $x = 2$.

(b) $f(x) = 2 + x$ *for* $x \geq 0$ *and* $f(x) = 2 - x$ *for* $x < 0$ *at the origin.*

Sol. *(a)* $\text{L.H.L.} = \lim_{x \to 2-0} f(x) = \lim_{x \to 2-0} (1 + x) \qquad | \textbf{ Put } \mathbf{x = 2 - h}$

$$= \lim_{h \to 0} [1 + (2-h)] = \lim_{h \to 0} (3-h) = 3$$

$$\text{R.H.L.} = \lim_{x \to 2+0} f(x) = \lim_{x \to 2+0} (5-x) \qquad | \textbf{ Put } \mathbf{x = 2 + h}$$

$$= \lim_{h \to 0} [5 - (2+h)] = \lim_{h \to 0} (3-h) = 3$$

$\because$ L.H.L. = R.H.L. = 3, $\therefore \lim_{x \to 2} f(x) = 3$,

Now putting $x = 2$ in $f(x) = 1 + x$, we get $f(2) = 1 + 2 = 3$.

As $\lim_{x \to 2} f(x) = 3 = f(2)$ $\therefore$ $f(x)$ **is continuous at x = 2**

Again, $\text{L.H.D.} = \lim_{x \to 2-0} \frac{f(x)-f(2)}{x-2} = \lim_{x \to 2-0} \frac{(1+x)-3}{x-2} \qquad | \textbf{ Put } \mathbf{x = 2 - h}$

$$= \lim_{h \to 0} \frac{(1+2-h)}{2-h-2} = \lim_{h \to 0} \frac{-h}{-h} = \lim_{h \to 0} (1) = 1$$

$$\text{R.H.D.} = \lim_{x \to 2+0} \frac{f(x)-f(2)}{x-2} = \lim_{x \to 2+0} \frac{(5-x)-3}{x-2} = \lim_{x \to 2+0} \frac{2-x}{x-2} \qquad | \textbf{ Put } \mathbf{x = 2 + h}$$

$$= \lim_{h \to 0} \frac{2-(2+h)}{2+h-2} = \lim_{h \to 0} \frac{-h}{h} = \lim_{h \to 0} (-1) = -1$$

$\because$ L.H.D. $\neq$ R.H.D.

$\therefore$ $f(x)$ is **not derivable** at $x = 2$.

Hence $f(x)$ is **continuous but not derivable** at $x = 2$.

(b) Please try yourself.

Example 12. *A function f(x) is defined as follows :*

$$f(x) = 1 + \sin x \qquad \text{for } 0 < x < \frac{\pi}{2}$$

$$f(x) = 2 + \left(x - \frac{\pi}{2}\right)^2 \quad \text{for } x \geq \frac{\pi}{2}.$$

Examine its continuity and derivability at $x = \frac{\pi}{2}$.

Sol. Continuity at $\mathbf{x = \frac{\pi}{2}}$

$$\text{L.H.L.} = \lim_{x \to \frac{\pi}{2} - 0} f(x) = \lim_{x \to \frac{\pi}{2} - 0} (1 + \sin x) \qquad \Big| \text{ Put } x = \frac{\pi}{2} - h$$

$$= \lim_{h \to 0} \left[1 + \sin\left(\frac{\pi}{2} - h\right)\right] = \lim_{h \to 0} (1 + \cos h) = 1 + 1 = 2$$

$$\text{R.H.L.} = \lim_{x \to \frac{\pi}{2} + 0} f(x) = \lim_{x \to \frac{\pi}{2} + 0} \left[2 + \left(x - \frac{\pi}{2}\right)^2\right] \qquad \Big| \text{ Put } x = \frac{\pi}{2} + h$$

$$= \lim_{h \to 0} \left[2 + \left(\frac{\pi}{2} + h - \frac{\pi}{2}\right)^2\right] = \lim_{h \to 0} (2 + h^2) = 2$$

$\because$ L.H.L. = R.H.L. = 2 $\therefore$ $\lim_{x \to \frac{\pi}{2}} f(x)$ exists and is = 2

Also $f\left(\frac{\pi}{2}\right) = 2 + \left(\frac{\pi}{2} - \frac{\pi}{2}\right)^2 = 2$

$\because$ $\lim_{x \to \frac{\pi}{2}} f(x) = f\left(\frac{\pi}{2}\right)$ $\therefore$ $f(x)$ is continuous at $x = \frac{\pi}{2}$.

Differentiability at $\mathbf{x = \frac{\pi}{2}}$.

$$\text{L.H.D.} = \lim_{x \to \frac{\pi}{2} - 0} \frac{f(x) - f\left(\frac{\pi}{2}\right)}{x - \frac{\pi}{2}} = \lim_{x \to \frac{\pi}{2} - 0} \frac{1 + \sin x - 2}{x - \frac{\pi}{2}} = \lim_{x \to \frac{\pi}{2} - 0} \frac{\sin x - 1}{x - \frac{\pi}{2}}$$

$$\Big| \text{ Put } x = \frac{\pi}{2} - h$$

$$= \lim_{h \to 0} \frac{\sin\left(\frac{\pi}{2} - h\right) - 1}{\left(\frac{\pi}{2} - h\right) - \frac{\pi}{2}} = \lim_{h \to 0} \frac{\cos h - 1}{-h}$$

$$= \lim_{h \to 0} \frac{1 - \cos h}{h} \qquad \Big| \text{ Form } \frac{0}{0}$$

$$= \lim_{h \to 0} \frac{\sin h}{1} \qquad | \text{ L' Hospital's Rule}$$

$$= 0$$

$$\text{R.H.D.} = \lim_{x \to \frac{\pi}{2} + 0} \frac{f(x) - f\left(\frac{\pi}{2}\right)}{x - \frac{\pi}{2}} = \lim_{x \to \frac{\pi}{2} + 0} \frac{2 + \left(x - \frac{\pi}{2}\right)^2 - 2}{x - \frac{\pi}{2}} = \lim_{x \to \frac{\pi}{2} + 0} \left(x - \frac{\pi}{2}\right)$$

$$\Big| \text{ Put } x = \frac{\pi}{2} + h$$

$$= \lim_{h \to 0} \left(\frac{\pi}{2} + h - \frac{\pi}{2}\right) = \lim_{h \to 0} h = 0$$

$\because$ L.H.D. = R.H.D. = 0

$\therefore$ $f(x)$ is differentiable at $x = \frac{\pi}{2}$ and $f'\left(\frac{\pi}{2}\right) = 0$.

Example 13. *Show that the function defined by* $f(x) = |x| + |x - 1|$ *is continuous but not derivable at* $x = 0$ *and* $x = 1$.

Sol. When $x < 0$, $\quad f(x) = -x - (x - 1) = 1 - 2x$

when $x = 0$, $\quad f(x) = 0 + 1 = 1$

when $0 < x < 1$, $\quad f(x) = x - (x - 1) = 1$

when $x = 1$, $\quad f(x) = 1 + 0 = 1$

when $x > 1$, $\quad f(x) = x + x - 1 = 2x - 1$

$$\therefore \quad f(x) = \begin{cases} 1 - 2x & \text{if } x < 0 \\ 1 & \text{if } 0 \leq x \leq 1 \\ 2x - 1 & \text{if } x > 1 \end{cases}$$

Continuity at x = 0

$$\text{L.H.L.} = \lim_{x \to 0-0} f(x) = \lim_{x \to 0-0} (1 - 2x) = \lim_{h \to 0} 1 - 2(0 - h) = 1.$$

$$\text{R.H.L.} = \lim_{x \to 0+0} f(x) = \lim_{x \to 0+0} 1 = 1$$

$\therefore$ $\lim_{x \to 0} f(x) = 1 = f(0)$ $\Rightarrow$ $f(x)$ is continuous at $x = 0$.

Continuity at x = 1

$$\text{L.H.L.} = \lim_{x \to 1-0} f(x) = \lim_{x \to 1-0} 1 = 1$$

$$\text{R.H.L.} = \lim_{x \to 1+0} f(x) = \lim_{x \to 1+0} (2x - 1) = \lim_{h \to 0} 2(1 + h) - 1 = 1$$

$\therefore$ $\lim_{x \to 1} f(x) = 1 = f(1)$ $\Rightarrow$ $f(x)$ is continuous at $x = 1$.

Derivability at x = 0

$$\text{L.H.D.} = \lim_{x \to 0-0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0-0} \frac{(1 - 2x) - 1}{x} = \lim_{x \to 0-0} -2 = -2$$

$$\text{R.H.D.} = \lim_{x \to 0+0} \frac{f(x)-f(0)}{x-0} = \lim_{x \to 0+0} \frac{1-1}{x} = \lim_{x \to 0+0} 0 = 0$$

L.H.D. ≠ R.H.D. ⇒ $f(x)$ is not derivable at $x = 0$.

Derivability at x = 1

$$\text{L.H.D.} = \lim_{x \to 1-0} \frac{f(x)-f(1)}{x-1} = \lim_{x \to 1-0} \frac{1-1}{x-1} = \lim_{x \to 1-0} 0 = 0$$

$$\text{R.H.D.} = \lim_{x \to 1+0} \frac{f(x)-f(1)}{x-1} = \lim_{x \to 1+0} \frac{(2x-1)-1}{x-1} = \lim_{x \to 1+0} \frac{2(x-1)}{x-1} = \lim_{x \to 1+0} 2 = 2$$

L.H.D. ≠ R.H.D. ⇒ $f(x)$ is not derivable at $x = 1$.

Example 14. *(i) Show that the function $f(x)$ defined by $f(x) = |x-1| + 2|x-2|$ is continuous but not derivable at 1 and 2.*

(ii) Show that $f(x) = |x+1| + |x-1|$ is continuous but not derivable at $x = -1, 1$.

(iii) Let f be defined on R by setting $f(x) = |x-2| + |x| + |x+2|$ for all $x \in R$.

Show that f is not derivable at $x = -2$, $x = 0$ and $x = 2$, and is derivable at every other point.

Sol. *Please try yourself.*

Hint. *(i)* when $x < 1$

	$f(x) = (1-x) + 2(2-x) = 5-3x$
when $x = 1$,	$f(x) = 0 + 2 \times 1 = 2$
when $1 < x < 2$,	$f(x) = (x-1) + 2(2-x) = 3-x$
when $x = 2$,	$f(x) = 1 + 0 = 1$
when $x > 2$,	$f(x) = (x-1) + 2(x-2) = 3x-5$

$$\therefore \quad f(x) = \begin{cases} 5-3x, & \text{if } x \le 1 \\ 3-x, & \text{if } 1 < x \le 2 \\ 3x-5, & \text{if } x > 2 \end{cases}$$

(ii) when $x < -1$,	$f(x) = -(x+1) + (1-x) = -2x$
when $x = -1$,	$f(x) = 0 + 2 = 2$
when $-1 < x < 1$,	$f(x) = (x+1) + (1-x) = 2$
when $x = 1$,	$f(x) = 2 + 0 = 2$
when $x > 1$,	$f(x) = (x+1) + (x-1) = 2x$

$$\therefore \quad f(x) = \begin{cases} -2x, & \text{if } x \le -1 \\ 2, & \text{if } -1 < x < 1 \\ 2x, & \text{if } x \ge 1 \end{cases}$$

(iii) when $x < -2$,	$f(x) = (2-x) + (-x) + (-x-2) = -3x$
when $x = -2$,	$f(x) = 4 + 2 + 0 = 6$
when $-2 < x < 0$,	$f(x) = (2-x) + (-x) + (x+2) = 4-x$
when $x = 0$,	$f(x) = 2 + 0 + 2 = 4$
when $0 < x < 2$,	$f(x) = (2-x) + (x) + (x+2) = 4+x$
when $x = 2$,	$f(x) = 0 + 2 + 4 = 6$
when $x > 2$,	$f(x) = (x-2) + (x) + (x+2) = 3x$

$$\therefore \quad f(x) = \begin{cases} -3x, & \text{if } x \le -2 \\ 4-x, & \text{if } -2 < x < 0 \\ 4+x, & \text{if } 0 \le x \le 2 \\ 3x, & \text{if } x > 2 \end{cases}$$

Example 15. *Let $f(x) = x \cdot \dfrac{e^{\frac{1}{x}} - e^{-\frac{1}{x}}}{e^{\frac{1}{x}} + e^{-\frac{1}{x}}}$; $x \ne 0$, $f(0) = 0$. Show that f is continuous but not differentiable at $x = 0$.*

Sol. Continuity at x = 0

$$\text{L.H.L.} = \lim_{x \to 0-0} f(x) = \lim_{x \to 0-0} x \frac{e^{\frac{1}{x}} - e^{-\frac{1}{x}}}{e^{\frac{1}{x}} + e^{-\frac{1}{x}}} = \lim_{h \to 0} -h \cdot \frac{e^{-\frac{1}{h}} - e^{\frac{1}{h}}}{e^{-\frac{1}{h}} + e^{\frac{1}{h}}}$$

| Putting $x = 0 - h = -h$

$$= \lim_{h \to 0} -h \cdot \frac{e^{-\frac{2}{h}} - 1}{e^{-\frac{2}{h}} + 1} = 0 \times \frac{0-1}{0+1} = 0 \qquad \left[\because \lim_{h \to 0} e^{-\frac{2}{h}} = 0\right]$$

$$\text{R.H.L.} = \lim_{x \to 0+0} f(x) = \lim_{x \to 0+0} x \frac{e^{\frac{1}{x}} - e^{-\frac{1}{x}}}{e^{\frac{1}{x}} + e^{-\frac{1}{x}}} = \lim_{h \to 0} h \frac{e^{\frac{1}{h}} - e^{-\frac{1}{h}}}{e^{\frac{1}{h}} + e^{-\frac{1}{h}}}$$

| Putting $x = 0 + h = h$

$$= \lim_{h \to 0} h \frac{1 - e^{-\frac{2}{h}}}{1 + e^{-\frac{2}{h}}} = 0 \times \frac{1-0}{1+0} = 0$$

$\therefore \quad \lim_{x \to 0} f(x) = 0 = f(0)$. ⇒ $f(x)$ is continuous at $x = 0$.

Differentiability at x = 0

$$\text{L.H.D.} = \lim_{x \to 0-0} \frac{f(x)-f(0)}{x-0} = \lim_{x \to 0-0} \frac{x \cdot \dfrac{e^{\frac{1}{x}} - e^{-\frac{1}{x}}}{e^{\frac{1}{x}} + e^{-\frac{1}{x}}} - 0}{x}$$

$$= \lim_{x \to 0-0} \frac{e^{\frac{1}{x}} - e^{-\frac{1}{x}}}{e^{\frac{1}{x}} + e^{-\frac{1}{x}}} = \lim_{h \to 0} \frac{e^{-\frac{1}{h}} - e^{\frac{1}{h}}}{e^{-\frac{1}{h}} + e^{\frac{1}{h}}} = \lim_{h \to 0} \frac{e^{-\frac{2}{h}} - 1}{e^{-\frac{2}{h}} + 1} = \frac{0-1}{0+1} = -1$$

$$\text{R.H.D.} = \lim_{x \to 0+0} \frac{f(x)-f(0)}{x-0} = \lim_{x \to 0+0} \frac{x \cdot \dfrac{e^{\frac{1}{x}} - e^{-\frac{1}{x}}}{e^{\frac{1}{x}} + e^{-\frac{1}{x}}} - 0}{x}$$

$$= \lim_{x \to 0+0} \frac{e^{\frac{1}{x}} - e^{-\frac{1}{x}}}{e^{\frac{1}{x}} + e^{-\frac{1}{x}}} = \lim_{h \to 0} \frac{e^{\frac{1}{h}} - e^{-\frac{1}{h}}}{e^{\frac{1}{h}} + e^{-\frac{1}{h}}} = \lim_{h \to 0} \frac{1 - e^{-\frac{2}{h}}}{1 + e^{-\frac{2}{h}}} = \frac{1-0}{1+0} = 1$$

L.H.D. $\neq$ R.H.D. $\Rightarrow$ $f(x)$ is not differentiable at $x = 0$.

Example 16. *Examine for derivability at $x = 0$ the function f defined by*

$$f(x) = \begin{cases} x \cdot \dfrac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Sol. *Please try yourself.*

Example 17. *Discuss the continuity and differentiability of the function*

$f(x) = |x - 1| + |x - 2|$ *in the interval* $[0, 3]$.

Sol. According to the definition of f, we have

$$f(x) = \begin{cases} 1 - x + 2 - x = 3 - 2x & \text{if } 0 \le x \le 1 \\ x - 1 + 2 - x = 1 & \text{if } 1 \le x \le 2 \\ x - 1 + x - 2 = 2x - 3 & \text{if } 2 \le x \le 3 \end{cases}$$

f is a linear function or a constant function over the various sub-intervals.

$\Rightarrow$ f is continuous and differentiable over each sub-interval.

The only doubtful points are the breaking points $x = 1$ and $x = 2$.

At x = 1 $\quad f(1) = 3 - 2 \times 1 = 1$

$$\lim_{x \to 1-} f(x) = \lim_{x \to 1-} (3 - 2x) \qquad | \textbf{ Put } x = 1 - h.$$

$$= \lim_{h \to 0} [3 - 2(1 - h)] = 1$$

$$\lim_{x \to 1+} f(x) = \lim_{x \to 1+} 1 = 1$$

Since $\lim_{x \to 1-} f(x) = f(1) = \lim_{x \to 1+} f(x)$

$\therefore$ f is continuous at $x = 1$.

Also $\quad Lf'(1) = \lim_{x \to 1-} \dfrac{f(x) - f(1)}{x - 1} = \lim_{x \to 1-} \dfrac{(3 - 2x) - 1}{x - 1} = \lim_{x \to 1-} \dfrac{2(1 - x)}{x - 1} = \lim_{x \to 1-} (-2) = -2$

$Rf'(1) = \lim_{x \to 1+} \dfrac{f(x) - f(1)}{x - 1} = \lim_{x \to 1+} \dfrac{1 - 1}{x - 1} = 0$

Since $\quad Lf'(1) \neq Rf'(1)$

$\therefore$ f is not derivable at $x = 1$.

At x = 2 $\quad f(2) = 1$

$$\lim_{x \to 2-} f(x) = \lim_{x \to 2-} 1 = 1$$

$$\lim_{x \to 2+} f(x) = \lim_{x \to 2+} (2x - 3) \qquad | \textbf{ Put } x = 2 + h$$

$$= \lim_{h \to 0} [2(2 + h) - 3] = 1$$

Since $\quad \lim_{x \to 2-} f(x) = 1 = \lim_{x \to 2+} f(x)$

$\therefore$ f is continuous at $x = 2$

Also $\quad Lf'(2) = \lim_{x \to 2-} \dfrac{f(x) - f(2)}{x - 2} = \lim_{x \to 2-} \dfrac{1 - 1}{x - 2} = 0$

$Rf'(2) = \lim_{x \to 2+} \dfrac{f(x) - f(2)}{x - 2} = \lim_{x \to 2+} \dfrac{(2x - 3) - 1}{x - 2} = \lim_{x \to 2+} \dfrac{2(x - 2)}{x - 2} = \lim_{x \to 2+} 2 = 2$

Since $\quad Lf'(2) \neq Rf'(2)$

$\therefore$ f is not derivable at $x = 2$.

Hence f is continuous on $[0, 3]$. Also f is differentiable on $[0, 3]$ except at $x = 1$ and $x = 2$.

Example 18. *Discuss the continuity and differentiability of the function*

$f(x) = |x - 2| + 2|x - 3|$ *in* $[1, 4]$.

Sol. *Please try yourself.* **[Ans.** f is continuous on $[1, 4]$

Also f is differentiable on $[1, 4]$ except at $x = 2$ and $x = 3$.]

$$\left[\textbf{Hint. } f(x) = \begin{cases} -3x + 8 & \text{if } 1 \le x \le 2 \\ -x + 4 & \text{if } 2 \le x \le 3 \\ 3x - 8 & \text{if } 3 \le x \le 4 \end{cases} \right]$$

Example 19. *Show that the function $f(x) = |x - 1| + 2|x - 2| + 3|x - 3|$ is continuous but not differentiable at $x = 1, 2, 3$.*

Sol. *Please try yourself.*

Example 20. *Discuss the continuity and differentiability of the following functions at $x = a$.*

$$(i)\ f(x) = \begin{cases} (x - a) \sin \dfrac{1}{x - a} & \text{when } x \neq a \\ 0 & \text{when } x = a \end{cases}$$

$$(ii)\ f(x) = \begin{cases} (x - a)^2 \cos \dfrac{1}{x - a} & \text{when } x \neq a \\ 0 & \text{when } x = a \end{cases}$$

Sol. (*i*) Here $f(a) = 0$

Continuity at x = a

$$\lim_{x \to a-} f(x) = \lim_{x \to a-} (x - a) \sin \frac{1}{x - a} \qquad | \textbf{ Put } x = a - h$$

$$= \lim_{h \to 0} -h \sin \frac{1}{-h} = \lim_{h \to 0} h \sin \frac{1}{h} = 0 \times \text{a finite quantity} = 0$$

$$\lim_{x \to a+} f(x) = \lim_{x \to a+} (x - a) \sin \frac{1}{x - a} \qquad | \textbf{ Put } x = a + h$$

$$= \lim_{h \to 0} h \sin \frac{1}{h} = 0 \times \text{a finite quantity} = 0$$

Since $\quad \lim_{x \to a-} f(x) = 0 = \lim_{x \to a+} f(x)$

$\therefore$ f is continuous at $x = a$.

Differentiability at x = a

$$Lf'(a) = \lim_{x \to a-} \frac{f(x)-f(a)}{x-a} = \lim_{x \to a-} \frac{(x-a)\sin\frac{1}{x-a} - 0}{x-a}$$

$$= \lim_{x \to a-} \sin\frac{1}{x-a} \qquad \text{| Put x = a – h}$$

$$= \lim_{h \to 0} \sin\frac{1}{-h} = \lim_{h \to 0} -\sin\frac{1}{h} \text{ which does not exist.}$$

$\Rightarrow$ f is not differentiable at $x = a$.

(ii) Please try yourself.

Example 21. *A function f is defined as follows :*

$$f(x) = \begin{cases} x^p \cos\frac{1}{x}, & \text{when } x \neq 0 \\ 0, & \text{when } x = 0 \end{cases}$$

What conditions should be imposed on p so that

(i) f may be continuous at x = 0 *(ii) f may be derivable at x = 0 ?*

Sol. (i) Here $f(0) = 0$

$$\lim_{x \to 0-} f(0) = \lim_{x \to 0-} x^p \cos\frac{1}{x} \qquad \text{| Put x = 0 – h.}$$

$$= \lim_{h \to 0} (0-h)^p \cos\frac{1}{0-h} = \lim_{h \to 0} (-h)^p \cos\frac{1}{h} \qquad ...(1)$$

Also $\lim_{x \to 0+} f(0) = \lim_{x \to 0+} x^p \cos\frac{1}{x}$ | **Put x = 0 + h**

$$= \lim_{h \to 0} h^p \cos\frac{1}{h} \qquad ...(2)$$

In order that f may be continuous at $x = 0$, the limits (1) and (2) both must be zero. This is possible only when $p > 0$.

Hence the required condition for continuity of f at $x = 0$ is $p > 0$.

(ii) $$Lf'(0) = \lim_{x \to 0-} \frac{f(x)-f(0)}{x-0} = \lim_{x \to 0-} \frac{x^p \cos\frac{1}{x} - 0}{x} = \lim_{x \to 0-} x^{p-1} \cos\frac{1}{x}$$

| **Put x = 0 – h**

$$= \lim_{h \to 0} (-h)^{p-1} \cos\frac{1}{-h} = \lim_{h \to 0} (-h)^{p-1} \cos\frac{1}{h} \qquad ...(3)$$

$$Rf'(0) = \lim_{x \to 0+} \frac{f(x)-f(0)}{x-0} = \lim_{x \to 0+} \frac{x^p \cos\frac{1}{x} - 0}{x} = \lim_{x \to 0+} x^{p-1} \cos\frac{1}{x}$$

| **Put x = 0 + h**

$$= \lim_{h \to 0} h^{p-1} \cos\frac{1}{h} \qquad ...(4)$$

Now in order that f may be differentiable at $x = 0$, the limits (3) and (4) both must be equal. This is possible only when $p > 1$, for in that case,

$$Lf'(0) = 0 = Rf'(0).$$

Hence the required condition for differentiability of f at $x = 0$ is $p > 1$.

Example 22. *A function f is defined as follows :*

$$f(x) = \begin{cases} x^m \sin\frac{1}{x}, & \text{when } x \neq 0 \\ 0, & \text{when } x = 0 \end{cases}$$

What conditions should be imposed on m so that

(i) f may be continuous at x = 0

(ii) f may be differentiable at x = 0 ?

Sol. *Please try yourself.* **[Ans.** (i) $m > 0$, (ii) $m > 1$]

Example 23. *Prove that the greatest integer function [x] is not differentiable at x = 1.*

Sol. Here $f(x) = [x]$ $\therefore$ $f(1) = [1] = 1$

$$\lim_{x \to 1-} f(x) = \lim_{x \to 1-} [x] \qquad \text{| Put x = 1 – h, h > 0}$$

$$= \lim_{h \to 0+} [1-h] \qquad \because 0 < 1-h < 1$$

$$= \lim_{h \to 0+} 0 = 0 \qquad \therefore [1-h] = 0$$

$$\lim_{x \to 1+} f(x) = \lim_{x \to 1+} [x] \qquad \text{| Put x = 1 + h, h > 0}$$

$$= \lim_{h \to 0+} [1+h] \qquad \because 1 < 1+h < 2$$

$$= \lim_{h \to 0+} 1 = 1 \qquad \therefore [1+h] = 1$$

Since $\lim_{x \to 1-} f(x) \neq \lim_{x \to 1+} f(x)$

$\therefore$ $\lim_{x \to 1} f(x)$ does not exist.

$\Rightarrow$ f is not continuous at $x = 1$ $\Rightarrow$ f is not differentiable at $x = 1$.

Example 24. *The function f defined by* $f(x) = \begin{cases} x^2 + 3x + a, & \text{if } x \leq 1 \\ bx + 2, & \text{if } x > 1 \end{cases}$ *is given to be derivable for every x. Find a and b.*

Sol. Since f is derivable for every x, f must be derivable at $x = 1$ and, hence, f must be continuous at $x = 1$.

Now $\lim_{x \to 1-} f(x) = \lim_{x \to 1-} (x^2 + 3x + a)$ | **Put x = 1 – h**

$$= \lim_{h \to 0} (1-h)^2 + 3(1-h) + a = 4 + a$$

$\lim_{x \to 1+} f(x) = \lim_{x \to 1+} (bx + 2)$ | **Put x = 1 + h**

$$= \lim_{h \to 0} b(1+h) + 2 = b + 2$$

Also $f(1) = 1 + 3 + a = 4 + a$

Since f is continuous at $x = 1$

$\therefore \quad \lim_{x\to 1-} f(x) = \lim_{x\to 1+} f(x) = f(1)$

$\Rightarrow \quad 4 + a = b + 2 = 4 + a \quad \Rightarrow a - b + 2 = 0$ ...(1)

Now $\quad Lf'(1) = \lim_{x\to 1-} \dfrac{f(x) - f(1)}{x-1}$

$$= \lim_{x\to 1-} \frac{(x^2+3x+a)-(4+a)}{x-1} = \lim_{x\to 1-} \frac{x^2+3x-4}{x-1} \qquad | \textbf{ Put } x = 1-h$$

$$= \lim_{h\to 0} \frac{(1-h)^2+3(1-h)-4}{-h} = \lim_{h\to 0} \frac{h^2-5h}{-h} = \lim_{h\to 0} \frac{h(h-5)}{-h}$$

$$= \lim_{h\to 0} (5-h) = 5$$

$$Rf'(1) = \lim_{x\to 1+} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1+} \frac{(bx+2)-(4+a)}{x-1}$$

$$= \lim_{x\to 1+} \frac{(bx+2)-(b+2)}{x-1} \qquad \text{[Using (1)]}$$

$$= \lim_{x\to 1+} \frac{b(x-1)}{x-1} = \lim_{x\to 1+} b = b$$

Since f is derivable at $x = 1$

$\therefore \quad Lf'(1) = Rf'(1) \quad \Rightarrow \quad b = 5$

$\therefore$ From (1), $\quad a - 5 + 2 = 0 \quad \Rightarrow \quad a = 3$

Hence $\quad a = 3,\ b = 5.$

Example 25. *For what choice of a and b, if any, will the function*

$$f(x) = \begin{cases} ax - 6, & \text{if } x > 1 \\ bx^2, & \text{if } x \le 1 \end{cases} \text{ become differentiable at } x = 1\,?$$

Sol. *Please try yourself.* **[Ans.** $a = 12,\ b = 6$]

Example 26. *If f is differentiable at $x = a$, find* $\lim_{x\to a} \dfrac{x^2 f(a) - a^2 f(x)}{x-a}$.

Sol. f is differentiable at $x = a$

$\Rightarrow \quad \lim_{x\to a} \dfrac{f(x)-f(a)}{x-a}$ exists and is equal to $f'(a)$ ...(1)

Now $\quad \lim_{x\to a} \dfrac{x^2 f(a) - a^2 f(x)}{x-a} = \lim_{x\to a} \dfrac{(x^2-a^2)f(a) + a^2 f(a) - a^2 f(x)}{x-a}$

$$= \lim_{x\to a} \frac{(x^2-a^2)f(a) - a^2[f(x)-f(a)]}{x-a} = \lim_{x\to a} \left\{ \frac{(x^2-a^2)f(a)}{x-a} - \frac{a^2[f(x)-f(a)]}{x-a} \right\}$$

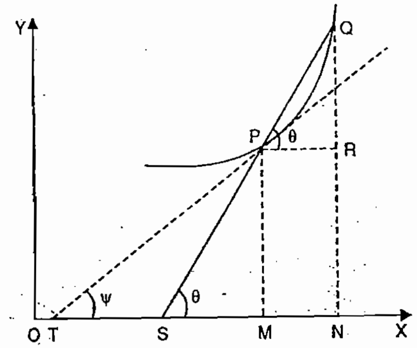
$$= \lim_{x\to a} (x+a) f(a) - a^2 \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = (a+a) f(a) - a^2 f'(a) \qquad \text{[using (1)]}$$

$$= 2a\, f(a) - a^2 f'(a).$$

Example 27. *If f is differentiable at $x = a$, find* $\lim_{x\to a} \dfrac{xf(a) - af(x)}{x-a}$.

Sol. *Please try yourself.* **[Ans.** $f(a) - af'(a)$]

7.5. GEOMETRICAL MEANING OF THE DERIVATIVE



Let $f : [a, b] \to \mathbb{R}$ be derivable at $c \in (a, b)$. Then $P(c, f(c))$ is a point on the graph of $y = f(x)$. Let $Q\,(c+h, f(c+h))$ be a point in the nbd. of P such that $a < c + h < b$. Join PQ and produce it to meet the x-axis at S. Let PQ make an angle θ with x-axis. Draw the tangent line at P which makes an angle ψ with x-axis. Draw PM and QN perpendiculars from P and Q on the x-axis and $PR \perp QN$.

Now $\quad QR = QN - RN = QN - PM$

$\qquad$ = difference in ordinates of Q

and $\quad P = f(c+h) - f(c)$

and $\quad PR = MN = ON - OM$

$\qquad$ = difference in abscissae of Q and $P = (c+h) - c = h$

$\therefore$ Slope of $\quad PQ = \tan\theta = \dfrac{QR}{PR} = \dfrac{f(c+h)-f(c)}{h}$

As $h \to 0$, the point Q approaches the point P along the graph of f and, consequently, the chord PQ becomes the tangent to the curve at P. Thus $\theta \to \psi$.

$$\therefore \qquad \tan\psi = \lim_{Q\to P} \tan\theta = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = f'(c)$$

Hence the derivative of f at c is the slope of the tangent to the curve $y = f(x)$ at the point $(c, f(c))$.

7.6. ALGEBRA OF DERIVATIVES

Theorem 1. *Let $f : [a, b] \to R$ and $g : [a, b] \to R$ be derivable at $c \in [a, b]$. Then*

(i) αf is derivable at c and $(\alpha f)'(c) = \alpha f'(c)$ where $\alpha \in R$

(ii) $f + g$ is derivable at c and $(f+g)'(c) = f'(c) + g'(c)$

(iii) fg is derivable at c and $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$

(iv) f/g is derivable at c and $\left(\dfrac{f}{g}\right)(c) = \dfrac{f'(c)g(c) - f(c)g'(c)}{[g(c)]^2}$, *provided $g(c) \ne 0$*

(v) $\dfrac{1}{g}$ *is derivable at c and* $\left(\dfrac{1}{g}\right)(c) = -\dfrac{g'(c)}{[g(c)]^2}$, *provided $g(c) \ne 0$.*

Proof. Since f and g are differentiable at $x = c$

$$\therefore \quad \lim_{x\to c} \frac{f(x)-f(c)}{x-c} = f'(c) \text{ and } \lim_{x\to c} \frac{g(x)-g(c)}{x-c} = g'(c) \qquad \text{...(1)}$$

(*i*) $\quad (\alpha f)'(c) = \lim_{x\to c} \dfrac{(\alpha f)(x) - (\alpha f)(c)}{x-c} = \lim_{x\to c} \dfrac{\alpha f(x) - \alpha f(c)}{x-c} = \lim_{x\to c} \dfrac{\alpha[f(x)-f(c)]}{x-c}$

$$= \alpha \lim_{x\to c} \frac{f(x)-f(c)}{x-c} = \alpha f'(c) \qquad \text{[by (1)]}$$

(ii) $(f+g)'(c) = \lim\limits_{x \to c} \dfrac{(f+g)(x)-(f+g)(c)}{x-c}$

$= \lim\limits_{x \to c} \dfrac{[f(x)+g(x)]-[f(c)+g(c)]}{x-c} = \lim\limits_{x \to c} \dfrac{[f(x)-f(c)]+[g(x)-g(c)]}{x-c}$

$= \lim\limits_{x \to c} \left[\dfrac{f(x)-f(c)}{x-c} + \dfrac{g(x)-g(c)}{x-c}\right] = \lim\limits_{x \to c} \dfrac{f(x)-f(c)}{x-c} + \lim\limits_{x \to c} \dfrac{g(x)-g(c)}{x-c}$

$= f'(c) + g'(c)$ [by (1)]

(iii) $(fg)'(c) = \lim\limits_{x \to c} \dfrac{(fg)(x)-(fg)(c)}{x-c}$

$= \lim\limits_{x \to c} \dfrac{f(x)g - f(c)g(c)}{x-c} = \lim\limits_{x \to c} \dfrac{f(x)g(x)-f(c)g(x)+f(c)g(x)-f(c)g(c)}{x-c}$

$= \lim\limits_{x \to c} \left[g(x)\dfrac{f(x)-f(c)}{x-c} + f(c)\dfrac{g(x)-g(c)}{x-c}\right]$

$= \lim\limits_{x \to c} g(x) \,.\, \lim\limits_{x \to c} \dfrac{f(x)-f(c)}{x-c} + f(c) \lim\limits_{x \to c} \dfrac{g(x)-g(c)}{x-c} = g(c)f'(c) + f(c)g'(c)$

$\left[\text{since } g \text{ is derivable at } c, g \text{ is continuous at } c \text{ so that } \lim\limits_{x \to c} g(x) = g(c)\right]$

(iv) $\left(\dfrac{f}{g}\right)'(c) = \lim\limits_{x \to c} \dfrac{\left(\frac{f}{g}\right)(x) - \left(\frac{f}{g}\right)(c)}{x-c} = \lim\limits_{x \to c} \dfrac{\frac{f(x)}{g(x)} - \frac{f(c)}{g(c)}}{x-c} = \lim\limits_{x \to c} \dfrac{f(x)g(c)-g(x)f(c)}{g(x)g(c).(x-c)}$

$= \dfrac{1}{g(c)} \lim\limits_{x \to c} \dfrac{f(x)g(c)-f(c)g(c)+f(c)g(c)-g(x)f(c)}{g(x).(x-c)}$

$= \dfrac{1}{g(c)} \lim\limits_{x \to c} \dfrac{g(c)[f(x)-f(c)]-f(c)[g(x)-g(c)]}{g(x).(x-c)}$

$= \dfrac{1}{g(c)} \lim\limits_{x \to c} \dfrac{1}{g(x)} \times \lim\limits_{x \to c} \left[g(c)\dfrac{f(x)-f(c)}{x-c} + f(c)\dfrac{g(x)-g(c)}{x-c}\right]$

$= \dfrac{1}{g(c)} \times \dfrac{1}{g(c)} \left[g(c) \lim\limits_{x \to c} \dfrac{f(x)-f(c)}{x-c} - f(c) \lim\limits_{x \to c} \dfrac{g(x)-g(c)}{x-c}\right]$

[since g is derivable at c, g is continuous at c. Also $g(c) \neq 0$

$\therefore \dfrac{1}{g}$ is continuous at c and $\lim\limits_{x \to c} \dfrac{1}{g(x)} = \dfrac{1}{g(c)}\Big]$

$= \dfrac{g(c)f'(c) - f(c)\,g'(c)}{[g(c)]^2}$ [by (1)]

(v) $\left(\dfrac{1}{g}\right)'(c) = \lim\limits_{x \to c} \dfrac{\left(\frac{1}{g}\right)(x) - \left(\frac{1}{g}\right)(c)}{x-c} = \lim\limits_{x \to c} \dfrac{\frac{1}{g(x)} - \frac{1}{g(c)}}{x-c} = \lim\limits_{x \to c} \dfrac{g'(c)-g(x)}{g(x)g(c).(x-c)}$

$= \dfrac{-1}{g(c)} \lim\limits_{x \to c} \dfrac{1}{g(x)} \times \lim\limits_{x \to c} \dfrac{g(x)-g(c)}{x-c}$

$= \dfrac{-1}{g(c)} \times \dfrac{1}{g(c)} \times g'(c) = -\dfrac{g'(c)}{[g(c)]^2}$ [by (1)]

Theorem 2. *(Chain Rule). If f and g are two functions such that*
(i) range of $f \subset$ domain of g *(ii) f is derivable at c*
(iii) g is derivable at $f(c)$
then the composite function gof is derivable at c and $(gof)'(c) = g'(f(c)) \,.\, f'(c)$.

Proof. f is derivable at c $\Rightarrow \lim\limits_{x \to c} \dfrac{f(x)-f(c)}{x-c} = f'(c)$ ...(1)

g is derivable at $f(c)$ $\Rightarrow \lim\limits_{f(x) \to f(c)} \dfrac{g(f(x))-g(f(c))}{f(x)-f(c)} = g'(f(c))$ ...(2)

$\therefore \quad (gof)'(c) = \lim\limits_{x \to c} \dfrac{(gof)(x)-(gof)(c)}{x-c} = \lim\limits_{x \to c} \dfrac{g(f(x))-g(f(c))}{f(x)-f(c)} \times \dfrac{f(x)-f(c)}{x-c}$

$= \lim\limits_{x \to c} \dfrac{g(f(x))-g(f(c))}{f(x)-f(c)} \times \lim\limits_{x \to c} \dfrac{f(x)-f(c)}{x-c}$ ...(3)

Now f is derivable at c $\Rightarrow$ f is continuous at c

$\therefore \quad x \to c \Rightarrow f(x) \to f(c)$

Thus, from (3), we have

$(gof)'(c) = \lim\limits_{f(x) \to f(c)} \dfrac{g(f(x))-g(f(c))}{f(x)-f(c)} \times \lim\limits_{x \to c} \dfrac{f(x)-f(c)}{x-c} = g'(f(c)) \,.\, f'(c)$

[by (1) and (2)]

7.7. ROLLE'S THEOREM

Statement. *If a function f is defined on $[a, b]$ is*
(i) continuous in the closed interval $[a, b]$ (ii) derivable in the open interval (a, b)
(iii) $f(a) = f(b)$
then there exists atleast one real number $c \in (a, b)$ such that $f'(c) = 0$.

Proof. Since f is continuous on the closed interval $[a, b]$, therefore, it is bounded and attains its bounds in $[a, b]$.

Thus if M and m are the l.u.b. and the g.l.b. of f in $[a, b]$, then there exist points c and d in $[a, b]$ such that $f(c) = \text{M}$ and $f(d) = m$

There are two possibilities : either $\text{M} = m$ or $\text{M} \neq m$.

Case 1. If $\text{M} = m$, then $f(x) = m = \text{M}$ throughout $[a, b]$

$\Rightarrow$ f is a constant function on $[a, b]$

$\Rightarrow \quad f'(x) = 0 \quad \forall\, x \in [a, b]$

In particular, $f'(c) = 0$ when $x \in (a, b)$

Hence the theorem is true for any $c \in (a, b)$.

Case 2. If $\text{M} \neq m$, then since $f(a) = f(b)$, at least one of the numbers M and m will be different from the equal values of $f(a)$ and $f(b)$.

Suppose $M = f(c)$ is different from $f(a)$ and $f(b)$.

Now $M = f(c) \neq f(a) \Rightarrow c \neq a$

$M = f(c) \neq f(b) \Rightarrow c \neq b$

$\therefore$ $c \neq a$ and $c \neq b \Rightarrow c \in (a, b)$.

Since f is derivable in (a, b) and $c \in (a, b)$

$\therefore$ f is derivable at c.

$$\Rightarrow \quad Lf'(c) = \lim_{x \to c-} \frac{f(x) - f(c)}{x - c} \qquad | \text{ Put } x = c - h, h > 0$$

$$= \lim_{h \to 0-} \frac{f(c-h) - f(c)}{-h}$$

and

$$Rf'(c) = \lim_{x \to c+} \frac{f(x) - f(c)}{x - c} \qquad | \text{ Put } x = c + h, h > 0$$

$$= \lim_{h \to 0-} \frac{f(c+h) - f(c)}{h}$$ both exist and are equal, each $= f'(c)$.

Since $f(c) = M = \text{l.u.b. of } f$

$\therefore$ $f(c-h) \leq f(c)$ and $f(c+h) \leq f(c)$

$\Rightarrow$ $f(c-h) - f(c) \leq 0$ and $f(c+h) - f(c) \leq 0$

$\Rightarrow$ $Lf'(c) \geq 0$ and $Rf'(c) \leq 0$

The two derivatives will be equal if each $= 0$

$\therefore$ $f'(c) = 0$

Similarly, when $m = f(d)$ is different from $f(a)$ and $f(b)$, we can prove that

$f'(d) = 0$ where $d \in (a, b)$.

7.8. GEOMETRICAL INTERPRETATION OF ROLLE'S THEOREM

Let A and B be the points on the graph of the function $y = f(x)$ corresponding to $x = a$ and $x = b$ respectively.

(*i*) $f(x)$ is continuous on $[a, b] \Rightarrow$ the graph of $f(x)$ is without gaps or jumps from A to B.

(*ii*) $f(x)$ is derivable on $(a, b) \Rightarrow$ the graph of $f(x)$ has a tangent at every point between A and B.

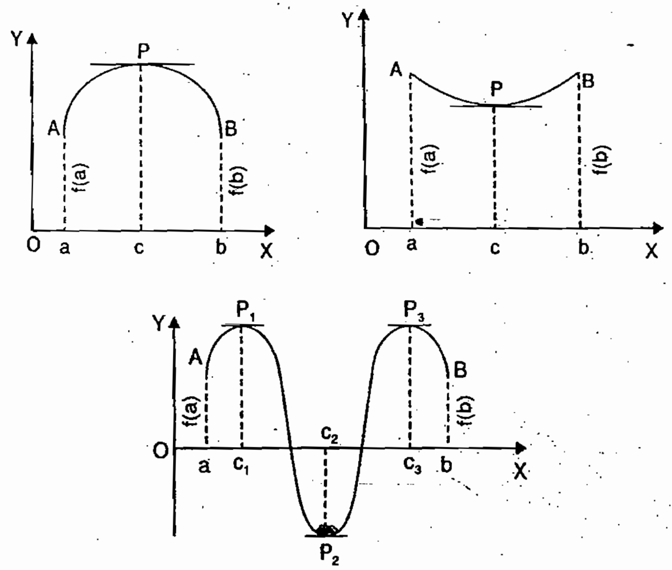
(*iii*) $f(a) = f(b) \Rightarrow$ the ordinates of A and B are equal so that the points A and B are equidistant from the x-axis.

Graphs of functions satisfying all the above three conditions are as shown in the figures. From the figures, it is clear that there exists at least one point P on the curve between A and B where the tangent is parallel to the x-axis.

$\Rightarrow$ Slope of tangent at $P = 0 \Rightarrow f'(x) = 0$ at P

If c is the abscissa of P, then $f'(c) = 0$ where $a < c < b$.

Hence, if all the three conditions are satisfied, then Rolle's Theorem confirms the existence of at least one point $c \in (a, b)$ where the tangent is parallel to x-axis.



7.9. FAILURE OF ROLLE'S THEOREM

Rolle's theorem is not applicable to the function f if the hypothesis is weakened. That is, Rolle's theorem fails to hold good for a function which does not satisfy *all the three* conditions of the theorem. The theorem is not applicable if either

(*i*) f is discontinuous at a point in $[a, b]$

or (*ii*) f is not derivable at a point in (a, b)

or (*iii*) $f(a) \neq f(b)$.

Note 1. The converse of Rolle's theorem is not true *i.e.*, $f'(x)$ may vanish at a point $c \in (a, b)$ without $f(x)$ satisfying all the three conditions of Rolle's theorem.

For example, consider $f(x) = \begin{cases} 0 & \text{when } 0 \leq x \leq 1 \\ x+1 & \text{when } 1 < x \leq 2 \end{cases}$

Clearly f is not continuous and derivable at $x = 1$

$\therefore$ f is not continuous on $[0, 2]$; f is not derivable on $(0, 2)$

Also $f(0) \neq f(2)$

But $f'(0) = 0 \;\; \forall x \in (0, 1) \subset (0, 2)$.

Hence the conditions in the theorem are only sufficient and not necessary for the conclusion.

Note 2. Another Form of Rolle's Theorem

If a function f defined on $[a, a + h]$ is

(*i*) continuous on the closed interval $[a, a + h]$

(*ii*) derivable on the open interval $(a, a + h)$

(*iii*) $f(a) = f(a + h)$

then there exists at least one real number θ in $(0, 1)$ such that $f'(a + \theta h) = 0$

Here b is replaced by $a + h$, $h > 0$ and $c = a + \theta h$.

Since $a < c = a + \theta h < a + h$

$\therefore$ $0 < \theta h < h \Rightarrow 0 < \theta < 1$.

ILLUSTRATIVE EXAMPLES

Example 1. *Verify Rolle's Theorem in the following cases :*

(a) $f(x) = (x-a)^m (x-b)^n$, *where m and n are +ve integers, in the interval* $[a, b]$.

(b) $f(x) = (x-a)^3 (x-b)^4$ *in* $[a, b]$.

Sol. *(a)* We have $f(x) = (x-a)^m (x-b)^n$.

(i) As m and n are +ve integers $f(x) = (x-a)^m (x-b)^n$ will be a polynomial in x on expansion by Binomial Theorem. Since every polynomial is a continuous function of x for every value of x.

$\therefore$ $f(x)$ **is continuous in the closed interval** $[a, b]$.

(ii) Also
$$f'(x) = m(x-a)^{m-1}(x-b)^n + n(x-b)^{n-1}(x-a)^m$$
$$= (x-a)^{m-1}(x-b)^{n-1}[m(x-b) + n(x-a)]$$
$$= (x-a)^{m-1}(x-b)^{n-1}[(m+n)x - (mb+na)]$$

which exists *i.e.*, has a unique and definite value for any x in (a, b). Thus $f(x)$ **is derivable in the open interval** (a, b).

(iii) Also $f(a) = 0 = f(b)$.

Hence $f(x)$ satisfies all the three conditions of Rolle's Theorem. Therefore, there exists at least one value c of x in (a, b) such that $f'(c) = 0$, *i.e.*
$$f'(c) = (c-a)^{m-1}(c-b)^{n-1}[c(m+n) - (mb+na)] = 0$$

or $c(m+n) - (mb+na) = 0$ $\quad [\because\ c \neq a, c \neq b]$

or
$$c = \frac{mb+na}{m+n},$$

which is a point **within** the open interval (a, b) because it divides a and b internally in the ratio $m : n$.

Hence Rolle's Theorem is verified.

(b) Please try yourself.

Example 2. *Verify Rolle's Theorem for the following functions :*

(a) $f(x) = 2 + (x-1)^{2/3}$ *in* $[0, 2]$

(b) $f(x) = e^x \sin x$ *in* $[0, \pi]$ *(c)* $f(x) = x(x+3)\, e^{-x/2}$ *in* $[-3, 0]$.

Sol. *(a)*
$$f'(x) = \frac{2}{3}(x-1)^{-1/3} = \frac{2}{3(x-1)^{1/3}}$$

which does not exist (*i.e.*, is not finite) at $x = 1 \in (0, 2)$ and so $f'(x)$ does not exist at every point of $(0, 2)$ *i.e.*, $f(x)$ **is not derivable in the open interval** $(0, 2)$.

Hence Rolle's Theorem is **not applicable** to $f(x)$ in the interval $[0, 2]$.

(b) $\mathbf{f(x) = e^x \sin x}$ in $[0, \pi]$.

(i) Since e^x and $\sin x$ are both continuous for every value of x, their product $e^x \sin x = f(x)$ is also continuous for every value of x and in particular $f(x)$ is continuous in the closed interval $[0, \pi]$.

(ii)
$$f'(x) = e^x \cos x + \sin x \cdot e^x = e^x(\cos x + \sin x)$$

and this does not become infinite or indeterminate for any value of x in the open interval $(0, \pi)$, it follows that $f(x)$ is derivable in the open interval $(0, \pi)$.

(iii)
$$f(0) = e^0 \sin 0 = 1 \times 0 = 0$$
$$f(\pi) = e^\pi \sin \pi = e^\pi \times 0 = 0.$$

$\therefore$ $f(0) = f(\pi)$.

Hence $f(x)$ satisfies all the three conditions of Rolle's Theorem in the interval $[0, \pi]$, therefore, there exists at least one value 'c' of x in the open interval $(0, \pi)$ such that $f'(c) = 0$.

i.e.,
$$f'(c) = e^c(\sin c + \cos c) = 0$$

But $e^c \neq 0$ for any finite value of c.

$\therefore$ $\sin c + \cos c = 0$ or $\sin c = -\cos c$ or $\tan c = -1$

or
$$\tan c = -1 = -\tan\frac{\pi}{4} = \tan\left(\pi - \frac{\pi}{4}\right)$$

$\therefore$
$$c = \frac{3\pi}{4}.$$

Clearly, the value $c = \frac{3\pi}{4}$ lies **within** the open interval $(0, \pi)$ which verifies Rolle's Theorem.

(c) $\mathbf{f(x) = x(x+3)e^{-x/2}}$ in $[-3, 0]$.

(i) Since $x(x+3)$ being a polynomial is continuous for all x and $e^{-x/2}$ is also continuous for all x, therefore their product $f(x) = x(x+3)\, e^{-x/2}$ is also continuous for every value of x and in particular $f(x)$ **is continuous in the closed interval** $[-3, 0]$.

(ii)
$$f'(x) = (2x+3)e^{-x/2} + x(x+3)e^{-x/2}\left(-\tfrac{1}{2}\right)$$
$$= e^{-x/2}\left[2x + 3 - \frac{x^2+3x}{2}\right] = e^{-x/2}\left[\frac{6+x-x^2}{2}\right]$$

which does not become infinite or indeterminate at any point of the interval $(-3, 0)$ and **thus** $\mathbf{f(x)}$ **is derivable in the open interval** $(-3, 0)$.

(iii) Now $f(-3) = 0 = f(0)$.

Thus $f(x)$ satisfies all the three conditions of Rolle's Theorem.

Hence there must exist at least one point $c \in (-3, 0)$ such that $f'(c) = 0$, *i.e.*,
$$f'(c) = \tfrac{1}{2}\, e^{-c/2}(6 + c - c^2) = 0$$

$\because$ $e^{-c/2}$ is not zero for any finite value of c.

$\therefore$ $6 + c - c^2 = 0$ or $c^2 - c - 6 = 0$

or $(c-3)(c+2) = 0$ $\therefore$ $c = 3, -2$.

Of these two values of c for which $f'(c) = 0$, -2 belongs to the interval $(-3, 0)$.

Hence the verification.

Example 3. *Discuss the applicability of Rolle's Theorem to the following functions :*

(a) $f(x) = |x|$ *in* $[-1, 1]$ *(b)* $f(x) = \log\left[\frac{x^2+ab}{x(a+b)}\right]$ *in* $[a, b]$, $0 \notin [a, b]$

(c) $f(x) = \log\left(\frac{x^2+3}{4x}\right)$ *in* $[1, 3]$.

Sol. *(a)* *(i)* $f(x) = |x|$ is continuous for every value of x and so **in particular it is continuous in the closed interval** $[-1, 1]$.

(ii) $f(x)$ is not derivable at $x = 0$, [prove it]

$\therefore$ $f(x)$ **is not derivable in the open interval** $(-1, 1)$.

Hence Rolle's Theorem is **not applicable** to $f(x) = |x|$ in $[-1, 1]$.

(*b*) $\mathbf{f(x) = \log\left[\frac{x^2+ab}{x(a+b)}\right]}$ in $[a, b]$

(*i*) $$f(x) = \log(x^2+ab) - \log x - \log(a+b)$$
being a composite function of continuous functions in $[a, b]$ is a **continuous** function of x in $[a, b]$.

(*ii*) $$f'(x) = \frac{2x}{x^2+ab} - \frac{1}{x} = \frac{x^2-ab}{x(x^2+ab)}$$
which does not become infinite or indeterminate for $a < x < b$ and so $f(x)$ **is derivable in the open interval** (a, b).

(*iii*) $$f(a) = \log\frac{a^2+ab}{a(a+b)} = \log 1 = 0.$$
$$f(b) = \log\frac{b^2+ab}{b(a+b)} = \log 1 = 0.$$

Thus $f(x)$ satisfies all the three conditions of Rolle's Theorem and therefore, there must exist at least one value c of x in $a < x < b$ such that $f'(c) = 0$.

$\therefore$ $$f'(c) = \frac{c^2-ab}{c(c^2+ab)} = 0 \quad\text{or}\quad \frac{c^2-ab}{c(c^2+ab)} = 0$$

or $$c^2 = ab \quad \therefore \quad c = \pm\sqrt{ab}.$$

Of these two values of c, clearly $c = \sqrt{ab}$ lies between a and b, being the geometric mean of a and b. Hence Rolle's Theorem is **applicable** to $f(x)$ in the interval $[a, b]$.

(*c*) *Please try yourself.* [Same as part (*b*) with $a = 1, b = 3$]

Example 4. *Verify Rolle's theorem for the following functions* :

(*a*) $f(x) = x^2 - 6x + 8$ *in* $[2, 4]$ (*b*) $f(x) = 8x - x^2$ *in* $[2, 6]$

(*c*) $f(x) = x^3 - 4x$ *in* $[-2, 2]$ (*d*) $f(x) = \begin{cases} x^2+1 & \text{for } 0 \le x \le 1 \\ 3-x & \text{for } 1 \le x \le 2. \end{cases}$

Sol. (*a*) $f(x) = x^2 - 6x + 8$ is a polynomial in x.

(*i*) $\therefore$ It is continuous for every value of x.

In particular, $f(x)$ is continuous in $[2, 4]$.

(*ii*) $f'(x) = 2x - 6$ exists for every $x \in (2, 4) \Rightarrow f(x)$ is derivable in $(2, 4)$.

(*iii*) $f(2) = 4 - 12 + 8 = 0$; $f(4) = 16 - 24 + 8 = 0 \;\therefore\; f(2) = f(4)$

$\therefore$ $f(x)$ satisfies all the three conditions of Rolle's Theorem.

$\Rightarrow$ There must exist at least one $c \in (2, 4)$ s.t. $f'(c) = 0$

Now $f'(c) = 0 \Rightarrow 2c - 6 = 0 \Rightarrow c = 3 \in (2, 4)$

Hence Rolle's Theorem is verified.

(*b*) *Please try yourself.*

(*c*) *Please try yourself.*

(*d*) Here $f(x)$ is defined on $[0, 2]$, $f(x) = x^2 + 1$ for $0 \le x \le 1$ which being a polynomial is continuous and derivable for all $x \in [0, 1]$. Also $f(x) = 3 - x$ for $1 \le x \le 2$ which being a polynomial is continuous and derivable for all $x \in [1, 2]$.

Since the domain of definition is partitioned at $x = 1$ while defining $f(x)$, we are not sure about the continuity and derivability of $f(x)$ at $x = 1$.

Now $$\text{L.H.L.} = \lim_{x\to 1-0} f(x) = \lim_{x\to 1-0}(x^2+1) = \lim_{h\to 0}(1-h)^2 + 1 = 2$$
$$\text{R.H.L.} = \lim_{x\to 1+0} f(x) = \lim_{x\to 1+0}(3-x) = \lim_{h\to 0} 3 - (1+h) = 2$$

$\therefore$ $\lim_{x\to 1} f(x) = 2$ Also $f(1) = 1^2 + 1 = 2$

$\therefore$ $\lim_{x\to 1} f(x) = f(1) \;\therefore\; f(x)$ is continuous at $x = 1$

Hence $f(x)$ is continuous for all values of x in $[0, 2]$ ...(*i*)

$$\text{L.H.D.} = \lim_{x\to 1-0}\frac{f(x)-f(1)}{x-1} = \lim_{x\to 1-0}\frac{(x^2+1)-2}{x-1}$$
$$= \lim_{h\to 0}\frac{(1-h)^2-1}{1-h-1} = \lim_{h\to 0}\frac{-2h+h^2}{-h} = \lim_{h\to 0}(2-h) = 2$$
$$\text{R.H.D.} = \lim_{x\to 1+0}\frac{f(x)-f(1)}{x-1} = \lim_{x\to 1+0}\frac{(3-x)-2}{x-1}$$
$$= \lim_{h\to 0}\frac{1-(1+h)}{1+h-1} = \lim_{h\to 0}\frac{-h}{h} = \lim_{h\to 0} -1 = -1$$
$$\text{L.H.D.} \ne \text{R.H.D.} \Rightarrow f(x) \text{ is not derivable at } x = 1$$

$\Rightarrow$ $f(x)$ is not derivable at every point of the open inverval $(0, 2)$.

$\therefore$ $f(x)$ does not satisfy the condition of derivability in the open interval $(0, 2)$, Rolle's theorem is not applicable to $f(x)$ in $[0, 2]$.

Example 5. *Verify Rolle's theorem for the following functions* :

(*i*) $f(x) = x(x-3)^2$ *on* $[0, 3]$ (*ii*) $f(x) = \frac{\sin x}{e^x}$ *on* $[0, \pi]$

(*iii*) $f(x) = \cos 2\left(x - \frac{\pi}{4}\right)$ *on* $\left[0, \frac{\pi}{2}\right]$ (*iv*) $f(x) = (x^2-1)(x-2)$ *on* $[-1, 2]$.

Sol. (*i*) $$f(x) = x(x-3)^2 = x(x^2 - 6x + 9) = x^3 - 6x^2 + 9x$$

$\Rightarrow$ $$f'(x) = 3x^2 - 12x + 9$$

which exists for all x so that f is derivable and hence, continuous for all x.

$\Rightarrow$ f is continuous in $[0, 3]$; f is derivable in $(0, 3)$

Also $f(0) = 0 = f(3)$.

Thus all the three conditions of Rolle's theorem are satisfied. Therefore, there exists at least one value c of x in $(0, 3)$ such that

$$f'(c) = 0 \quad i.e., \quad 3(c^2 - 4c + 3) = 0 \quad\text{or}\quad (c-1)(c-3) = 0$$

$\therefore$ $c = 1, 3.$

But $c = 3 \notin (0, 3) \;\therefore\; c = 1 \in (0, 3)$.

Hence Rolle's theorem is verified.

(*ii*) $$f(x) = \frac{\sin x}{e^x}$$

$$\Rightarrow \qquad f'(x) = \frac{e^x \cos x - \sin x\, e^x}{e^{2x}} = \frac{e^x(\cos x - \sin x)}{e^{2x}} = \frac{\cos x - \sin x}{e^x}$$

[$\because$ $e^x \neq 0$ for any $x \in$ R]

which exists for all x so that f is derivable and, hence, continuous for all x.

$\Rightarrow$ f is continuous in $[0, \pi]$; f is derivable in $(0, \pi)$.

Also $\qquad f(0) = 0 = f(\pi)$

Thus all the three conditions of Rolle's theorem are satisfied. Therefore, there exists at least one value c of x in $(0, \pi)$ such that $f'(c) = 0$

i.e., $\qquad \dfrac{\cos c - \sin c}{e^c} = 0$ or $\cos c - \sin c = 0$ or $\cos c = \sin c$

$\therefore \qquad \tan c = 1 \Rightarrow c = \dfrac{\pi}{4} \in (0, \pi)$

Hence Rolle's theorem is verified.

(*iii*) $\qquad f(x) = \cos 2\left(x - \dfrac{\pi}{4}\right) = \cos\left(2x - \dfrac{\pi}{2}\right) = \cos\left(\dfrac{\pi}{2} - 2x\right) = \sin 2x$

Proceed further yourself. $\qquad \left[\textbf{Ans.}\ c = \dfrac{\pi}{4}\right]$

(*iv*) $\qquad f(x) = (x^2 - 1)(x - 2) = x^3 - 2x^2 - x + 2$

$\Rightarrow \qquad f'(x) = 3x^2 - 4x - 1$

which exists for all x so that f is derivable and, hence, continuous for all x.

$\Rightarrow$ f is continuous in $[-1, 2]$; f is derivable in $(-1, 2)$

Also $\qquad f(-1) = 0 = f(2)$.

Thus all the three conditions of Rolle's theorem are satisfied.

Therefore, there exists at least one value c of x in $(-1, 2)$ such that $f'(c) = 0$

i.e., $\qquad 3c^2 - 4c - 1 = 0$ or $c = \dfrac{4 \pm 2\sqrt{7}}{6} = \dfrac{2 \pm \sqrt{7}}{3} = \dfrac{2 \pm 2.64}{3} = 1.55$ or -0.21

Both these values lie in $(-1, 2)$

$\therefore \qquad c = \dfrac{2 \pm \sqrt{7}}{3}$ such that $f'(c) = 0$

Hence Rolle's theorem is verified.

Example 6. *Verify Rolle's theorem for the following functions :*

(*i*) *$f(x) = x^3 - 6x^2 + 11x - 6$ on $[1, 3] = (x - 1)(x - 2)(x - 3)$*

(*ii*) *$f(x) = \sin x - \sin 2x$ on $[0, \pi]$* $\qquad$ (*iii*) *$f(x) = \log(x^2 + 2) - \log 3$ on $[-1, 1]$*

(*iv*) *$f(x) = \sqrt{1 - x^2}$ on $[-1, 1]$* $\qquad$ (*v*) *$f(x) = \sqrt{x(1 - x)}$ on $[0, 1]$*

(*vi*) *$f(x) = e^x(\sin x - \cos x)$ on $\left\{\dfrac{\pi}{4}, \dfrac{5\pi}{4}\right\}$* $\qquad$ (*vii*) *$f(x) = (x - 1)(x - 4)e^{-x}$ on $[1, 4]$.*

Sol. (*i*) $\qquad f(x) = x^3 - 6x^2 + 11x - 6 \Rightarrow f'(x) = 3x^2 - 12x + 11$

which exists for all x so that f is derivable and, hence, continuous for all x.

$\Rightarrow$ f is continuous in $[1, 3]$; f is derivable in $(1, 3)$.

Also $\qquad f(1) = 0 = f(3)$

Thus all the three conditions of Rolle's theorem are satisfied. Therefore, there exists atleast one value c of x in $(1, 3)$ such that $f'(c) = 0$

i.e., $\qquad 3c^2 - 12c + 11 = 0$ or $c = \dfrac{12 \pm 2\sqrt{3}}{6} = 2 \pm \dfrac{1}{\sqrt{3}} = 2 \pm 0.57 = 2.57$ or 1.43

Both these values lie in $(1, 3)$

$\therefore \qquad c = 2 \pm \dfrac{1}{\sqrt{3}}$ such that $f'(c) = 0$

Hence Rolle's theorem is verified.

(*ii*) $f(x) = \sin x - \sin 2x \Rightarrow f'(x) = \cos x - 2\cos 2x$

which exists for all x so that f is derivable and, hence, continuous for all x.

$\Rightarrow$ f is continuous in $[0, \pi]$; f is derivable in $(0, \pi)$

Also $\qquad f(0) = 0 = f(\pi)$

Thus all the three conditions of Rolle's theorem are satisfied. Therefore, there exists at least one value c of x in ~~$(0, \pi)$ such that~~ $f'(c) = 0$.

i.e., $\qquad \cos c - 2\cos 2c = 0$ *i.e.,* $\cos c - 2(2\cos^2 c - 1) = 0$ or $4\cos^2 c - \cos c - 2 = 0$

$\therefore \qquad \cos c = \dfrac{1 \pm \sqrt{33}}{8} = \dfrac{1 \pm 5.744}{8} = 0.84$ or -0.59

Since $\left|\dfrac{1 \pm \sqrt{33}}{8}\right| < 1$, $\exists\, c_1, c_2$ such that $0 < c_1 < \dfrac{\pi}{2}$, $\cos c_1 = \dfrac{1 + \sqrt{33}}{8}$

$\qquad \dfrac{\pi}{2} < c_2 < \pi$, $\cos c_2 = \dfrac{1 - \sqrt{33}}{8}$

Thus $\quad c = \cos^{-1}\left(\dfrac{1 \pm \sqrt{33}}{8}\right)$ such that $f'(c) = 0$

Hence Rolle's theorem is verified.

(*iii*) *Please try yourself.* $\qquad$ [**Ans.** $c = 0$]

(*iv*) $f(x) = \sqrt{1 - x^2}$

Since $f(x)$ has a unique finite value for each x in $[-1, 1]$

$\therefore$ f is continuous in $[-1, 1]$

$$f'(x) = \frac{-2x}{2\sqrt{1 - x^2}} = \frac{-x}{\sqrt{1 - x^2}} \text{ which exists for all } x \text{ in } (-1, 1).$$

$\therefore$ f is derivable in $(-1, 1)$.

Note that $f'(x)$ is not defined at $x = \pm 1$ so that f is not derivable in $[-1, 1]$.

Also $\qquad f(-1) = 0 = f(1)$.

Thus all the three conditions of Rolle's theorem are satisfied. Therefore, there exists at least one value c of x in $(-1, 1)$ such that $f'(c) = 0$

i.e., $\qquad \dfrac{-c}{\sqrt{1 - c^2}} = 0 \quad \therefore \quad c = 0 \in (-1, 1)$

Hence Rolle's theorem is verified.

(v) *Please try yourself.* [Ans. $c = \frac{1}{2}$]

(vi) *Please try yourself.* [Ans. $c = \pi$]

(vii) *Please try yourself.* $\left[\textbf{Ans. } c = \frac{7-\sqrt{13}}{2}\right]$

Example 7. *If $f(x)$, $g(x)$ and $h(x)$ have derivatives when $a \le x \le b$, show that there is a value c of x in (a, b) such that*

$$\begin{vmatrix} f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \\ f'(c) & g'(c) & h'(c) \end{vmatrix} = 0.$$

Sol. Consider $\quad F(x) = \begin{vmatrix} f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \\ f(x) & g(x) & h(x) \end{vmatrix}$

Expanding the determinant, $F(x)$ is of the form $A f(x) + Bg(x) + Ch(x)$ where A, B, C are constants.

$f(x), g(x), h(x)$ being derivable in $[a, b]$, $F(x)$ is also derivable in $[a, b]$.

Since derivability $\Rightarrow$ continuity, $F(x)$ is continuous in $[a, b]$ and derivable in (a, b) $\because (a, b) \subset [a, b]$

Again $\quad F(a) = 0 = F(b)$

[$\because$ on putting $x = a, b$ two rows become identical and so the determinant vanishes.]

$\therefore$ $F(x)$ satisfies all the three conditions of Rolle's theorem and so there exists a point $c \in (a, b)$ such that $F'(c) = 0$.

Now $\quad F'(x) = Af'(x) + Bg'(x) + Ch'(x) = \begin{vmatrix} f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \\ f'(x) & g'(x) & h'(x) \end{vmatrix}$

$\therefore \quad F'(c) = 0 \Rightarrow \begin{vmatrix} f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \\ f'(c) & g'(c) & h'(c) \end{vmatrix} = 0$ which proves the result.

Example 8. *Let $\frac{a_0}{n+1} + \frac{a_1}{n} + \frac{a_2}{n-1} + \ldots\ldots + \frac{a_{n-1}}{2} + a_n = 0$. Show that the function $a_0x^n + a_1x^{n-1} + \ldots\ldots + a_n$ vanishes at least once in $(0, 1)$.*

Sol. Consider the function

$$f(x) = a_0 \frac{x^{n+1}}{n+1} + a_1 \frac{x^n}{n} + \ldots\ldots + a_{n-1} \frac{x^2}{2} + a_n x, \text{ then}$$

$f(x)$ being a polynomial is continuous and derivable for all x

$\Rightarrow$ $f(x)$ is continuous in $[0, 1]$ and derivable in $(0, 1)$.

Also $\quad f(0) = 0$

$$f(1) = \frac{a_0}{n+1} + \frac{a_1}{n} + \ldots\ldots + \frac{a_{n-1}}{2} + a_n = 0 \text{ (given)}$$

$\therefore$ By Rolle's Theorem, $\exists$ at least one $c \in (0, 1)$ s.t. $f'(c) = 0$

Now $\quad f'(x) = a_0x^n + a_1x^{n-1} + \ldots\ldots + a_{n-1}x + a_n$

$\therefore \quad f'(c) = 0$ for at least one c in $(0, 1)$.

$\Rightarrow \quad a_0x^n + a_1x^{n-1} + \ldots\ldots + a_{n-1}x + a_n$ vanishes at least once in $(0, 1)$.

Example 9. *Show that for no value of k, $f(x) = x^3 - 3x + k$ has two distinct zeros in $(0, 1)$.*

Sol. If possible, let α, β be two distinct zeros of $f(x)$ where $0 < \alpha < \beta < 1$.

$f(x)$ being a polynomial is continuous and derivable for all values of x

$\Rightarrow$ $f(x)$ is continuous in $[\alpha, \beta]$ and derivable in (α, β).

Also $\quad f(\alpha) = 0 = f(\beta)$

$\therefore$ By Rolle's Theorem, there must exist a value c of x in (α, β) s.t.

$$f'(c) = 0 \text{ or } 3c^2 - 3 = 0 \qquad | \because f'(x) = 3x^2 - 3$$

which gives $\quad c = \pm 1$

which contradicts the fact that $\quad 0 < \alpha < c < \beta < 1$

Hence the result.

Example 10. *By considering the function $(x - 4) \log x$, show that the equation $x \log x = 4 - x$ is satisfied by at least one value of x lying between 1 and 4.*

Sol. Let $\quad f(x) = (x - 4) \log x$.

(i) Since the functions $(x - 4)$ and $\log x$ are continuous for all values of x in $1 \le x \le 4$, therefore their product $f(x) = (x - 4) \log x$ **is also continuous in the closed interval** $[1, 4]$.

(ii) $\quad f'(x) = (x - 4) \cdot \frac{1}{x} + 1 \cdot \log x$

which is neither infinite nor indeterminate for any value of x in the open interval $(1, 4)$, so $f(x)$ **is derivable in the open interval** $(1, 4)$.

(iii) Also $\quad f(1) = (1 - 4) \log 1 = -3 \times 0 = 0$

$f(4) = 0 \times \log 4 = 0$.

$\therefore \quad f(1) = f(4)$

Hence $f(x)$ satisfies all the three conditions of Rolle's Theorem. Thus $f'(x)$ must be zero for at least one value of x in $(1, 4)$.

$\therefore \quad f'(x) = \frac{x-4}{x} + \log x = 0$ for at least one value of x in $(1, 4)$

or $\quad x \log x = 4 - x$ for at least one value of x in $(1, 4)$.

i.e., the equation $x \log x = 4 - x$ is satisfied by at least one value of x which lies between 1 and 4. Hence the result.

Example 11. *Show that between any two roots of $e^x \cos x = 1$, there exists at least one root of $e^x \sin x - 1 = 0$.*

Sol. Let $x = a$ and $x = b$ be two distinct roots of the given equation

$$e^x \cos x = 1$$

Then $\quad e^a \cos a = 1$ and $e^b \cos b = 1$

$\Rightarrow \quad \cos a = e^{-a}$ and $\cos b = e^{-b}$ $\quad$ ...(1)

Consider the function f defined on $[a, b]$ as follows :

$$f(x) = e^{-x} - \cos x$$

We find that

(i) f is continuous on $[a, b]$ because e^{-x} and $\cos x$ are continuous.

(ii) $f'(x) = -e^{-x} + \sin x$, which exists for all x in (a, b) so that f is derivable in (a, b).

(iii) $f(a) = e^{-a} - \cos a = e^{-a} - e^{-a} = 0$ $\quad$ [by (1)]

$$f(b) = e^{-b} - \cos b = e^{-b} - e^{-b} = 0 \qquad \text{[by (1)]}$$

$\therefore \qquad f(a) = f(b)$

Thus f satisfies all the three conditions of Rolle's theorem in $[a, b]$.

$\therefore \quad \exists$ at least one $c \in (a, b)$ such that $f'(c) = 0$ *i.e.*, $-e^{-c} + \sin c = 0$ or $e^c \sin c - 1 = 0$

$\Rightarrow \quad c \in (a, b)$ is a root of the equation $e^x \sin x - 1 = 0$

Hence $e^x \sin x - 1 = 0$ has at least one root between any two roots of the equation $e^x \cos x = 1$.

Example 12. *Prove that between any two real roots of $e^x \sin x = 1$, there is at least one real root of*

$$e^x \cos x + 1 = 0.$$

Sol. *Please try yourself.*

[**Hint.** Apply Rolle's theorem to the function $f(x) = e^{-x} - \sin x$.]

7.10. LAGRANGE'S MEAN VALUE THEOREM

(*First Mean Value Theorem of Differential Calculus*)

Statement. If a function f defined on $[a, b]$ is

(*i*) continuous on the closed interval $[a, b]$, and

(*ii*) derivable on the open interval (a, b)

then there exists at least one real number $c \in (a, b)$ such that $\dfrac{f(b) - f(a)}{b - a} = f'(c)$.

Proof. Consider the function $\phi(x) = f(x) + Ax$

where A is a constant to be determined such that $\phi(a) = \phi(b)$

Now $\qquad \phi(a) = \phi(b) \Rightarrow f(a) + Aa = f(b) + Ab \Rightarrow A(a - b) = f(b) - f(a)$

$$\Rightarrow \qquad A = -\frac{f(b) - f(a)}{b - a} \qquad \text{...(1)}$$

Since f is continuous on $[a, b]$ and Ax is continuous on R

$\therefore \quad \phi$ is continuous on $[a, b]$

Also f is derivable on (a, b) and Ax is derivable on R

$\therefore \quad \phi$ is derivable on (a, b)

Further, from the definition of ϕ, $\phi(a) = \phi(b)$

$\therefore$ The function ϕ satisfies all the three conditions of Rolle's theorem. Therefore, there exists at least one real number $c \in (a, b)$ such that $\phi'(c) = 0$.

But $\qquad \phi'(x) = f'(x) + A$

$$\therefore \qquad \phi'(c) = 0 \Rightarrow f'(c) + A = 0 \Rightarrow A = -f'(c) \qquad \text{...(2)}$$

From (1) and (2), we have $\dfrac{f(b) - f(a)}{b - a} = f'(c)$.

Note. $f(b) = f(a) + (b - a) f'(c)$.

Cor. When $f(a) = f(b)$, it reduces to Rolle's theorem.

Another Form of the statement

If a function f defined on $[a, a + h]$ is

(*i*) continuous on $[a, a + h]$, and (*ii*) derivable on $(a, a + h)$

then there exists at least one real number $\theta \in (0, 1)$ such that $f(a + h) = f(a) + hf'(a + \theta h)$.

[**Hint for Proof.** Define $\phi(x) = f(x) + Ax$ such that $\phi(a) = \phi(a + h)$

$$\text{Then} \qquad A = -\frac{f(a + h) - f(a)}{h} \qquad \text{...(1)}$$

Since ϕ satisfies all the three conditions of Rolle's theorem, therefore, there exists at least one real number $\theta \in (0, 1)$ such that

$$\phi'(a + \theta h) = 0 \Rightarrow A = -f'(a + \theta h) \qquad \text{...(2)}$$

From (1) and (2), we have $\dfrac{f(a + h) - f(a)}{h} = f'(a + \theta h)$ or $f(a + h) = f(a) + hf'(a + \theta h)$.]

7.11. GEOMETRICAL INTERPRETATION

Let A and B be the points on the graph of the function $y = f(x)$ corresponding to $x = a$ and $x = b$, then the co-ordinates of A and B are $(a, f(a))$ and $(b, f(b))$ respectively.

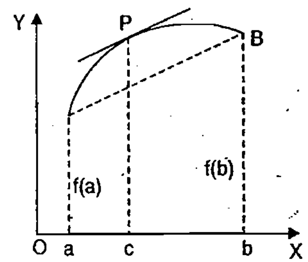
Now slope of chord $AB = \dfrac{f(b) - f(a)}{b - a}$ $\qquad \left[\because \text{ Slope} = \dfrac{y_2 - y_1}{x_2 - x_1}\right]$

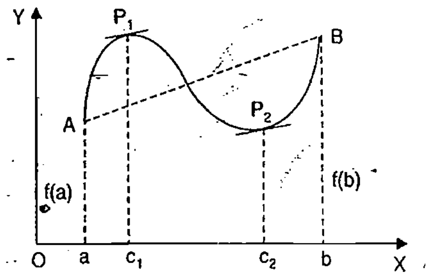
Since f is continuous on $[a, b]$, the graph of f has no gap or jump from A to B. Also f is derivable on (a, b), therefore, at every point between a and b, the curve has a unique tangent. It is evident from the figures that there is at least one point between A and B the tangent at which is parallel to the chord AB.

If 'c' be the abscissa of P, then the slope of tangent at P is $f'(c)$.

$\therefore$ Chord AB is parallel to the tangent at P

$$\Rightarrow \qquad \frac{f(b) - f(a)}{b - a} = f'(c).$$





7.12. IMPORTANT DEDUCTIONS FROM LAGRANGE'S MEAN VALUE THEOREM

1. *If a function f is (i) continuous on $[a, b]$ and (ii) derivable on (a, b), then*

$$f'(x) = 0 \ \forall \ x \in [a, b] \Rightarrow f \text{ is constant on } [a, b].$$

Proof. Let x_1, x_2 (where $x_1 < x_2$) be any two distinct points of $[a, b]$ so that $[x_1, x_2] \subset [a, b]$. Then f satisfies both the conditions of Lagrange's Mean Value Theorem on $[x_1, x_2]$.

$$\therefore \quad \exists \ c \in (x_1, x_2) \text{ such that } \frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) \qquad \text{...(1)}$$

But $\qquad f'(x) = 0 \ \forall \ x \in [a, b]$ and $x_1 < c < x_2 \Rightarrow f'(c) = 0$

$\therefore$ From (1), $f(x_2) - f(x_1) = 0 \Rightarrow f(x_1) = f(x_2)$.

Since x_1 and x_2 are any two distinct points of $[a, b]$, it follows that f keeps the same value for every $x \in [a, b]$.

Hence f is constant on $[a, b]$.

2. *If two functions f and g are (i) continuous on [a, b] and (ii) derivable on (a, b), then*

$$f'(x) = g'(x) \ \forall\, x \in [a, b] \Rightarrow f - g \text{ is constant on } [a, b].$$

Proof. Consider $\phi(x) = f(x) - g(x) \ \forall\, x \in [a, b]$

Clearly, ϕ is continuous on $[a, b]$; ϕ is derivable on (a, b)

$$\phi'(x) = f'(x) - g'(x) = 0 \ \forall\, x \in [a, b]$$

$\therefore$ By deduction 1 above, ϕ is constant on $[a, b]$

$\Rightarrow$ $f - g$ is constant on $[a, b]$.

3. *If a function f is (i) continuous on [a, b] and (ii) derivable on (a, b), then*

$$f'(x) > 0 \ \forall\, x \in [a, b] \Rightarrow f \text{ is strictly increasing on } [a, b].$$

Proof. Let x_1, x_2 (where $x_1 < x_2$) be any two distinct points of $[a, b]$ so that $[x_1, x_2] \subset [a, b]$. Then f satisfies both the conditions of Lagrange's mean value theorem on $[x_1, x_2]$

$$\therefore \ \exists\, c \in (x_1, x_2) \text{ such that } \frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) \text{ or } f(x_2) - f(x_1) = (x_2 - x_1) f'(c) \qquad ...(1)$$

Now $x_1 < x_2 \Rightarrow x_2 - x_1 > 0$

$f'(x) > 0 \ \forall\, x \in [a, b]$ and $x_1 < c < x_2 \Rightarrow f'(c) > 0$

$\therefore$ From (1), $f(x_2) - f(x_1) > 0$ or $f(x_1) < f(x_2)$

Since $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

$\therefore$ f is strictly increasing on $[a, b]$.

4. *If a function f is (i) continuous on [a, b] and (ii) derivable on (a, b) then*

$$f'(x) < 0 \ \forall\, x \in [a, b] \Rightarrow f \text{ is strictly decreasing on } [a, b].$$

Proof. Proceeding as in deduction 3 above, we have

$$f(x_2) - f(x_1) = (x_2 - x_1) f'(c) \qquad ...(1)$$

Now $x_1 < x_2 \Rightarrow x_2 - x_1 > 0$

$f'(x) < 0 \ \forall\, x \in [a, b]$ and $x_1 < c < x_2 \Rightarrow f'(c) < 0$

$\therefore$ From (1), $f(x_2) - f(x_1) < 0$ or $f(x_1) > f(x_2)$

Since $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$

$\therefore$ f is strictly decreasing on $[a, b]$.

ILLUSTRATIVE EXAMPLES

Example 1. *Verify mean value theorem for the following functions in the specified intervals :*

(a) $f(x) = x(x-1)(x-2)$ *in* $[0, \frac{1}{2}]$

(b) $f(x) = x^2 - 3x + 2$ *in* $[-2, 3]$.

Sol. (a) $\mathbf{f(x) = x(x-1)(x-2)}$ **in** $[0, \frac{1}{2}]$

(i) $f(x) = x(x-1)(x-2) = x^3 - 3x^2 + 2x$ being a polynomial in x is continuous in the closed interval $[0, \frac{1}{2}]$.

(ii) $f'(x) = 3x^2 - 6x + 2$ and $f'(x)$ is always unique and definite for every value of x in $(0, \frac{1}{2})$. Hence $f(x)$ is derivable in $(0, \frac{1}{2})$.

$\therefore$ $f(x)$ satisfies the two conditions of Mean Value Theorem and so there must exist a point c **within** $(0, \frac{1}{2})$ such that

$$\frac{f(\frac{1}{2}) - f(0)}{\frac{1}{2} - 0} = f'(c) \qquad \left| \because \ \frac{f(b) - f(a)}{b - a} = f'(c) \right.$$

$$\frac{\frac{3}{8} - 0}{\frac{1}{2}} = 3c^2 - 6c + 2 \text{ or } 3c^2 - 6c + 2 = \frac{3}{4} \text{ or } 12c^2 - 24c + 5 = 0$$

$$\therefore \quad c = \frac{24 \pm \sqrt{576 - 240}}{24} = \frac{24 \pm \sqrt{336}}{24} = \frac{24 \pm 4\sqrt{21}}{24} = \frac{6 \pm \sqrt{21}}{6}$$

Now the two values of c are $1 + \frac{1}{6}\sqrt{21}$ and $1 - \frac{1}{6}\sqrt{21}$ and of these two, the second value lies between 0 and $\frac{1}{2}$. Hence there exists at least one value $c = 1 - \frac{1}{6}\sqrt{21}$ within $(0, \frac{1}{2})$ such that

$$\frac{f(\frac{1}{2}) - f(0)}{\frac{1}{2} - 0} = f'(c).$$

Hence the verification of mean value theorem.

(b) $\mathbf{f(x) = x^2 - 3x + 2}$ **in** $[-2, 3]$.

Hence $f(a) = f(-2) = 4 + 6 + 2 = 12$

$f(b) = f(3) = 9 - 9 + 2 = 2$

Now $f(x)$ being a polynomial in x is continuous for every value of x and so it is also continuous in the closed interval $[-2, 3]$.

Again, $f'(x) = 2x - 3$ which is finite and unique for every x in $(-2, 3)$ and so $f(x)$ is derivable in $(-2, 3)$.

Thus $f(x)$ satisfies both the conditions of Mean Value Theorem.

$\therefore$ there must exist a value c of x within $(-2, 3)$ for which

$$\frac{f(b) - f(a)}{b - a} = f'(c) \quad \text{or} \quad \frac{2 - 12}{3 - (-2)} = 2c - 3$$

or $\qquad 10c - 15 = -10$ or $10c = 5$ $\therefore$ $c = \frac{1}{2}$

which is clearly within the interval $(-2, 3)$.

Hence the verification of Mean Value Theorem.

Example 2. *Discuss the applicability of Mean Value Theorem to the following functions in the specified intervals :*

(a) $f(x) = \begin{cases} 2 & \text{if } x = 1 \\ x^2 & \text{if } 1 < x < 2 \text{ in } [1, 2] \\ 1 & \text{if } x = 2 \end{cases}$ $\qquad$ (b) $f(x) = |x|$ *for all x in* $[-1, 2]$.

Sol. (a) We have $\quad f(x) = \begin{cases} 2 & \text{if } x = 1 \\ x^2 & \text{if } 1 < x < 2 \\ 1 & \text{if } x = 2. \end{cases}$

Thus $f(1) = 2$ and $f(2) = 1$.

Since $f(x) = x^2$ is a polynomial function in $1 < x < 2$, and every polynomial function is continuous at each point, therefore, $f(x)$ is continuous for each x in $1 < x < 2$.

Also $\lim_{x \to 1+0} f(x) = \lim_{x \to 1+0} x^2$ | **Put x = 1 + h**

$= \lim_{h \to 0} (1+h)^2 = \lim_{h \to 0} (1 + 2h + h^2) = 1$

and $\lim_{x \to 2-0} f(x) = \lim_{h \to 2-0} x^2$ | **Put x = 2 – h**

$= \lim_{h \to 0} (2-h)^2 = \lim_{h \to 0} (4 - 4h + h^2) = 4$

Since $\lim_{x \to 1+0} f(x) \neq f(1)$ and $\lim_{h \to 2-0} f(x) \neq f(2)$,

therefore $f(x)$ is not continuous at the end points $x = 1$ and $x = 2$.

$\therefore$ $f(x)$ is only continuous in the open interval $(1, 2)$ and not in the closed interval. Thus $f(x)$ **violates the first condition** of Lagrange's Mean Value Theorem and hence Mean Value Theorem is **not applicable** to $f(x)$.

(*b*) We have $f(x) = |x|$ for all x in $[-1, 2]$.

Here $f(x) = |x|$ is continuous in the closed interval $[-1, 2]$.

Also $f(x)$ is derivable at each point of the open interval $(-1, 2)$ except at $x = 0$. Thus $f(x)$ is not derivable in the open interval $(-1, 2)$ and hence the second condition of the Mean Value Theorem is not satisfied and therefore Mean Value Theorem is **not applicable** to $f(x)$ in $[-1, 2]$.

Note. The above example illustrates that the hypothesis of Lagrange's Mean Value Theorem cannot be weakened.

Example 3. *If $f(x) = (x-1)(x-2)(x-3)$; $a = 0, b = 4$, find c of Lagrange's Mean Value Theorem.*

Sol. $f(x) = (x-1)(x-2)(x-3) = x^3 - 6x^2 + 11x - 6$

$f(a) = f(0) = -6 \qquad f(b) = f(4) = (3)(2)(1) = 6$

$f'(x) = 3x^2 - 12x + 11 \quad \therefore \quad f'(c) = 3c^2 - 12c + 11$

$\therefore \quad \dfrac{f(b) - f(a)}{b - a} = f'(c) \quad \Rightarrow \quad \dfrac{6 - (-6)}{4 - 0} = 3c^2 - 12c + 11$

$\Rightarrow \quad 3 = 3c^2 - 12c + 11 \quad \Rightarrow \quad 3c^2 - 12c + 8 = 0$

$\Rightarrow \quad c = \dfrac{12 \pm \sqrt{144 - 96}}{6} = \dfrac{12 \pm \sqrt{48}}{6} = \dfrac{12 \pm 4\sqrt{3}}{6} = 2 \pm \dfrac{2}{\sqrt{3}}$

Both the values lie in the open interval $(0, 4)$

Hence $c = 2 \pm \dfrac{2}{\sqrt{3}}$.

Example 4. *Verify Lagrange's Mean Value Theorem for $f(x) = lx^2 + mx + n$, $x \in [a, b]$.*

Sol. $f(x) = lx^2 + mx + n$ being a polynomial is continuous over R and hence in $[a, b]$.

$f'(x) = 2lx + m$ which exists finitely for all x in (a, b).

$\therefore$ $f(x)$ satisfies both the conditions of Lagrange's Mean Value Theorem.

$\therefore \quad \exists\, c \in (a, b)$ s.t. $\dfrac{f(b) - f(a)}{b - a} = f'(c)$

$\Rightarrow \quad \dfrac{(lb^2 + mb + n) - (la^2 + ma + n)}{b - a} = 2lc + m \quad \Rightarrow \quad \dfrac{l(b^2 - a^2) + m(b - a)}{b - a} = 2lc + m$

$\Rightarrow \quad \dfrac{(b - a)[l(b + a) + m]}{b - a} = 2lc + m \quad \Rightarrow \quad l(b + a) + m = 2lc + m \quad (\because \; a \neq b)$

$\Rightarrow \quad l(b + a) = 2lc \quad \Rightarrow \quad c = \dfrac{a + b}{2} \quad (\because \; l \neq 0)$

Since $c = \dfrac{a+b}{2} \in (a, b)$, Lagrange's Mean Value Theorem is verified.

Example 5. *Discuss the applicability of Lagrange's Mean Value Theorem to*

(*i*) $f(x) = \dfrac{1}{x}$ *in* $[-1, 1]$ (*ii*) $f(x) = x^{1/3}$ *in* $[-1, 1]$.

Sol. (*i*) $f(x) = \dfrac{1}{x}$ $f(0)$ is not finite while $0 \in [-1, 1]$

L.H.S. $= -\infty$ R.H.L. $= +\infty$ (verify)

$\Rightarrow$ $f(x)$ is not continuous at $x = 0$

$\therefore$ Lagrange's Mean Value Theorem is not applicable.

(*ii*) $f(x) = x^{1/3}$ $f'(x) = \dfrac{1}{3} x^{-2/3} = \dfrac{1}{3x^{2/3}}$ which does not exist finitely at $x = 0 \in (-1, 1)$.

$\Rightarrow$ $f(x)$ is not differentiable in $(-1, 1)$.

$\therefore$ Lagrange's Mean Value Theorem is not applicable.

However $\dfrac{f(1) - f(-1)}{1 - (-1)} = f'(c)$

$\Rightarrow \quad \dfrac{1 - (-1)}{2} = \dfrac{1}{3c^{2/3}} \quad \Rightarrow \quad c^{2/3} = \dfrac{1}{3} \quad \Rightarrow \quad c = (\tfrac{1}{3})^{3/2} = \dfrac{1}{3\sqrt{3}}$ and $\dfrac{1}{3\sqrt{3}} \in (-1, 1)$

Hence the hypothesis of Lagrange's Mean Value theorem is not valid but the conclusion is. In other words, the two conditions of L.M.V. theorem are sufficient but not necessary.

Example 6. *Verify Lagrange's mean value theorem for the following functions :*

(*i*) $f(x) = 2x^2 - 10x + 29$ *on* $[2, 7]$ (*ii*) $f(x) = x^3 + x^2 - 6x$ *on* $[-1, 4]$

(*iii*) $f(x) = \log x$ *on* $[1, e]$ (*iv*) $f(x) = \sqrt{x^2 - 4}$ *on* $[2, 4]$

(*v*) $f(x) = e^x$ *on* $[0, 1]$.

Sol. *Please try yourself.* [**Ans.** (*i*) $c = 4.5$, (*ii*) $c = 2$, (*iii*) $c = e - 1$, (*iv*) $c = \sqrt{6}$, (*v*) $c = \log(e - 1)$.]

Example 7. *Show that if $x > 0$, $\log(1 + x) > \dfrac{x}{1 + x}$ and hence prove that $x^{-1} \log(1 + x)$ decreases monotonically as x increases from 0 to ∞.*

Sol. Let $f(x) = \log(1 + x) - \dfrac{x}{1 + x}$

$\therefore \quad f'(x) = \dfrac{1}{1+x} - \dfrac{(1+x).1 - x.1}{(1+x)^2} = \dfrac{1}{1+x} - \dfrac{1}{(1+x)^2} = \dfrac{1 + x - 1}{(1+x)^2} = \dfrac{x}{(1+x)^2}$

which is positive, because $x > 0$

$\therefore$ $f(x)$ is monotonic increasing when $x > 0 \Rightarrow f(x) > f(0)$

Now $f(0) = \log 1 - 0 = 0$

$\therefore$ $f(x) > 0 \Rightarrow \log(1+x) - \frac{x}{1-x} > 0 \Rightarrow \log(1+x) > \frac{x}{1+x}$

Now consider $F(x) = x^{-1}\log(1+x) = \frac{\log(1+x)}{x}$

$$\therefore \quad F'(x) = \frac{x \cdot \frac{1}{1+x} - \log(1+x) \cdot 1}{x^2} = -\frac{\log(1+x) - \frac{x}{1+x}}{x^2}$$

$$= -\frac{f(x)}{x^2} < 0 \text{ for } x > 0 \qquad [\because f(x) > 0, \text{ proved above}]$$

$\Rightarrow$ $F(x) = x^{-1}\log(1+x)$ is a monotonic decreasing function in $(0, \infty)$.

Example 8. *If $x > 0$, show that $x - \frac{x^2}{2} < \log(1+x) < x - \frac{x^2}{2(1+x)}$.*

Sol. Let $f(x) = x - \frac{x^2}{2} - \log(1+x)$

$$\therefore \quad f'(x) = 1 - x - \frac{1}{1+x} = \frac{1-x^2-1}{1+x} = -\frac{x^2}{1+x} < 0 \text{ for } x > 0$$

$\Rightarrow$ $f(x)$ is monotonic decreasing for $x > 0 \Rightarrow f(x) < f(0)$

But $f(0) = 0 - 0 - \log 1 = 0$

$$\therefore \quad f(x) < 0 \Rightarrow x - \frac{x^2}{2} - \log(1+x) < 0 \Rightarrow x - \frac{x^2}{2} < \log(1+x) \qquad ...(1)$$

Now let $g(x) = \log(1+x) - x + \frac{x^2}{2(1+x)}$

$$\therefore \quad g'(x) = \frac{1}{1+x} - 1 + \frac{1}{2}\cdot\frac{(1+x).2x - x^2}{(1+x)^2} = \frac{1-1-x}{1-x} + \frac{1}{2}\cdot\frac{2x+x^2}{(1+x)^2} = -\frac{x}{1+x} + \frac{2x+x^2}{2(1+x)^2}$$

$$= \frac{-2x(1+x)+2x+x^2}{2(1+x)^2} = -\frac{x^2}{2(1+x)^2} < 0 \text{ for } x > 0$$

$\Rightarrow$ $g(x)$ is monotonic decreasing for $x > 0 \Rightarrow g(x) < g(0)$

But $g(0) = 0 \therefore g(x) < 0 \Rightarrow \log(1+x) - x + \frac{x^2}{2(1+x)} < 0$

$$\therefore \quad \log(1+x) < x - \frac{x^2}{2(1+x)} \qquad ...(2)$$

Combining (1) and (2), $x - \frac{x^2}{2} < \log(1+x) < x - \frac{x^2}{2(1+x)}$.

Example 9. *If $x > 0$, show that $\frac{x}{1+x} < \log(1+x) < x$.*

Sol. *Please try yourself.*

Example 10. *Show that $\frac{x^2}{2} > x - \log(1+x) > \frac{x^2}{2(1+x)}$ for $x > 0$.*

Sol. *Please try yourself.*

Example 11. *Prove that $x - \frac{x^2}{2} + \frac{x^3}{3(1+x)} < \log(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3}$ for $x > 0$.*

Sol. Let $f(x) = x - \frac{x^2}{2} + \frac{x^3}{3(1+x)} - \log(1+x)$

$$\therefore \quad f'(x) = 1 - x + \frac{1}{3}\cdot\frac{(1+x).3x^2 - x^3.1}{(1+x)^2} - \frac{1}{1+x}$$

$$= 1 - x + \frac{2x^3+3x^2}{3(1+x)^2} - \frac{1}{1+x} = \frac{3(1-x)(1+x)^2 + 2x^3 + 3x^2 - 3(1+x)}{3(1+x)^2}$$

$$= \frac{3+3x-3x^2-3x^3+2x^3+3x^2-3-3x}{3(1+x)^2} = -\frac{x^3}{3(1+x)^2} < 0 \text{ for } x > 0$$

$\Rightarrow$ $f(x)$ is monotonic decreasing for $x > 0 \Rightarrow f(x) < f(0)$

But $f(0) = 0 \therefore f(x) < 0$

$$\Rightarrow \quad x - \frac{x^2}{2} + \frac{x^3}{3(1+x)} - \log(1+x) < 0$$

$$\Rightarrow \quad x - \frac{x^2}{2} + \frac{x^3}{3(1+x)} < \log(1+x) \qquad ...(1)$$

Now let $g(x) = \log(1+x) - x - \frac{x^2}{2} - \frac{x^3}{3}$

$$\therefore \quad g'(x) = \frac{1}{1+x} - 1 + x - x^2 = \frac{1}{1+x} - (1 - x + x^2)$$

$$= \frac{1-(1+x^3)}{1+x} = -\frac{x^3}{1+x} < 0 \text{ for } x > 0$$

$\Rightarrow$ $g(x)$ is monotonic decreasing for $x > 0 \Rightarrow g(x) < g(0)$

But $g(0) = 0 \therefore g(x) < 0$

$$\Rightarrow \quad \log(1+x) - x + \frac{x^2}{2} - \frac{x^3}{3} < 0$$

$$\Rightarrow \quad \log(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3} \qquad ...(2)$$

Combining (1) and (2), we have

$$x - \frac{x^2}{2} + \frac{x^3}{3(1+x)} < \log(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3}.$$

Example 12. *Apply Lagrange's mean value theorem to the function $\log(1+x)$ to show that*

$$0 < [\log(1+x)]^{-1} - x^{-1} < 1 \ \forall\, x > 0.$$

Sol. We know that for +ve real values of x, the function f defined by $f(x) = \log(1+x)$ is continuous on $[0, x]$ and differentiable on $(0, x)$.

$\therefore$ By Lagrange's mean value theorem, $\exists$ a real number $\theta \in (0, 1)$ such that

$$f(x) - f(0) = (x-0) f'(\theta x)$$

$$\Rightarrow \quad \log(1+x) - \log 1 = \frac{x}{1+\theta x} \qquad \left[\because f'(x) = \frac{1}{1+x}\right]$$

$$\Rightarrow \quad \log(1+x) = \frac{x}{1+\theta x} \qquad \ldots(1)$$

Now $\quad x > 0$ and $0 < \theta < 1 \quad \Rightarrow \quad 0 < \theta x < x$

$$\Rightarrow \quad 1 < 1 + \theta x < 1 + x \quad \Rightarrow \quad 1 > \frac{1}{1+\theta x} > \frac{1}{1+x}$$

$$\Rightarrow \quad \frac{x}{1+x} < \frac{x}{1+\theta x} < x \quad \Rightarrow \quad \frac{x}{1+x} < \log(1+x) < x \quad \text{[using (1)]}$$

$$\Rightarrow \quad \frac{1+x}{x} > \frac{1}{\log(1+x)} > \frac{1}{x} \quad \Rightarrow \quad 1 + \frac{1}{x} > [\log(1+x)]^{-1} > \frac{1}{x}$$

$$\Rightarrow \quad 1 > [\log(1+x)]^{-1} - \frac{1}{x} > 0 \quad \Rightarrow \quad 0 < [\log(1+x)]^{-1} - x^{-1} < 1.$$

Example 13. *Use Lagrange's mean value theorem to prove that*

$$1 + x < e^x < 1 + xe^x \quad \forall\, x > 0.$$

Sol. Consider the function $f(x) = e^x$ on $[0, x]$.

Clearly, f is continuous on $[0, x]$ and derivable on $(0, x)$.

$\therefore$ By Lagrange's mean value theorem, $\exists$ a real number $c \in (0, x)$ such that

$$\frac{f(x) - f(0)}{x - 0} = f'(c) \quad \Rightarrow \quad \frac{e^x - e^0}{x} = e^c \quad \Rightarrow \quad \frac{e^x - 1}{x} = e^c. \qquad \ldots(1)$$

Since $0 < c < x$ and e^x is an increasing function on R

$$\therefore \quad e^0 < e^c < e^x \quad \Rightarrow \quad 1 < \frac{e^x - 1}{x} < e^x \qquad \text{[using (1)]}$$

$$\Rightarrow \quad x < e^x - 1 < xe^x \quad \Rightarrow \quad 1 + x < e^x < 1 + xe^x.$$

Example 14. *Show that* $\dfrac{v-u}{1+v^2} < \tan^{-1} v - \tan^{-1} u < \dfrac{v-u}{1+u^2}$ *if* $0 < u < v$ *and deduce that*

$$\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}.$$

Sol. Let $f(x) = \tan^{-1} x$, then $f'(x) = \dfrac{1}{1+x^2}$

Applying Lagrange's mean value theorem to f on $[u, v]$, we get

$$\frac{f(v) - f(u)}{v - u} = f'(c) \text{ where } u < c < v$$

$$\Rightarrow \quad \frac{\tan^{-1} v - \tan^{-1} u}{v - u} = \frac{1}{1+c^2} \qquad \ldots(1)$$

Now $\quad c > u \;\Rightarrow\; \dfrac{1}{1+c^2} < \dfrac{1}{1+u^2}$ and $c < v \;\Rightarrow\; \dfrac{1}{1+c^2} > \dfrac{1}{1+v^2}$

$$\therefore \quad \frac{1}{1+v^2} < \frac{1}{1+c^2} < \frac{1}{1+u^2}$$

$$\Rightarrow \quad \frac{1}{1+v^2} < \frac{\tan^{-1} v - \tan^{-1} u}{v-u} < \frac{1}{1+u^2} \qquad \text{[using (1)]}$$

$$\Rightarrow \quad \frac{v-u}{1+v^2} < \tan^{-1} v - \tan^{-1} u < \frac{v-u}{1+u^2}$$

Taking $v = \dfrac{4}{3}$ and $u = 1$, we have

$$\frac{\frac{1}{3}}{1 + \frac{16}{9}} < \tan^{-1}\frac{4}{3} - \tan^{-1} 1 < \frac{\frac{1}{3}}{1+1}$$

$$\Rightarrow \quad \frac{3}{25} < \tan^{-1}\frac{4}{3} - \frac{\pi}{4} < \frac{1}{6} \quad \Rightarrow \quad \frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}.$$

Example 15. *If a function f is such that its derivative f' is continuous on $[a, b]$ and derivable on (a, b), then show that there exists a number c between a and b such that*

$$f(b) = f(a) + (b-a) f'(a) + \tfrac{1}{2}(b-a)^2 f''(c).$$

Sol. f' is continuous on $[a, b] \Rightarrow f'$ exists on $[a, b]$

$\Rightarrow$ f is derivable on $[a, b] \quad \Rightarrow$ f is continuous on $[a, b]$

$\therefore$ The functions f and f' are continuous on $[a, b]$ and derivable on (a, b).

Consider the function ϕ on $[a, b]$ defined by

$$\phi(x) = f(x) + (b-x) f'(x) + (b-x)^2 . A$$

where A is a constant to be determined such that $\phi(a) = \phi(b)$

$$\therefore \quad f(a) + (b-a) f'(a) + (b-a)^2 . A = f(b)$$

or $\quad f(b) = f(a) + (b-a) f'(a) + (b-a)^2 A \qquad \ldots(1)$

Now f and f' are continuous on $[a, b]$.

$(b-x)$ and $(b-x)^2 . A$ are continuous on R

$\Rightarrow$ ϕ is continuous on $[a, b]$

f and f' are derivable on (a, b)

$(b-x)$ and $(b-x)^2 . A$ are derivable on R

$\Rightarrow$ ϕ is derivable on (a, b)

Also $\quad \phi(a) = \phi(b)$

Thus ϕ satisfies all the three conditions of Rolle's theorem.

$\because \quad \exists\, c \in (a, b)$ such that $\phi'(c) = 0$.

But $\quad \phi'(x) = f'(x) - f'(x) + (b-x) f''(x) - 2(b-x) . A = (b-x)[f''(x) - 2A]$

$\therefore \quad \phi'(c) = 0 \Rightarrow (b-c)[f''(c) - 2A] = 0 \Rightarrow f''(c) - 2A = 0 \quad [\because a < c < b, b - c \neq 0]$

$$\Rightarrow \quad A = \tfrac{1}{2} f''(c)$$

$\therefore$ From (1), we have $f(b) = f(a) + (b-a) f'(a) + \frac{1}{2}(b-a)^2 f''(c)$ where $a < c < b$.

Example 16. *If a function f is twice differentiable on an interval $[a, a+h]$, then show that*

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a+\theta h) \text{ for some real number } \theta \text{ where } 0 < \theta < 1.$$

Sol. f is twice differentiable on $[a, a+h]$

$\Rightarrow$ f', f'' exist on $[a, a+h]$

$\Rightarrow$ f, f' are derivable on $[a, a+h]$ and hence continuous on $[a, a+h]$.

Consider the function ϕ on $[a, a+h]$ defined by

$$\phi(x) = f(x) + (a+h-x) f'(x) + \frac{(a+h-x)^2}{2!} . A$$

where A is a constant to be determined such that $\phi(a) = \phi(a+h)$

$$\therefore \quad f(a) + hf'(a) + \frac{h^2}{2!} A = f(a+h)$$

or $$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} . A \quad ...(1)$$

Now f and f' are continuous on $[a, a+h]$

$(a+h-x)$ and $\frac{(a+h-x)^2}{2!} . A$ are continuous on R

$\Rightarrow$ ϕ is continuous on $[a, a+h]$

f and f' are derivable on $(a, a+h)$

$(a+h-x)$ and $\frac{(a+h-x)^2}{2!} . A$ are derivable on R

$\Rightarrow$ ϕ is derivable on $(a, a+h)$

Also $\phi(a) = \phi(a+h)$

Thus ϕ satisfies all the three conditions of Rolle's theorem.

$\therefore$ $\exists$ a real number $\theta \in (0, 1)$ such that $\phi'(a+\theta h) = 0$

But $\phi'(x) = f'(x) - f'(x) + (a+h-x) f''(x) - (a+h-x) . A = (a+h-x)[f''(x) - A]$

$\therefore$ $\phi'(a+\theta h) = 0 \Rightarrow (a+h-a-\theta h)[f''(a+\theta h) - A] = 0$

$\Rightarrow$ $h(1-\theta)[f''(a+\theta h) - A] = 0 \Rightarrow f''(a+\theta h) - A = 0 \quad [\because h > 0 \text{ and } \theta < 1]$

$\Rightarrow$ $A = f''(a+\theta h)$

$\therefore$ From (1), we have $f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a+\theta h)$ where $0 < \theta < 1$.

Example 17. *A twice differentiable function f on $[a, b]$ is such that $f(a) = f(b) = 0$ and $f(c) > 0$ for $a < c < b$. Prove that there is atleast one value ξ between a and b for which $f''(\xi) < 0$.*

Sol. f is twice differentiable on $[a, b]$.

$\Rightarrow$ f', f'' exist on $[a, b]$.

$\Rightarrow$ f, f' are derivable on $[a, b]$ and hence continuous on $[a, b]$.

Since $a < c < b$, applying Lagrange's mean value theorem to f on the intervals $[a, c]$ and $[c, b]$, we get

$$\frac{f(c) - f(a)}{c - a} = f'(\xi_1) \quad \text{where} \quad a < \xi_1 < c$$

and $$\frac{f(b) - f(c)}{b - c} = f'(\xi_2) \text{ where } c < \xi_2 < b$$

But $f(a) = f(b) = 0$

$$\therefore \quad f'(\xi_1) = \frac{f(c)}{c-a} \quad \text{and} \quad f'(\xi_2) = -\frac{f(c)}{b-c} \text{ where } a < \xi_1 < c < \xi_2 < b.$$

Again f' is continuous and derivable on $[\xi_1, \xi_2]$.

$\therefore$ By Lagrange's mean value theorem, we have

$$\frac{f'(\xi_2) - f'(\xi_1)}{\xi_2 - \xi_1} = f''(\xi) \text{ where } \xi_1 < \xi < \xi_2$$

Substituting the values of $f'(\xi_1)$ and $f'(\xi_2)$, we get

$$f''(\xi) = \frac{1}{\xi_2 - \xi_1}\left[-\frac{f(c)}{b-c} - \frac{f(c)}{c-a}\right] = \frac{-f(c)}{\xi_2 - \xi_1}\left(\frac{1}{b-c} + \frac{1}{c-a}\right) = \frac{-(b-a) f(c)}{(\xi_2 - \xi_1)(b-c)(c-a)}$$

Since $a < \xi_1 < c < \xi_2 < b$ and $f(c) > 0$

$\therefore$ $f''(\xi) < 0$ where $a < \xi < b$.

Example 18. *Prove that $|\sin x - \sin y| \le |x - y| \ \forall x, y \in R$.*

Sol. If $x = y$, there is nothing to prove.

If $x > y$, then consider the function $f(x) = \sin x$ on $[y, x]$

Clearly, f is continuous on $[y, x]$ and derivable on (y, x).

$\therefore$ By Lagrange's mean value theorem, there exists $\alpha \in (y, x)$ such that

$$\frac{f(x) - f(y)}{x - y} = f'(\alpha) \Rightarrow \frac{\sin x - \sin y}{x - y} = \cos \alpha \quad [\because f'(x) = \cos x]$$

Since $|\cos \alpha| \le 1$

$$\therefore \quad \left|\frac{\sin x - \sin y}{x - y}\right| \le 1 \Rightarrow |\sin x - \sin y| \le |x - y|$$

If $y > x$, then in a similar manner, we have

$$|\sin y - \sin x| \le |y - x| \Rightarrow |\sin x - \sin y| \le |x - y| \quad [\because |-a| = |a|]$$

Hence for all $x, y \in R$, $|\sin x - \sin y| \le |x - y|$.

7.13. CAUCHY'S MEAN VALUE THEOREM (*Second Mean Value Theorem of Differential Calculus*)

Statement. *If two functions f and g defined on $[a, b]$ are*

(i) continuous on the closed interval $[a, b]$ (ii) derivable on the open interval (a, b)

(iii) $g'(x) \ne 0$ for any $x \in (a, b)$

then there exists atleast one real number $c \in (a, b)$ such that $\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$.

Proof. First we notice that $g(a) \ne g(b)$, for otherwise, as a consequence of conditions (*i*) and (*ii*) of the theorem, the function g would satisfy all the conditions of Rolle's theorem and, hence, for some $x \in (a, b)$ we would have $g'(x) = 0$ which contradicts condition (*iii*) of theorem.

Consider the function ϕ on $[a, b]$ defined by $\phi(x) = f(x) + Ag(x)$

where A is a constant to be determined such that $\phi(a) = \phi(b)$

$$\therefore \quad f(a) + Ag(a) = f(b) + Ag(b) \quad \text{or} \quad A[g(a) - g(b)] = f(b) - f(a)$$

or $$A = -\frac{f(b)-f(a)}{g(b)-g(a)} \qquad ...(1)$$

which exists since $g(a) \neq g(b)$.

Now ϕ is continuous on $[a, b]$, being the sum of two continuous functions on $[a, b]$.

ϕ is derivable on (a, b), being the sum of two derivable functions on (a, b).

Also $\phi(a) = \phi(b)$

$\therefore$ By Rolle's theorem, $\exists$ a real number $c \in (a, b)$ such that $\phi'(c) = 0$

But $\phi'(x) = f'(x) + Ag'(x)$

$\therefore$ $\phi'(c) = 0 \Rightarrow f'(c) + Ag'(c) = 0$

$$\Rightarrow \quad A = -\frac{f'(c)}{g'(c)} \qquad ...(2)$$

From (1) and (2), we have $\dfrac{f(b)-f(a)}{g(b)-g(a)} = \dfrac{f'(c)}{g'(c)}$.

Cor. When $g(x) = x$, Cauchy's mean value theorem reduces to Lagrange's mean value theorem.

Another Form of the Statement

If two functions f and g defined on $[a, a+h]$ are

(*i*) continuous on $[a, a+h]$ (*ii*) derivable on $(a, a+h)$

(*iii*) $g'(x) \neq 0$ for any $x \in (a, a+h)$

then there exists atleast one real number $\theta \in (0, 1)$ such that $\dfrac{f(a+h)-f(a)}{g(a+h)-g(a)} = \dfrac{f'(a+\theta h)}{g'(a+\theta h)}$.

Geometrical Interpretation

We may write the conclusion of Cauchy's mean value theorem as $\dfrac{f(b)-f(a)}{g(b)-g(a)} g'(c) = f'(c)$.

Hence there is an ordinate $x = c$ between $x = a$ and $x = b$ such that the tangents at the points where $x = c$ cuts the graphs of the functions $f(x)$ and $\dfrac{f(b)-f(a)}{g(b)-g(a)} g(x)$ are mutually parallel.

Physical Interpretation

We may write the conclusion of Cauchy's mean value theorem as

$$\frac{f(b)-f(a)}{b-a} \Big/ \frac{g(b)-g(a)}{b-a} = \frac{f'(c)}{g'(c)}.$$

Hence the ratio of the mean rates of increases of two functions in an interval is equal to the ratio of the actual rates of increases of the functions at some point within the interval.

7.14. GENERALISED MEAN VALUE THEOREM

Statement. *If three functions f, g and h defined on $[a, b]$ are*

(i) continuous on $[a, b]$ *(ii) derivable on (a, b)*

then there exists a real number $c \in (a, b)$ such that $\begin{vmatrix} f'(c) & g'(c) & h'(c) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} = 0.$

Proof. Consider the function ϕ on $[a, b]$ defined by

$$\phi(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix}$$

Then $\phi(x) = Af(x) + Bg(x) + Ch(x)$ where A, B and C are constants.

Since each of the functions f, g and h is continuous on $[a, b]$ and derivable on (a, b).

$\therefore$ ϕ, which is a linear combination of f, g and h is also continuous on $[a, b]$ and derivable on (a, b).

Also $\phi(a) = \phi(b)$, because two rows become identical when $x = a$ or b.

Thus ϕ satisfies all the three conditions of Rolle's theorem on $[a, b]$. Consequently, there exists a real number $c \in (a, b)$ such that $\phi'(c) = 0$.

Now $$\phi'(x) = Af'(x) + Bg'(x) + Ch'(x) = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix}$$

$$\therefore \quad \phi'(c) = 0 \Rightarrow \begin{vmatrix} f'(c) & g'(c) & h'(c) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} = 0, \text{ where } c \in (a, b).$$

Corollary 1. *When h is a constant function, the above theorem reduces to Cauchy's mean value theorem.*

Let $h(x) = k$, a constant then $h(a) = h(b) = k$ and $h'(c) = 0$.

$\therefore$ The conclusion of the above theorem takes the form

$$\begin{vmatrix} f'(c) & g'(c) & 0 \\ f(a) & g(a) & k \\ f(b) & g(b) & k \end{vmatrix} = 0$$

$\Rightarrow$ $kf'(c)[g(a) - g(b)] - kg'(c)[f(a) - f(b)] = 0$

$\Rightarrow$ $f'(c)[g(b) - g(a)] = g'(c)[f(b) - f(a)]$

$\Rightarrow$ $\dfrac{f(b)-f(a)}{g(b)-g(a)} = \dfrac{f'(c)}{g'(c)}$

which is Cauchy's mean value theorem.

Corollary 2. *When $g(x) = x$ and $h(x) = k$, the above theorem reduces to Lagrange's mean value theorem.*

$g(x) = x$ and $h(x) = k \Rightarrow g'(x) = 1$ and $h'(x) = 0$

$\therefore$ $g'(c) = 1$, $h(a) = h(b) = k$ and $h'(c) = 0$

The conclusion of the above theorem takes the form

$$\begin{vmatrix} f'(c) & 1 & 0 \\ f(a) & a & k \\ f(b) & b & k \end{vmatrix} = 0$$

$\Rightarrow$ $kf'(c).(a-b) - k[f(a) - f(b)] = 0$ $\Rightarrow$ $f'(c).(b-a) = f(b) - f(a)$

$\Rightarrow$ $\dfrac{f(b)-f(a)}{b-a} = f'(c)$ which is Lagrange's mean value theorem.

ILLUSTRATIVE EXAMPLES

Example 1. *Verify Cauchy's mean value theorem for the functions x^2 and x^3 in the interval [1, 2].*

Sol. Let $\quad f(x) = x^2$ and $g(x) = x^3$

f and g are continuous and derivable for all values of x

$\Rightarrow$ f and g are continuous in [1, 2]

f and g are derivable in (1, 2)

Also $g'(x) = 3x^2 \neq 0$ for any $x \in (1, 2)$

Thus f and g satisfy the conditions of Cauchy's mean value theorem. Consequently, $\exists$ a point $c \in (1, 2)$ s.t.

$$\frac{f(2)-f(1)}{g(2)-g(1)} = \frac{f'(c)}{g'(c)} \qquad \ldots(i)$$

Now $\quad f(2) = 4, \quad f(1) = 1, \quad g(2) = 8, \; g(1) = 1$

$f'(x) = 2x \qquad \therefore \; f'(c) = 2c$

$g'(x) = 3x^2 \qquad \therefore \; g'(c) = 3c^2$

$\therefore$ From (i), $\quad \dfrac{4-1}{8-1} = \dfrac{2c}{3c^2}$ or $\dfrac{3}{7} = \dfrac{2c}{3c^2}$

$\Rightarrow \quad 9c^2 - 14c = 0 \quad \Rightarrow \quad c(9c-14) = 0 \quad \Rightarrow \quad c = 0$ or $\dfrac{14}{9}$

$$c = 0 \notin (1, 2) \text{ whereas } c = \frac{14}{9} \in (1, 2).$$

Hence Cauchy's mean value theorem is verified.

Example 2. *Find 'c' of Cauchy's Mean Value Theorem for the following pairs of functions :*

(i) $f(x) = e^x, \quad g(x) = e^{-x}$ in [a, b] $\qquad$ (ii) $f(x) = x^2, \quad g(x) = x$ in [a, b]

(iii) $f(x) = \sqrt{x}, \quad g(x) = \dfrac{1}{\sqrt{x}}$ in [a, b] $\qquad$ (iv) $f(x) = \dfrac{1}{x^2}, \quad g(x) = \dfrac{1}{x}$ in [a, b]

(v) $f(x) = \sin x, \quad g(x) = \cos x$ in $\left[-\dfrac{\pi}{2}, 0\right]$.

Sol. (i) $\quad f(x) = e^x, \quad \Rightarrow \quad f(a) = e^a, \quad f(b) = e^b$

$g(x) = e^{-x} \quad \Rightarrow \quad g(a) = e^{-a}, \quad g(b) = e^{-b}$

$f'(x) = e^x \quad \Rightarrow \quad f'(c) = e^c$

$g'(x) = -e^{-x} \quad \Rightarrow \quad g'(c) = -e^{-c}$

$\therefore \quad \dfrac{f(b)-f(a)}{g(b)-g(a)} = \dfrac{f'(c)}{g'(c)} \quad \Rightarrow \quad \dfrac{e^b - e^a}{e^{-b} - e^{-a}} = \dfrac{e^c}{-e^{-c}}$

$\Rightarrow \quad \dfrac{e^b - e^a}{e^{1/b} - e^{1/a}} = -e^{2c} \quad \Rightarrow \quad \dfrac{(e^b - e^a) \times e^a e^b}{e^a - e^b} = -e^{2c}$

$\Rightarrow \quad -e^{a+b} = -e^{2c} \quad \Rightarrow \quad a + b = 2c$

$\therefore \quad c = \dfrac{a+b}{2}$.

(ii) *Please try yourself.* $\qquad \left[\textbf{Ans. } c = \dfrac{a+b}{2}\right]$

(iii) $\quad f(x) = \sqrt{x} \quad \Rightarrow \quad f(a) = \sqrt{a}, \quad f(b) = \sqrt{b}$

$g(x) = \dfrac{1}{\sqrt{x}} \quad \Rightarrow \quad g(a) = \dfrac{1}{\sqrt{a}}, \quad g(b) = \dfrac{1}{\sqrt{b}}$

$f'(x) = \dfrac{1}{2\sqrt{x}} \quad \Rightarrow \quad f'(c) = \dfrac{1}{2\sqrt{c}}$

$g'(x) = -\dfrac{1}{2x^{3/2}} \quad \Rightarrow \quad g'(c) = -\dfrac{1}{2c\sqrt{c}}$

$\therefore \quad \dfrac{f(b)-f(a)}{g(b)-g(a)} = \dfrac{f'(c)}{g'(c)} \quad \Rightarrow \quad \dfrac{\sqrt{b}-\sqrt{a}}{\dfrac{1}{\sqrt{b}} - \dfrac{1}{\sqrt{a}}} = \dfrac{\dfrac{1}{2\sqrt{c}}}{-\dfrac{1}{2c\sqrt{c}}}$

$\Rightarrow (\sqrt{b} - \sqrt{a}) \cdot \dfrac{\sqrt{a}\,.\sqrt{b}}{\sqrt{a} - \sqrt{b}} = -c \quad \Rightarrow \quad -\sqrt{ab} = -c \quad \therefore \quad c = \sqrt{ab}$.

(iv) *Please try yourself.* $\qquad \left[\textbf{Ans. } c = \dfrac{2ab}{a+b}\right]$

(v) *Please try yourself.* $\qquad \left[\textbf{Ans. } c = -\dfrac{\pi}{4}\right]$

Example 3. *Show that* $\dfrac{\sin \alpha - \sin \beta}{\cos \beta - \cos \alpha} = \cot \theta$, *where* $0 < \alpha < \theta < \beta < \dfrac{\pi}{2}$.

Sol. Let $f(x) = \sin x$ and $g(x) = \cos x$

Then $f'(x) = \cos x$ and $g'(x) = -\sin x$

Clearly f and g are continuous on $[\alpha, \beta]$.

f and g are derivable on (α, β)

$$g'(x) = -\sin x \neq 0 \text{ for any } x \in (\alpha, \beta)$$

$\therefore$ By Cauchy's mean value theorem on $[\alpha, \beta]$, we have

$$\frac{f(\beta)-f(\alpha)}{g(\beta)-g(\alpha)} = \frac{f'(\theta)}{g'(\theta)}, \; \alpha < \theta < \beta$$

$\Rightarrow \quad \dfrac{\sin \beta - \sin \alpha}{\cos \beta - \cos \alpha} = \dfrac{\cos \theta}{-\sin \theta} \quad \Rightarrow \quad \dfrac{\sin \alpha - \sin \beta}{\cos \beta - \cos \alpha} = \cot \theta, \; \alpha < \theta < \beta.$

7.15. TAYLOR'S THEOREM (*with Schlomilch and Roche's form of remainder*) [First Form]

Statement. *If a function f defined on [a, a + h] is such that*

(i) the (n – 1)th derivative f^{n-1} is continuous on [a, a + h]

(ii) the nth derivative f^n exists on (a, a + h)

(iii) $p \in N$

then there exists atleast one real number $\theta \in (0, 1)$ such that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots\ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n (1-\theta)^{n-p}}{p\,(n-1)!} f^n(a + \theta h).$$

Proof. First of all we observe that condition (*i*) in the statement of the theorem implies that $f, f', f'', \ldots\ldots, f^{n-1}$ are all defined (*i.e.*, exist) and continuous on $[a, a+h]$.

Consider the function ϕ defined on $[a, a+h]$ as

$$\phi(x) = f(x) + (a+h-x) f'(x) + \frac{(a+h-x)^2}{2!} f''(x) + \ldots\ldots + \frac{(a+h-x)^{n-1}}{(n-1)!} f^{n-1}(x) + A(a+h-x)^p$$

where A is a constant to be determined such that $\phi(a) = \phi(a+h)$.

But $\phi(a) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots\ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + Ah^p$

and $\phi(a+h) = f(a+h)$

$\therefore$ $\phi(a) = \phi(a+h)$

$$\Rightarrow \quad f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots\ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + Ah^p \qquad \ldots(1)$$

Now, (*i*) Since $f, f', f'', \ldots\ldots, f^{n-1}$ are all continuous on $[a, a+h]$ and $(a+h-x)^r$, $r \in N$ is continuous on R.

$\therefore$ ϕ is continuous on $[a, a+h]$.

(*ii*) Since $f, f', f'', \ldots\ldots, f^{n-1}$ are all derivable on $(a, a+h)$ and $(a+h-x)^r$, $r \in N$ is derivable on R.

$\therefore$ ϕ is derivable on $(a, a+h)$.

(*iii*) Also $\phi(a) = \phi(a+h)$

Thus the function ϕ satisfies all the three conditions of Rolle's theorem on $[a, a+h]$ and hence, there exists a real number $\theta \in (0, 1)$ such that $\phi'(a+\theta h) = 0$.

But $\phi'(x) = f'(x) - f'(x) + (a+h-x) f''(x) - (a+h-x) f''(x)$

$$+ \ldots\ldots + \frac{(a+h-x)^{n-1}}{(n-1)!} f^n(x) - pA(a+h-x)^{p-1}$$

$$= \frac{(a+h-x)^{n-1}}{(n-1)!} f^n(x) - pA(a+h-x)^{p-1}$$

[other terms cancel in pairs]

$$\Rightarrow \quad \phi'(a+\theta h) = \frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!} f^n(a+\theta h) - pAh^{p-1}(1-\theta)^{p-1}$$

$\therefore$ $\phi'(a+\theta h) = 0$

$$\Rightarrow \quad \frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!} f^n(a+\theta h) = pAh^{p-1}(1-\theta)^{p-1}$$

$$\Rightarrow \quad A = \frac{h^{n-p}(1-\theta)^{n-p}}{p(n-1)!} f^n(a+\theta h) \qquad [\because \ h \neq 0, \theta \neq 1]$$

Putting this value of A in (1), we have

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots\ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n (1-\theta)^{n-p}}{p(n-1)!} f^n(a+\theta h)$$

The term $\quad R_n = \frac{h^n (1-\theta)^{n-p}}{p(n-1)!} f^n(a+\theta h)$

which occurs after n terms is known as Taylor's remainder after n terms. The theorem with the above form of remainder is known as Taylor's Theorem with Schlomilch and Roche's form of remainder.

Note 1. (*i*) For $p = 1$, we get $R_n = \frac{h^n (1-\theta)^{n-1}}{(n-1)!} f^n(a+\theta h)$ called **Cauchy's form of remainder.**

2. For $p = n$, we get $R_n = \frac{h^n}{n!} f^n(a+\theta h)$ called **Lagrange's form of remainder.**

7.16. TAYLOR'S THEOREM (*with Schlomilch and Roche's form of remainder*) [Second Form]

Statement. *If a function f defined on [a, b] is such that*

(*i*) *the (n – 1)th derivative f^{n-1} is continuous on [a, b]*

(*ii*) *the nth derivative f^n exists on (a, b)*

(*iii*) $p \in N$

then there exists atleast one real number $c \in (a, b)$ such that

$$f(b) = f(a) + (b-a) f'(a) + \frac{(b-a)^2}{2!} f''(a) + \ldots\ldots + \frac{(b-a)^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{(b-a)^p (b-c)^{n-p}}{p(n-1)!} f^n(c).$$

Proof. First of all we observe that condition (*i*) in the statement of the theorem implies that $f, f', f'', \ldots\ldots, f^{n-1}$ are all defined (*i.e.*, exist) and continuous on $[a, b]$.

Consider the function ϕ defined on $[a, b]$ as

$$\phi(x) = f(x) + (b-x) f'(x) + \frac{(b-x)^2}{2!} f''(x) + \ldots\ldots + \frac{(b-x)^{n-1}}{(n-1)!} f^{n-1}(x) + A(b-x)^p$$

where A is a constant to be determined such that $\phi(a) = \phi(b)$.

Put $\phi(a) = f(a) + (b-a) f'(a) + \frac{(b-a)^2}{2!} f''(a) + \ldots\ldots + \frac{(b-a)^{n-1}}{(n-1)!} f^{n-1}(a) + A(b-a)^p$

and $\phi(b) = f(b)$

$\therefore$ $\phi(a) = \phi(b)$

$$\Rightarrow \quad f(b) = f(a) + (b-a) f'(a) + \frac{(b-a)^2}{2!} f''(a) + \ldots\ldots + \frac{(b-a)^{n-1}}{(n-1)!} f^{n-1}(a) + A(b-a)^p \quad \ldots(1)$$

Now, (*i*) Since $f, f', f'', \ldots\ldots, f^{n-1}$ are all continuous on $[a, b]$ and $(b-x)^r$, $r \in N$ is continuous on R.

$\therefore$ ϕ is continuous on $[a, b]$.

(*ii*) Since $f, f', f'', \ldots\ldots, f^{n-1}$ are all derivable on (a, b) and $(b-x)^r$, $r \in N$ is derivable on R.

$\therefore$ ϕ is derivable on (a, b).

(*iii*) Also $\phi(a) = \phi(b)$.

Thus the function ϕ satisfies all the three conditions of Rolle's theorem on $[a, b]$ and, hence, there exists a real number $c \in (a, b)$ such that $\phi'(c) = 0$.

But $\phi'(x) = f'(x) - f'(x) + (b-x)f''(x) - (b-x)f''(x) + \ldots\ldots$

$$+ \frac{(b-x)^{n-1}}{(n-1)!} f^n(x) - pA(b-x)^{p-1}$$

$$= \frac{(b-x)^{n-1}}{(n-1)!} f^n(x) - pA(b-x)^{p-1} \quad \text{[others terms cancel in pairs]}$$

$\Rightarrow$ $$\phi'(c) = \frac{(b-c)^{n-1}}{(n-1)!} f^n(c) - pA(b-c)^{p-1}$$

$\therefore$ $\phi'(c) = 0$

$\Rightarrow$ $$\frac{(b-c)^{n-1}}{(n-1)!} f^n(c) = pA(b-c)^{p-1} \Rightarrow A = \frac{(b-c)^{n-p}}{p(n-1)!} f^n(c) \quad [\because c \neq b]$$

Putting this value of A in (1), we get

$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!} f''(a) + \ldots\ldots$$

$$+ \frac{(b-a)^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{(b-a)^p (b-c)^{n-p}}{p(n-1)!} f^n(c)$$

Schlomilch and Roche's form of remainder after n terms is

$$R_n = \frac{(b-a)^p (b-c)^{n-p}}{p(n-1)!} f^n(c).$$

Note. (*i*) For $p = 1$, we get $R_n = \dfrac{(b-a)(b-c)^{n-1}}{(n-1)!} f^n(c)$ called Cauchy's form of remainder.

(*ii*) For $p = n$, we get $R_n = \dfrac{(b-a)^n}{n!} f^n(c)$ called *Lagrange's form of remainder*.

7.17. TAYLOR'S THEOREM (*with Cauchy's form of remainder*) (First Form)

Statement. *If a function f defined on $[a, a+h]$ is such that*

(i) the $(n-1)$th derivative f^{n-1} is continuous on $[a, a+h]$

(ii) the nth derivative f^n exists on $(a, a+h)$

then there exists atleast one real number $\theta \in (0, 1)$ such that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots\ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n (1-\theta)^{n-1}}{(n-1)!} f^n(a+\theta h).$$

Proof. First of all we observe that condition (*i*) in the statement of the theorem implies that $f, f', f'', \ldots\ldots, f^{n-1}$ are all defined (*i.e.*, exist) and continuous on $[a, a+h]$.

Consider the function ϕ defined on $[a, a+h]$ as

$$\phi(x) = f(x) + (a+h-x)f'(x) + \frac{(a+h-x)^2}{2!} f''(x) + \ldots\ldots$$

$$+ \frac{(a+h-x)^{n-1}}{(n-1)!} f^{n-1}(x) + A(a+h-x)$$

where A is a constant to be determined such that $\phi(a) = \phi(a+h)$.

But $$\phi(a) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots\ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + Ah$$

and $\phi(a+h) = f(a+h)$

$\therefore$ $\phi(a) = \phi(a+h)$

$\Rightarrow$ $$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots\ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + Ah \quad \ldots(1)$$

Now, (*i*) Since $f, f', f'', \ldots\ldots, f^{n-1}$ are all continuous on $[a, a+h]$ and $(a+h-x)^r, r \in N$ is continuous on R.

$\therefore$ ϕ is continuous on $[a, a+h]$.

(*ii*) Since $f, f', f'', \ldots\ldots, f^{n-1}$ are all derivable on $(a, a+h)$ and $(a+h-x)^r, r \in N$ is derivable on R.

$\therefore$ ϕ is derivable on $(a, a+h)$.

(*iii*) Also $\phi(a) = \phi(a+h)$

Thus the function ϕ satisfies all the three conditions of Rolle's theorem on $[a, a+h]$ and, hence, there exists a real number $\theta \in (0, 1)$ such that $\phi'(a+\theta h) = 0$.

But $$\phi'(x) = f'(x) - f'(x) + (a+h-x)f''(x) - (a+h-x)f''(x) + \ldots\ldots$$

$$+ \frac{(a+h-x)^{n-1}}{(n-1)!} f^n(x) - A$$

$$= \frac{(a+h-x)^{n-1}}{(n-1)!} - f^n(x) - A \quad \text{[other terms cancel in pairs]}$$

$\Rightarrow$ $$\phi'(a+\theta h) = \frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!} f^n(a+\theta h) - A$$

$\therefore$ $\phi'(a+\theta h) = 0$

$\Rightarrow$ $$A = \frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!} f^n(a+\theta h)$$

Putting this value of A in (1), we get

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots\ldots$$

$$+ \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n (1-\theta)^{n-1}}{(n-1)!} f^n(a+\theta h).$$

7.18. TAYLOR'S THEOREM (*with Cauchy's form of remainder*) (Second Form)

Statement. *If a function f defined on $[a, b]$ is such that*

(i) the $(n-1)$th derivative f^{n-1} is continuous on $[a, b]$

(ii) the nth derivative f^n exists on (a, b)

then there exists atleast one real number $c \in (a, b)$ such that.

$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!} f''(a) + \ldots\ldots + \frac{(b-a)^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{(b-a)(b-c)^{n-1}}{(n-1)!} f^n(c).$$

Proof. First of all we observe that condition (*i*) in the statement of the theorem implies that $f, f', f'', \ldots\ldots, f^{n-1}$ are all defined (*i.e.*, exist) and continuous on $[a, b]$.

Consider the function ϕ defined on $[a, b]$ as

$$\phi(x) = f(x) + (b-x) f'(x) + \frac{(b-x)^2}{2!} f''(x) + \ldots\ldots + \frac{(b-x)^{n-1}}{(n-1)!} f^{n-1}(x) + A(b-x)$$

where A is a constant to be determined such that $\phi(a) = \phi(b)$.

But $\phi(a) = f(a) + (b-a) f'(a) + \frac{(b-a)^2}{2!} f''(a) + \ldots\ldots + \frac{(b-a)^{n-1}}{(n-1)!} f^{n-1}(a) + A(b-a)$

and $\phi(b) = f(b)$

$\therefore$ $\phi(a) = \phi(b)$

$$\Rightarrow \quad f(b) = f(a) + (b-a) f'(a) + \frac{(b-a)^2}{2!} f''(a) + \ldots\ldots + \frac{(b-a)^{n-1}}{(n-1)!} f^{n-1}(a) + A(b-a) \quad \ldots(1)$$

Now, (*i*) Since $f, f', f'', \ldots\ldots, f^{n-1}$ are all continuous on $[a, b]$ and $(b-x)^r, r \in N$ is continuous on R.

$\therefore$ ϕ is continuous on $[a, b]$.

(*ii*) Since $f, f', f'', \ldots\ldots, f^{n-1}$ are all derivable on (a, b) and $(b-x)^r, r \in N$ is derivable on R.

$\therefore$ ϕ is derivable on (a, b).

(*iii*) Also $\phi(a) = \phi(b)$

Thus the function ϕ satisfies all the three conditions of Rolle's theorem on $[a, b]$ and, hence, there exists a real number $c \in (a, b)$ such that $\phi'(c) = 0$.

But $\phi'(x) = f'(x) - f'(x) + (b-x) f''(x) - (b-x) f''(x) + \ldots\ldots + \frac{(b-x)^{n-1}}{(n-1)!} f^n(x) - A$

$$= \frac{(b-x)^{n-1}}{(n-1)!} f^n(x) - A \qquad \text{[other terms cancel in pairs]}$$

$$\Rightarrow \quad \phi'(c) = \frac{(b-c)^{n-1}}{(n-1)!} f^n(c) - A$$

$$\therefore \quad \phi'(c) = 0 \Rightarrow A = \frac{(b-c)^{n-1}}{(n-1)!} f^n(c)$$

Putting this value of A in (1), we get

$$f(b) = f(a) + (b-a) f'(a) + \frac{(b-a)^2}{2!} f''(a) + \ldots\ldots + \frac{(b-a)^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{(b-a)(b-c)^{n-1}}{(n-1)!} f^n(c).$$

7.19. TAYLOR'S THEOREM (*with Lagrange's form of remainder*) (First Form)

Statement. *If a function f defined on $[a, a+h]$ is such that*

(*i*) *the $(n-1)$th derivative f^{n-1} is continuous on $[a, a+h]$*

(*ii*) *the nth derivative f^n is derivable on $(a, a+h)$*

then there exists atleast one real number $\theta \in (0, 1)$ such that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots\ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{n!} f^n(a+\theta h).$$

Proof. Please try yourself by considering the function ϕ defined on $[a, a+h]$ as

$$\phi(x) = f(x) + (a+h-x) f'(x) + \frac{(a+h-x)^2}{2!} f''(x) + \ldots\ldots + \frac{(a+h-x)^{n-1}}{(n-1)!} f^{n-1}(x) + A(a+h-x)^n.$$

7.20. TAYLOR'S THEOREM (*with Lagrange's form of remainder*) (Second Form)

Statement. *If a function f defined on $[a, b]$ is such that*

(*i*) *the $(n-1)$th derivative f^{n-1} is continuous on $[a, b]$*

(*ii*) *the nth derivative f^n is derivable on (a, b)*

then there exists atleast one real number $c \in (a, b)$ such that

$$f(b) = f(a) + (b-a) f'(a) + \frac{(b-a)^2}{2!} f''(a) + \ldots\ldots + \frac{(b-a)^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{(b-a)^n}{n!} f^n(c).$$

Proof. Please try yourself by considering the functions ϕ defined on $[a, b]$ as

$$\phi(x) = f(x) + (b-x) f'(x) + \frac{(b-x)^2}{2!} f''(x) + \ldots\ldots + \frac{(b-x)^{n-1}}{(n-1)!} f^{n-1}(x) + A(b-x)^n.$$

7.21. DEDUCTION OF TAYLOR'S THEOREM, WITH CAUCHY'S FORM OF REMAINDER, FROM THE MEAN VALUE THEOREM

Let a function f be such that its $(n-1)$th derivative f^{n-1} is continuous on $[a, a+h]$ and its nth derivative f^n exists on $(a, a+h)$.

Consequently, $f, f', f'', \ldots\ldots, f^{n-1}$ are all defined (*i.e.*, exist) and continuous on $[a, a+h]$.

Consider the function

$$\phi(x) = f(x) + (a+h-x) f'(x) + \frac{(a+h-x)^2}{2!} f''(x) + \ldots\ldots + \frac{(a+h-x)^{n-1}}{(n-1)!} f^{n-1}(x)$$

which being the sum of continuous and derivable functions is itself continuous on $[a, a+h]$ and derivable on $(a, a+h)$.

$\therefore$ By Lagrange's mean value theorem, $\exists$ a real number $\theta \in (0, 1)$ such that

$$\phi(a+h) = \phi(a) + h\,\phi'(a+\theta h) \quad \ldots(1)$$

Now $\phi(a+h) = f(a+h)$

$$\phi(a) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots\ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a)$$

Also $\phi'(x) = f'(x) - f'(x) + (a+h-x) f''(x) - (a+h-x) f''(x) + \ldots\ldots + \frac{(a+h-x)^{n-1}}{(n-1)!} f^n(x)$

$$= \frac{(a+h-x)^{n-1}}{(n-1)!} f^n(x) \qquad \text{[other terms cancel in pairs]}$$

$$\Rightarrow \quad \phi'(a+\theta h) = \frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!} f^n(a+\theta h)$$

$\therefore$ From (1), we get

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots\ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n(1-\theta)^{n-1}}{(n-1)!} f^n(a+\theta h).$$

Note 1. For $n = 1$, Taylor's theorem reduces to the Mean Value Theorem. For this reason, Taylor's theorem is also called General Mean Value Theorem.

Note 2. If f satisfies the conditions of Taylor's Theorem on $[a, a + h]$ and x is any point of $[a, a + h]$, then it satisfies the conditions on $[a, x]$ also.

Replacing $a + h$ by x *i.e.*, h by $(x - a)$, we get

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \ldots\ldots + \frac{(x-a)^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{(x-a)^n (1-\theta)^{n-p}}{p(n-1)!} f^n(a + \theta(x-a))$$

$$R_n = \frac{(x-a)^n (1-\theta)^{n-p}}{p(n-1)!} f^n(a + \theta(x-a))$$

is Schlomilch and Roche's form of Taylor's remainder after n terms.

θ will be different for different values of x.

7.22. MACLAURIN'S THEOREM

Statement. *If a function f defined on $[0, x]$ is such that*

(i) the $(n-1)$th derivative f^{n-1} is continuous on $[0, x]$

(ii) the nth derivative f^n exists on $(0, x)$

(iii) $p \in N$

then there exists a real number $\theta \in (0, 1)$ such that

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \ldots\ldots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + \frac{x^n (1-\theta)^{n-p}}{p(n-1)!} f^n(\theta x).$$

Proof. First of all we observe that condition (*i*) in the statement of theorem implies that $f, f', f'', \ldots\ldots, f^{n-1}$ are all defined (*i.e.*, exist) and continuous on $[0, x]$.

Consider the function ϕ defined on $[0, x]$ as

$$\phi(t) = f(t) + (x-t) f'(t) + \frac{(x-t)^2}{2!} f''(t) + \ldots\ldots + \frac{(x-t)^{n-1}}{(n-1)!} f^{n-1}(t) + A(x-t)^p$$

where A is a constant to be determined such that $\phi(0) = \phi(x)$.

But $\qquad \phi(0) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \ldots\ldots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + Ax^p$

and $\qquad \phi(x) = f(x)$

$\therefore \qquad \phi(0) = \phi(x)$

$$\Rightarrow \quad f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \ldots\ldots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + Ax^p \qquad \ldots(1)$$

Now, (*i*) Since $f, f', f'', \ldots\ldots, f^{n-1}$ are all continuous on $[0, x]$ and $(x-t)^r, r \in N$ is continuous on R.

$\therefore$ ϕ is continuous on $[0, x]$.

(*ii*) Since $f, f', f'', \ldots\ldots, f^{n-1}$ are all derivable on $(0, x)$ and $(x-t)^r, r \in N$ is derivable on R.

$\therefore$ ϕ is derivable on $(0, x)$.

(*iii*) Also $\qquad \phi(0) = \phi(x)$

Thus the function ϕ satisfies all the three conditions of Rolle's theorem on $[0, x]$ and, hence, there exists a real number $\theta \in (0, 1)$ such that $\phi'(\theta x) = 0$.

But $\qquad \phi'(t) = f'(t) - f'(t) + (x-t) f''(t) - (x-t) f''(t) + \ldots\ldots + \frac{(x-t)^{n-1}}{(n-1)!} f^n(t) - pA(x-t)^{p-1}$

$$= \frac{(x-t)^{n-1}}{(n-1)!} f^n(t) - pA(x-t)^{p-1} \qquad \text{[other terms cancel in pairs]}$$

$$\Rightarrow \quad \phi'(\theta x) = \frac{(x-\theta x)^{n-1}}{(n-1)!} f^n(\theta x) - pA(x - \theta x)^{p-1}$$

$$= \frac{x^{n-1}(1-\theta)^{n-1}}{(n-1)!} f^n(\theta x) - pAx^{p-1}(1-\theta)^{p-1}$$

$\therefore \qquad \phi'(\theta x) = 0$

$$\Rightarrow \quad \frac{x^{n-1}(1-\theta)^{n-1}}{(n-1)!} f^n(\theta x) = pAx^{p-1}(1-\theta)^{p-1}$$

$$\Rightarrow \quad A = \frac{x^{n-p}(1-\theta)^{n-p}}{p(n-1)!} f^n(\theta x)$$

Putting this value of A in (1), we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \ldots\ldots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + \frac{x^n(1-\theta)^{n-p}}{p(n-1)!} f^n(\theta x)$$

The term $\qquad R_n = \frac{x^n(1-\theta)^{n-p}}{p(n-1)!} f^n(\theta x)$

which occurs after n terms in known as **Schlomilch and Roche's form of remainder.**

Note. (*i*) For $p = 1$, we get $R_n = \frac{x^n(1-\theta)^{n-1}}{(n-1)!} f^n(\theta x)$ called **Cauchy's form of remainder.**

(*ii*) For $p = n$, we get $R_n = \frac{x^n}{n!} f^n(\theta x)$ called **Lagrange's form of remainder.**

ILLUSTRATIVE EXAMPLES

Example 1. *If $f(x + h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x + \theta h)$, find the value of θ as $x \to a$ if $f(x) = (x-a)^{5/2}$.*

Sol. $\qquad f(x) = (x-a)^{5/2} \quad \Rightarrow \quad f(x+h) = (x+h-a)^{5/2}$

Also $\qquad f'(x) = \frac{5}{2}(x-a)^{3/2}$

$$f''(x) = \frac{15}{4}(x-a)^{1/2} \quad \Rightarrow \quad f''(x + \theta h) = \frac{15}{4}(x + \theta h - a)^{1/2}$$

$$\because \quad f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x + \theta h)$$

$$\Rightarrow \quad (x+h-a)^{5/2} = (x-a)^{5/2} + \frac{5}{2} h(x-a)^{3/2} + \frac{15}{8} h^2(x + \theta h - a)^{1/2}$$

When $x \to a$, we get

$$h^{5/2} = \frac{15}{8} h^2 . (\theta h)^{1/2} \Rightarrow \quad h^{5/2} = \frac{15}{8} h^{5/2} \theta^{1/2} \Rightarrow \quad 1 = \frac{15}{8} \theta^{1/2}$$

$$\therefore \quad \theta = \frac{64}{225}.$$

Example 2. *If $f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(\theta x)$, find the value of θ as $x \to 1$ if $f(x) = (1-x)^{5/2}$.*

Sol. *Please try yourself.* $\left[\textbf{Ans.}\ \frac{9}{25}\right]$

Example 3. *Find Lagrange's and Cauchy's remainders after n terms in the expansion of*

(i) $\log(1+x)$ *(ii)* $\cos x$

(iii) $\frac{1}{1+x}$ *(iv)* e^{kx}.

Sol. *(i)* $f(x) = \log(1+x) \quad \Rightarrow \quad f^n(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$

Using Maclaurin's expansion for $f(x)$, we have

$$\text{Lagrange's remainder} = \frac{x^n}{n!} f^n(\theta x) = \frac{x^n}{n!} . \frac{(-1)^{n-1}(n-1)!}{(1+\theta x)^n} = \frac{(-1)^{n-1}}{n} \left(\frac{x}{1+\theta x}\right)^n$$

$$\text{Cauchy's remainder} = \frac{x^n(1-\theta)^{n-1}}{(n-1)!} f^n(\theta x)$$

$$= \frac{x^n(1-\theta)^{n-1}}{(n-1)!} . \frac{(-1)^{n-1}(n-1)!}{(1+\theta x)^n} = (-1)^{n-1}(1-\theta)^{n-1} \left(\frac{x}{1+\theta x}\right)^n.$$

(ii) $f(x) = \cos x \quad \Rightarrow \quad f^n(x) = \cos\left(x + \frac{n\pi}{2}\right)$

Using Maclaurin's expansion for $f(x)$, we have

$$\text{Lagrange's remainder} = \frac{x^2}{n!} f^n(\theta x) = \frac{x^n}{n!} \cos\left(\theta x + \frac{n\pi}{2}\right)$$

$$\text{Cauchy's remainder} = \frac{x^n(1-\theta)^{n-1}}{(n-1)!} - f^n(\theta x) = \frac{x^n(1-\theta)^{n-1}}{(n-1)!} \cos\left(\theta x + \frac{n\pi}{2}\right).$$

(iii) $f(x) = \frac{1}{1+x} \quad \Rightarrow \quad f^n(x) = \frac{(-1)^n n!}{(1+x)^{n+1}}$

$$\therefore \quad \text{Lagrange's remainder} = \frac{x^n}{n!} f^n(\theta x) = \frac{x^n}{n!} . \frac{(-1)^n n!}{(1+\theta x)^{n+1}} = (-1)^n . \frac{x^n}{(1+\theta x)^{n+1}}$$

Cauchy's remainder

$$= \frac{x^n(1-\theta)^{n-1}}{(n-1)!} f^n(\theta x) = \frac{x^n(1-\theta)^{n-1}}{(n-1)!} . \frac{(-1)^n n!}{(1+\theta x)^{n+1}} = (-1)^n . \frac{nx^n(1-\theta)^{n-1}}{(1+\theta x)^{n+1}}.$$

(iv) Please try yourself. $\left[\textbf{Ans.}\ \frac{x^n}{n!} k^n e^{\theta kx}, \frac{x^n(1-\theta)^{n-1}}{(n-1)!} k^n e^{\theta kx}\right]$

Example 4. *Show that $\sin(\alpha + h)$ differs from $\sin\alpha + h\cos\alpha$ by not more than $\frac{h^2}{2}$.*

Sol. Let $f(\alpha + h) = \sin(\alpha + h)$, then $f(\alpha) = \sin\alpha$

$f'(\alpha) = \cos\alpha$, $f''(\alpha) = -\sin\alpha$

By Taylor's theorem with Lagrange's form of remainder, we have

$$f(\alpha + h) = f(\alpha) + hf'(\alpha) + \frac{h^2}{2!} f''(\alpha + \theta h) \text{ where } 0 < \theta < 1$$

$$\therefore \quad \sin(\alpha + h) = \sin\alpha + h\cos\alpha - \frac{h^2}{2!} \sin(\alpha + \theta h)$$

Since $\quad -1 \le \sin(\alpha + \theta h) \le 1$

$$\therefore \quad \frac{h^2}{2} \ge -\frac{h^2}{2} \sin(\alpha + \theta h) \ge -\frac{h^2}{2}$$

$$\Rightarrow \quad -\frac{h^2}{2} \le \sin(\alpha + h) - \sin\alpha - h\cos\alpha \le \frac{h^2}{2}$$

$$\Rightarrow \quad \left|\sin(\alpha + h) - \sin\alpha - h\cos\alpha\right| \le \frac{h^2}{2}$$

Hence the result.

Example 5. *Show that 'θ' which occurs in the Lagrange's mean value theorem approaches the limit $\frac{1}{2}$ as h approaches zero provided that $f''(a)$ is not zero. It is assumed that $f''(x)$ is continuous.*

Sol. By Lagrange's mean value theorem for $f(x)$ on $[a, a+h]$, we have

$$f(a + h) = f(a) + hf'(a + \theta h) \quad \text{...(1)}$$

where $\quad 0 < \theta < 1$.

Also, by Taylor's theorem for $f(x)$ on $[a, a+h]$, we have

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a + \theta_1 h) \quad \text{...(2)}$$

where $\quad 0 < \theta_1 < 1$.

Subtracting (2) from (1), we have

$$0 = h[f'(a + \theta h) - f'(a)] - \frac{h^2}{2!} f''(a + \theta_1 h)$$

or $\quad f'(a + \theta h) - f'(a) = \frac{h}{2} f''(a + \theta_1 h) \quad \text{...(3)}$

Now, by Lagrange's mean value theorem for $f'(x)$ on $[a, a + \theta h]$, we have

$$f'(a + \theta h) = f'(a) + \theta h f''(a + \theta_2 \theta h) \text{ where } 0 < \theta_2 < 1$$

or $\quad f'(a + \theta h) - f'(a) = \theta h f''(a + \theta_2 \theta h) \quad \text{...(4)}$

From (3) and (4), we have

$$\theta h f''(a + \theta_2 \theta h) = \frac{h}{2} f''(a + \theta_1 h) \text{ or } \theta = \frac{f''(a + \theta_1 h)}{2f''(a + \theta_2 \theta h)}$$

$$\therefore \quad \lim_{h \to 0} \theta = \lim_{h \to 0} \frac{1}{2} \frac{f''(a + \theta_1 h)}{f''(a + \theta_2 \theta h)} = \frac{1}{2} \frac{f''(a)}{f''(a)} \quad [\because f''(x) \text{ is continuous}]$$

$$= \frac{1}{2}. \quad [\because f''(x) \ne 0]$$

Example 6. *Prove that the number 'θ' which occurs in the Taylor's theorem with Lagrange's form of remainder after n terms approaches the limit* $\frac{1}{n+1}$ *as h approaches zero provided that* $f^{n+1}(x)$ *is continuous and different from zero at* $x = a$.

Sol. By Taylor's theorem with Lagrange's form of remainder after n terms and $(n + 1)$ terms successively, we have

$$f(a+h) = f(a) + hf'(a) + \ldots\ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{n!} f^n(a+\theta h) \quad \text{where } 0 < \theta < 1.$$

$$f(a+h) = f(a) + hf'(a) + \ldots\ldots + \frac{h^n}{n!} f^n(a) + \frac{h^{n+1}}{(n+1)!} f^{n+1}(a+\theta_1 h) \text{ where } 0 < \theta_1 < 1.$$

On subtraction, we get $0 = \frac{h^n}{n!}[f^n(a+\theta h) - f^n(a)] - \frac{h^{n+1}}{(n+1)!} f^{n+1}(a+\theta_1 h)$

or $$f^n(a+\theta h) - f^n(a) = \frac{h}{n+1} f^{n+1}(a+\theta_1 h) \qquad \ldots(1)$$

Using Lagrange's mean value theorem for $f^n(x)$ on $[a, a+\theta h]$, we have

$$f^n(a+\theta h) = f^n(a) + \theta h f^{n+1}(a+\theta_2\theta h) \text{ where } 0 < \theta_2 < 1$$

or $$f^n(a+\theta h) - f^n(a) = \theta h f^{n+1}(a+\theta_2\theta h) \qquad \ldots(2)$$

From (1) and (2), we have

$$\theta h f^{n+1}(a+\theta_2\theta h) = \frac{h}{n+1} f^{n+1}(a+\theta_1 h) \text{ or } \theta = \frac{1}{n+1} \cdot \frac{f^{n+1}(a+\theta_1 h)}{f^{n+1}(a+\theta_2\theta h)}$$

$$\therefore \quad \lim_{h\to 0} \theta = \lim_{h\to 0} \frac{1}{n+1} \cdot \frac{f^{n+1}(a+\theta_1 h)}{f^{n+1}(a+\theta_2\theta h)} = \frac{1}{n+1} \cdot \frac{f^{n+1}(a)}{f^{n+1}(a)} \quad [\because f^{n+1}(x) \text{ is continuous}]$$

$$= \frac{1}{n+1} \qquad [\because f^{n+1}(a) \neq 0]$$

Example 7. *Assuming the derivatives which occur are continuous, apply the mean value theorem to prove that*

$$\phi'(x) = F'\{f(x)\} f'(x) \text{ where } \phi(x) = F\{f(x)\}.$$

Sol. Let $f(x) = t$ so that $\phi(x) = F(t)$

Now $$\phi'(x) = \lim_{h\to 0} \frac{\phi(x+h) - \phi(x)}{h} = \lim_{h\to 0} \frac{F\{f(x+h)\} - F\{f(x)\}}{h}$$

$$= \lim_{h\to 0} \frac{F\{f(x) + hf'(x+\theta_1 h)\} - F\{f(x)\}}{h} \text{ where } \theta < \theta_1 < 1$$

$[\because f(x+h) = f(x) + hf'(x+\theta_1 h)$ by Mean Value Theorem$]$

$$= \lim_{h\to 0} \frac{F(t+H) - F(t)}{h} \text{ where } H = hf'(x+\theta_1 h)$$

$$= \lim_{h\to 0} \frac{F(t) + HF'(t+\theta_2 H) - F(t)}{h} \text{ where } 0 < \theta_2 < 1$$

$[\because F(t+H) = F(t) + HF'(t+\theta_2 H)$ by Mean Value Theorem$]$

$$= \lim_{h\to 0} \frac{HF'(t+\theta_2 H)}{h} = \lim_{h\to 0} \frac{hf'(x+\theta_1 h) F\{t+\theta_2 hf'(x+\theta_1 h)\}}{h}$$

$$= \lim_{h\to 0} f'(x+\theta_1 h) F'\{f(x) + \theta_2 hf'(x+\theta_1 h)\}$$

$$= f'(x) F'(f(x)) \qquad [\because f' \text{ and } F' \text{ are continuous}]$$
$$= F'\{f(x)\} f'(x).$$

Example 8. *If* $f''(x) > 0$ *for all values of x, prove that* $f\left(\frac{x_1+x_2}{2}\right) < \frac{f(x_1)+f(x_2)}{2}$.

Sol. By Taylor's theorem, we have

$$f(x_1) = f\left(\frac{x_1+x_2}{2} + \frac{x_1-x_2}{2}\right)$$

$$= f\left(\frac{x_1+x_2}{2}\right) + \frac{x_1-x_2}{2} f'\left(\frac{x_1+x_2}{2}\right) + \frac{1}{2!}\left(\frac{x_1-x_2}{2}\right)^2 f''\left\{\frac{x_1+x_2}{2} + \theta_1\left(\frac{x_1-x_2}{2}\right)\right\} \quad \ldots(1)$$

where $0 < \theta_1 < 1$

Also $f(x_2) = f\left(\frac{x_2+x_1}{2} + \frac{x_2-x_1}{2}\right)$

$$= f\left(\frac{x_2+x_1}{2}\right) + \frac{x_2-x_1}{2} f'\left(\frac{x_2+x_1}{2}\right) + \frac{1}{2!}\left(\frac{x_2-x_1}{2}\right)^2 f''\left\{\frac{x_2+x_1}{2} + \theta_2\left(\frac{x_2-x_1}{2}\right)\right\}$$

where $0 < \theta_2 < 1$

$$= f\left(\frac{x_1+x_2}{2}\right) - \frac{x_1-x_2}{2} f'\left(\frac{x_1+x_2}{2}\right) + \frac{1}{2!}\left(\frac{x_1-x_2}{2}\right)^2 f''\left\{\frac{x_1+x_2}{2} - \theta_2\left(\frac{x_1-x_2}{2}\right)\right\} \quad \ldots(2)$$

Adding (1) and (2), we get

$$f(x_1) + f(x_2) = 2f\left(\frac{x_1+x_2}{2}\right) + \frac{1}{2}\left(\frac{x_1-x_2}{2}\right)^2 \left[f''\left\{\frac{x_1+x_2}{2} + \theta_1\left(\frac{x_1-x_2}{2}\right)\right\} + f''\left\{\frac{x_1+x_2}{2} - \theta_2\left(\frac{x_1-x_2}{2}\right)\right\}\right]$$

$$> 2f\left(\frac{x_1+x_2}{2}\right)$$

since $\left(\frac{x_1-x_2}{2}\right)^2$ is positive and also $f''(x)$ is given to be positive for all values of x.

$$\Rightarrow \quad f\left(\frac{x_1+x_2}{2}\right) < \frac{f(x_1)+f(x_2)}{2}.$$

Example 9. *Using Taylor's theorem, show that*

(i) $\cos x \geq 1 - \frac{x^2}{2} \ \forall x \in R$ *(ii)* $1 + x + \frac{x^2}{2} < e^x < 1 + x + \frac{x^2}{2} e^x, x > 0$

(iii) $x - \frac{x^3}{3!} < \sin x < x, x > 0$ *(iv)* $x - \frac{x^3}{3!} \leq \sin x \leq x - \frac{x^3}{3!} + \frac{x^5}{5!}, x \geq 0.$

Sol. (*i*) **Case 1.** Let $x = 0$

Then $\cos x = 1, 1 - \frac{x^2}{2} = 1 \quad \therefore \quad \cos x = 1 - \frac{x^2}{2}.$

Case 2. Let $x > 0$ and $f(x) = \cos x$

Then $f'(x) = -\sin x, f''(x) = -\cos x$

Since $f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(\theta x)$ where $0 < \theta < 1$.

$\therefore$ $\cos x = 1 - \frac{x^2}{2} \cos \theta x$

But $\cos \theta x < 1 \ \forall\ \theta x, x > 0$

$\therefore$ $1 - \frac{x^2}{2} \cos \theta x > 1 - \frac{x^2}{2} \quad \Rightarrow \quad \cos x > 1 - \frac{x^2}{2}$.

Case 3. Let $x < 0$

Put $y = -x$ so that $y > 0$

By case 2, $\cos y > 1 - \frac{y^2}{2} \Rightarrow \cos(-x) > 1 - \frac{(-x)^2}{2} \Rightarrow \cos x > 1 - \frac{x^2}{2}$

Combining all cases, $\cos x \geq 1 - \frac{x^2}{2}\ \forall x \in \mathbb{R}$.

(*ii*) Let $f(x) = e^x, x > 0$, then $f'(x) = f''(x) = e^x$

Since $f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(\theta x)$ where $0 < \theta < 1$

$\therefore$ $e^x = 1 + x + \frac{x^2}{2} e^{\theta x}$

Now $0 < \theta < 1$ and $x > 0 \quad \Rightarrow \quad 0 < \theta x < x$

$\Rightarrow$ $e^0 < e^{\theta x} < e^x$ [$\because$ e^x is an increasing function]

$\Rightarrow$ $\frac{x^2}{2} < \frac{x^2}{2} e^{\theta x} < \frac{x^2}{2} e^x \Rightarrow 1 + x + \frac{x^2}{2} < 1 + x + \frac{x^2}{2} e^{\theta x} < 1 + x + \frac{x^2}{2} e^x$

$\Rightarrow$ $1 + x + \frac{x^2}{2} < e^x < 1 + x + \frac{x^2}{2} e^x$.

(*iii*) Let $f(x) = \sin x, x > 0$, then $f'(x) = \cos x, f''(x) = -\sin x$

$f'''(x) = -\cos x$

Since $f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(\theta x)$ where $0 < \theta < 1$

$\therefore$ $\sin x = x - \frac{x^3}{3!} \cos \theta x$

But $\cos \theta x < 1 \quad \forall\ \theta x, x > 0$

$\therefore$ $x - \frac{x^3}{3!} \cos \theta x > x - \frac{x^3}{3!}$

$\Rightarrow$ $\sin x > x - \frac{x^3}{3!}$ ...(1)

Also $f(x) = f(0) + xf'(\theta x)$ where $0 < \theta < 1$

$\therefore$ $\sin x = x \cos \theta x$

But $\cos \theta x < 1 \quad \forall\ \theta x, x > 0$

$\therefore$ $x \cos \theta x < x \quad \Rightarrow \quad \sin x < x$ ...(2)

Combining (1) and (2), we get $x - \frac{x^3}{3!} < \sin x < x$.

(*iv*) **Case 1.** Let $x = 0$

Then $x - \frac{x^3}{3!} = \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!}$ (each $= 0$).

Case 2. Let $x > 0$ and $f(x) = \sin x$

Then $f'(x) = \cos x, f''(x) = -\sin x, f'''(x) = -\cos x,$

$f^{iv}(x) = \sin x, f^{v}(x) = \cos x$

Since $f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(\theta x)$

$\therefore$ $\sin x = x - \frac{x^3}{3!} \cos \theta x$

But $\cos \theta x < 1\ \forall\ \theta x, x > 0$

$\therefore$ $x - \frac{x^3}{3!} \cos \theta x > x - \frac{x^3}{3!}$

$\Rightarrow$ $\sin x > x - \frac{x^3}{3!}$ ...(1)

Also $f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \frac{x^5}{5!} f^{v}(\theta x)$

$\therefore$ $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \cos \theta x$

But $\cos \theta x < 1\ \forall\ \theta x, x > 0$

$\therefore$ $x - \frac{x^3}{3!} + \frac{x^5}{5!} \cos \theta x < x - \frac{x^3}{3!} + \frac{x^5}{5!}$

$\Rightarrow$ $\sin x < x - \frac{x^3}{3!} + \frac{x^5}{5!}$ ...(2)

Combining (1) and (2), $x - \frac{x^3}{3!} < \sin x < x - \frac{x^3}{3!} + \frac{x^5}{5!}$

Combining the two cases, we have, $x - \frac{x^3}{3!} \leq \sin x \leq x - \frac{x^3}{3!} + \frac{x^5}{5!}$, $x > 0$.

7.23. TAYLOR'S AND MACLAURIN'S INFINITE SERIES

Let a function f possess continuous derivatives of every order in $[a, a + h]$, then for all $n \in \mathbb{N}$, we have by Taylor's theorem

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots\ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{n!} f^n(a + \theta h) \text{ where } 0 < \theta < 1$$

Let $S_n = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots\ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a)$

and $\quad R_n = \dfrac{h^n}{n!} f^n(a + \theta h)$ where R_n is Taylor's remainder after n terms.

Then $\quad f(a + h) = S_n + R_n$

Thus if $R_n \to 0$ as $n \to \infty$, we have $\lim_{n \to \infty} S_n = f(a + h)$

$\Rightarrow$ The infinite series $f(a) + hf'(a) + \dfrac{h^2}{2!} f''(a) + \ldots\ldots + \dfrac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \ldots\ldots$ converges to $f(a + h)$.

Thus, if

(*i*) f possesses continuous derivatives of every order in $[a, a + h]$

(*ii*) the Taylor's remainder after n terms *i.e.*, $R_n \to 0$ as $n \to \infty$

then, $$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots\ldots + \frac{h^n}{n!} f^n(a) + \ldots\ldots$$

The infinite series on R.H.S. is called **Taylor's Series.**

If we put $a = 0$ and replace h by x, we get

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \ldots\ldots + \frac{x^n}{n!} f^n(0) + \ldots\ldots$$

The infinite series on R.H.S. is called *Maclaurin's series*,

Note. In the above discussion, the remainder R_n can be of any form.

ILLUSTRATIVE EXAMPLES

Example 1. *Expand e^x as an infinite series.*

Sol. Let $f(x) = e^x$ so that $f^n(x) = e^x$ and $f^n(0) = e^0 = 1 \ \forall \ n \in N$

Clearly, f and all its derivatives exist and are continuous for every real value of x.

Lagrange's form of remainder is

$$R_n = \frac{x^n}{n!} f^n(\theta x),\ 0 < \theta < 1 = \frac{x^n}{n!} \cdot e^{\theta x}$$

Let $\quad a_n = \dfrac{x^n}{n!} \ \forall \ n \in N$ then $a_{n+1} = \dfrac{x^{n+1}}{(n+1)!}$

so that $\quad \lim_{n \to \infty} \dfrac{a_{n+1}}{a_n} = \lim_{n \to \infty} \dfrac{x}{n+1} = 0 < 1$

$\Rightarrow \quad \lim_{n \to \infty} a_n = 0 \qquad \left[\because \text{ if } \lim_{n \to \infty} \dfrac{a_{n+1}}{a_n} = l \text{ and } |l| < 1 \text{ then } \lim_{n \to \infty} a_n = 0\right]$

$\Rightarrow \quad \lim_{n \to \infty} \dfrac{x^n}{n!} = 0$

$\therefore \quad \lim_{n \to \infty} R_n = \lim_{n \to \infty} \dfrac{x^n}{n!} e^{\theta x} = \left(\lim_{n \to \infty} \dfrac{x^n}{n!}\right) e^{\theta x} = 0 \times e^{\theta x} = 0$

Thus the conditions of Maclaurin's infinite expansion are satisfied.

$\therefore$ For all $x \in R$, $\quad e^x = f(x)$

$$= f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \ldots\ldots + \frac{x^n}{n!} f^n(0) + \ldots\ldots$$

$$= 1 + x + \frac{x^2}{2!} + \ldots\ldots + \frac{x^n}{n!} + \ldots\ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Example 2. *Expand $\sin x$ as an infinite series.*

Sol. Let $\quad f(x) = \sin x$ so that $f^n(x) = \sin\left(\dfrac{n\pi}{2} + x\right)$

and $$f^n(0) = \sin \frac{n\pi}{2} \ \forall \ n \in N$$

$$\Rightarrow \quad f^n(0) = \begin{cases} 0, & \text{if } n \text{ is even} \\ (-1)^{\frac{n-1}{2}}, & \text{if } n \text{ is odd} \end{cases}$$

$$\Rightarrow \quad f''(0) = f^{iv}(0) = \ldots\ldots = 0,\ f'(0) = 1,\ f'''(0) = -1,\ f^v(0) = 1, \ldots\ldots$$

Clearly, f and all its derivatives exist and are continuous for every real value of x.

Lagrange's form of remainder is

$$R_n = \frac{x^n}{n!} f^n(\theta x),\ 0 < \theta < 1 = \frac{x^n}{n!} \sin\left(\frac{n\pi}{2} + \theta x\right)$$

$$\therefore \quad |R_n| = \left|\frac{x^n}{n!} \sin\left(\frac{n\pi}{2} + \theta x\right)\right|$$

$$= \left|\frac{x^n}{n!}\right| \left|\sin\left(\frac{n\pi}{2} + \theta x\right)\right| \le \left|\frac{x^n}{n!}\right| \qquad \left[\because \left|\sin\left(\frac{n\pi}{2} + \theta x\right)\right| \le 1\right]$$

But $\quad \lim_{n \to \infty} \left|\dfrac{x^n}{n!}\right| = 0 \ \forall \ x \in R \qquad$ [As in example 1]

$\therefore \quad \lim_{n \to \infty} |R_n| = 0$ and hence $\lim_{n \to \infty} R_n = 0$

Thus the conditions of Maclaurin's infinite expansion are satisfied.

$\therefore$ For all $x \in R$, $\sin x = f(x) = f(0) + xf'(0) + \dfrac{x^2}{2!} f''(0) + \ldots\ldots = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \ldots\ldots$

Example 3. *Expand $\cos x$ as an infinite series.*

Sol. *Please try yourself.*

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\ldots$$

Example 4. *Expand $(1 + x)^m$, $m \in R$.*

Sol. Two cases arise according as m is or is not a positive integer.

Case 1. *When m is a positive integer.*

Let $\quad f(x) = (1 + x)^m,\ x \in R$

Then $f^n(x)$ exists for all x and all n.

In fact, if $1 \le n \le m$, then $\quad f^n(x) = m(m-1)(m-2) \ldots.. (m - n + 1)(1 + x)^{m-n}$

so that $\quad f^m(x) = m!$ and $f^n(x) = 0$, if $n > m$

$$\Rightarrow \qquad f^n(0) = \begin{cases} m(m-1)\ldots\ldots(m-n+1) & \text{if } 1 \le n \le m \\ 0 & \text{if } n > m \end{cases}$$

Since $f^n(x) = 0$ for all $n > m$, it follows that $R_n \to 0$ as $n \to \infty$. Thus the conditions of Maclaurin's expansion are satisfied.

$$\therefore \quad (1+x)^m = f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \ldots\ldots + \frac{x^m}{m!} f^m(0) + 0 + 0 + \ldots\ldots$$

$$= 1 + mx + \frac{m(m-1)}{2!} x^2 + \ldots\ldots + x^m.$$

Case 2. *When m is not a positive integer.*

Let $\qquad f(x) = (1+x)^m, x \ne -1$

Taking Cauchy form of remainder, we have

$$R_n = \frac{x^n}{(n-1)!}(1-\theta)^{n-1} f^n(\theta x), \quad 0 < \theta < 1$$

$$= \frac{x^n}{(n-1)!}(1-\theta)^{n-1} \cdot m(m-1) \ldots\ldots (m-n+1)(1+\theta x)^{m-n}$$

$$= \left(\frac{m(m-1)\ldots\ldots(m-n+1)}{(n-1)!} x^n\right)\left(\frac{1-\theta}{1+\theta x}\right)^{n-1} (1+\theta x)^{m-1}$$

Now let $\qquad a_n = \dfrac{m(m-1)\ldots\ldots(m-n+1)}{(n-1)!} x^n$

so that $\qquad a_{n+1} = \dfrac{m(m-1)\ldots\ldots(m-n+1)(m-n)}{n!} x^{n+1}$

$$\therefore \quad \frac{a_{n+1}}{a_n} = \left(\frac{m-n}{n}\right)x = \left(\frac{m}{n} - 1\right)x$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\left(\frac{m}{n} - 1\right)x = -x$$

It follows that if $|-x| = |x| < 1$, then $\lim_{n\to\infty} a_n = 0$

$$\Rightarrow \quad \frac{m(m-1)\ldots\ldots(m-n+1)}{(n-1)!} x^n \to 0 \text{ for } |x| < 1 \qquad \ldots(1)$$

Since $\qquad 0 < \theta < 1 \qquad$ and $\qquad -1 < x < 1$

$\therefore \qquad -\theta < \theta x < \theta \qquad \Rightarrow \qquad 1 - \theta < 1 + \theta x < 1 + \theta$

$\Rightarrow \qquad 0 < 1 - \theta < 1 + \theta x \qquad \Rightarrow \qquad 0 < \dfrac{1-\theta}{1+\theta x} < 1$

$$\text{Consequently,} \quad \lim_{n\to\infty}\left(\frac{1-\theta}{1+\theta x}\right)^{n-1} = 0 \qquad \ldots(2)$$

Also, since $\; -|x| \le x \le |x| \;$ and $\; 0 < \theta < 1 \;\Rightarrow\; -\theta|x| \le \theta x \le \theta|x|$

$\Rightarrow \quad -|x| < -\theta|x| \le \theta x \le \theta|x| < |x| \;\Rightarrow\; 1 - |x| < 1 + \theta x < 1 + |x|$

$\therefore \quad$ If $m > 1$, then $(1+\theta x)^{m-1} < (1 + |x|)^{m-1}$

and if $\quad m < 1$, then $\quad (1+\theta x)^{m-1} = \dfrac{1}{(1+\theta x)^{1-m}}$

$$< \frac{1}{(1-|x|)^{1-m}} = (1 - |x|)^{m-1} \qquad \ldots(3)$$

From (1), (2) and (3), we find that for $|x| < 1$, $\lim_{n\to\infty} R_n = 0$.

Thus the conditions of Maclaurin's infinite expansion are satisfied and

$$(1+x)^m = f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \ldots\ldots + \frac{x^n}{n!} f^n(0) + \ldots\ldots$$

$$= 1 + mx + \frac{m(m-1)}{2!} x^2 + \ldots\ldots + \frac{m(m-1)\ldots\ldots(m-n+1)}{n!} x^n + \ldots\ldots$$

Example 5. *Expand log (1 + x) as an infinite series.*

Sol. Let $f(x) = \log(1+x)$ where $1 + x > 0$ i.e., $x > -1$

Then $\qquad f^n(x) = \dfrac{(-1)^{n-1}(n-1)!}{(1+x)^n} \; \forall n \in N$ and $x > -1$.

Case 1. When $0 \le x \le 1$

Writing Lagrange's remainder after n terms, we have

$$R_n = \frac{x^n}{n!} f^n(\theta x), \quad 0 < \theta < 1$$

$$= \frac{x^n}{n!} \cdot \frac{(-1)^{n-1}(n-1)!}{(1+\theta x)^n} = \frac{(-1)^{n-1}}{n} \cdot \left(\frac{x}{1+\theta x}\right)^n$$

If $x = 1$, then $\quad |R_n| = \dfrac{1}{n}\left(\dfrac{1}{1+\theta}\right)^n < \dfrac{1}{n}$ and $\dfrac{1}{n} \to 0$ as $n \to \infty$.

$\therefore \qquad \lim_{n\to\infty} R_n = 0$

If $\; 0 \le x < 1$, then since $0 < \theta < 1$

$\therefore \; 0 \le x < 1 + \theta x$

$\Rightarrow \qquad 0 \le \dfrac{x}{1+\theta x} < 1 \qquad \Rightarrow \qquad 0 \le \left(\dfrac{x}{1+\theta x}\right)^n < 1$

$\therefore \qquad |R_n| = \dfrac{1}{n}\left(\dfrac{x}{1+\theta x}\right)^n < \dfrac{1}{n}$ and $\dfrac{1}{n} \to 0$ as $n \to \infty$

$\therefore \qquad \lim_{n\to\infty} R_n = 0.$

Case 2. When $-1 < x < 0$

Since in this case, $\left|\dfrac{x}{1+\theta x}\right|$ need not be less than unity, therefore, it may not be easily shown that $R_n \to 0$ as $n \to \infty$ by considering Lagrange's remainder.

$\therefore$ Writing Cauchy's remainder after n terms, we have

$$R_n = \frac{x^n}{(n-1)!}(1-\theta)^{n-1} f^n(\theta x),\ 0<\theta<1$$

$$= \frac{x^n}{(n-1)!}(1-\theta)^{n-1}\cdot\frac{(-1)^{n-1}(n-1)!}{(1+\theta x)^n} = (-1)^{n-1}\cdot x^n\left(\frac{1-\theta}{1+\theta x}\right)^{n-1}\cdot\frac{1}{1+\theta x}$$

Now $\quad -1<x<0$ and $0<\theta<1 \Rightarrow -\theta<\theta x$

$\Rightarrow \quad 1-\theta<1+\theta x \quad \Rightarrow \quad 0<\dfrac{1-\theta}{1+\theta x}<1$

Also, $\quad -|x|\leq x \quad \Rightarrow \quad -\theta|x|\leq\theta x$

$\therefore \quad -|x|<-\theta|x|\leq\theta x$

$\Rightarrow \quad 1-|x|<1+\theta x \quad \Rightarrow \quad \dfrac{1}{1+\theta x}<\dfrac{1}{1-|x|}$

Consequently, $\quad |R_n| = |x|^n\left|\dfrac{1-\theta}{1+\theta x}\right|^{n-1}\left|\dfrac{1}{1+\theta x}\right|<\dfrac{|x|^n}{1-|x|}$

and $\quad |x|^n\to 0$ as $n\to\infty$ $\qquad$ (since $|x|<1$)

$\therefore \quad \lim\limits_{n\to\infty} R_n = 0$

Thus, we find that if $-1<x\leq 1$, then $\lim\limits_{n\to\infty} R_n = 0$.

Hence $\quad \log(1+x) = f(x) = f(0) + xf'(0) + \dfrac{x^2}{2!}f''(0) + \dfrac{x^3}{3!}f'''(0) + \ldots\ldots$

$$= \log 1 + x\cdot 1 + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}\cdot 2 \ldots\ldots = x - \frac{x^2}{2} + \frac{x^3}{3} \ldots\ldots$$

8

Arbitrary Series and Infinite Products

8.1. INTRODUCTION

So far we have been discussing series of positive terms or alternating series. In this chapter we shall discuss the convergence of series of arbitrary terms *i.e.*, series of terms having any sign. We shall also discuss rearrangement of terms of a series, insertion and removal of brackets, Cauchy product of two series and the convergence of infinite products.

8.2. ABEL'S LEMMA (OR ABEL'S INEQUALITY)

If the sequence $<S_n>$ of the partial sums of the series $\sum\limits_{n=1}^{\infty} a_n$ satisfies $m\leq S_n\leq M$, $(n\in N)$ and $<b_n>$ is a sequence of non-increasing, non-negative real numbers, then $mb_1\leq\sum\limits_{k=1}^{n} a_k b_k \leq Mb_1$.

Proof. Since $\quad S_1 = a_1, S_2 = a_1+a_2, S_3 = a_1+a_2+a_3, \ldots\ldots,$

$S_n = a_1+a_2+\ldots\ldots+a_n$

$\therefore \quad a_1 = S_1, a_2 = S_2 - S_1, a_3 = S_3 - S_2, \ldots\ldots, a_n = S_n - S_{n-1}$

$\therefore \quad \sum\limits_{k=1}^{n} a_k b_k = a_1b_1 + a_2b_2 + \ldots\ldots + a_nb_n$

$= S_1b_1 + (S_2-S_1)b_2 + \ldots\ldots + (S_n - S_{n-1})b_n$

$= S_1(b_1-b_2) + S_2(b_2-b_3) + \ldots\ldots + S_{n-1}(b_{n-1}-b_n) + S_nb_n \qquad \ldots(1)$

Now $\quad mb_1 = m[(b_1-b_2)+(b_2-b_3)+\ldots\ldots+(b_{n-1}-b_n)+b_n]$

$= m(b_1-b_2) + m(b_2-b_3) + \ldots\ldots + m(b_{n-1}-b_n) + mb_n$

$\leq S_1(b_1-b_2)+S_2(b_2-b_3)+\ldots\ldots+S_{n-1}(b_{n-1}-b_n)+S_nb_n \qquad [\because\ m\leq S_n, n\in N]$

$= \sum\limits_{k=1}^{n} a_kb_k \qquad$ [by (1)]

$= S_1(b_1-b_2)+S_2(b_2-b_3)+\ldots\ldots+S_{n-1}(b_{n-1}-b_n)+S_nb_n$

$\leq M(b_1-b_2)+M(b_2-b_3)+\ldots\ldots+M(b_{n-1}-b_n)+Mb_n \qquad [\because\ S_n\leq M, n\in N]$

$= Mb_1$

$\Rightarrow \quad mb_1 \leq \sum\limits_{k=1}^{n} a_kb_k \leq Mb_1.$

Remark. If $|S_n| \le M \ \forall n \in N$, then, $\left|\sum_{k=1}^{n} a_k b_k\right| \le Mb_1$

i.e., $|a_1 + a_2 + a_3 + \ldots + a_n| \le M \Rightarrow |a_1b_1 + a_2b_2 + a_3b_3 + \ldots + a_nb_n| \le Mb_1$

In particular, if $|a_{m+1} + a_{m+2} + \ldots + a_n| \le M$

then $|a_{m+1}b_{m+1} + a_{m+2}b_{m+2} + \ldots + a_nb_n| \le Mb_{m+1} \le Mb_1$ [$\because$ the sequence $<b_n>$ is non-increasing]

8.3. ABEL'S TEST

If $\sum_{n=1}^{\infty} a_n$ is convergent and the sequence $<b_n>$ is monotonic and bounded, then $\sum_{n=1}^{\infty} a_nb_n$ is convergent.

Proof. Since the sequence $<b_n>$ is monotonic and bounded, it is convergent. Let it converge to b.

Let $$u_n = \begin{cases} b - b_n \text{ if } <b_n> \text{ is increasing} \\ b_n - b \text{ if } <b_n> \text{ is decreasing} \end{cases}$$

When $<b_n>$ is increasing, b is the l.u.b. of the sequence so that $b_n \le b \ \forall n$

$\Rightarrow$ $b - b_n \ge 0 \quad \therefore \quad u_n \ge 0 \quad \forall n \in N$

and $u_n - u_{n+1} = (b - b_n) - (b - b_{n+1}) = b_{n+1} - b_n \ge 0$

$\Rightarrow$ $u_n \ge u_{n+1} \quad \forall n \in N$

$\Rightarrow <u_n>$ is a non-increasing sequence of non-negative numbers.

When $<b_n>$ is decreasing, b is the g.l.b. of the sequence so that

$b_n \ge b \ \forall n \Rightarrow b_n - b \ge 0 \quad \therefore \quad u_n \ge 0 \quad \forall n \in N$

and $u_n - u_{n+1} = (b_n - b) - (b_{n+1} - b) = b_n - b_{n+1} \ge 0$

$\Rightarrow$ $u_n \ge u_{n+1} \quad \forall n \in N$

$\Rightarrow$ $<u_n>$ is a non-increasing sequence of non-negative numbers.

$\therefore$ $u_n \ge u_{n+1} \ge 0 \ \forall n \in N$

Now $$b_n = \begin{cases} b - u_n \text{ if } <b_n> \text{ is increasing} \\ b + u_n \text{ if } <b_n> \text{ is decreasing} \end{cases}$$

$\Rightarrow$ $$a_nb_n = \begin{cases} ba_n - a_nu_n \text{ if } <b_n> \text{ is increasing} \\ ba_n + a_nu_n \text{ if } <b_n> \text{ is decreasing} \end{cases}$$

Since $\sum_{n=1}^{\infty} a_n$ is convergent $\Rightarrow \sum_{n=1}^{\infty} ba_n$ is convergent.

$\therefore$ $\sum_{n=1}^{\infty} a_nb_n$ will be convergent if $\sum_{n=1}^{\infty} a_nu_n$ is convergent.

Now, since $\sum_{n=1}^{\infty} a_n$ is convergent.

$\therefore$ By Cauchy's general principle of convergence, given $\varepsilon > 0$, $\exists$ ~~a positive~~ integer m such that

$$|a_{m+1} + a_{m+2} + \ldots + a_n| < \varepsilon \text{ whenever } n > m$$

Applying Abel's Lemma, we have

$$|a_{m+1}u_{m+1} + a_{m+2}u_{m+2} + \ldots + a_nu_n| \le \varepsilon u_{m+1} \le \varepsilon u_1$$ [$\because$ $<u_n>$ is decreasing]

$\therefore$ By Cauchy's general principle of convergence, $\sum_{n=1}^{\infty} a_nu_n$ is convergent.

Hence $\sum_{n=1}^{\infty} a_nb_n$ is convergent.

8.4. DIRICHLET'S TEST

If $\sum_{n=1}^{\infty} a_n$ has bounded partial sums and $<b_n>$ is a monotonic sequence converging to zero, then $\sum_{n=1}^{\infty} a_nb_n$ is convergent.

Proof. Replacing b_n by $-b_n$, if necessary, we assume that $<b_n>$ is a monotonically decreasing sequence converging to 0.

Let $S_n = a_1 + a_2 + \ldots + a_n, n \in N.$

Since $\sum_{n=1}^{\infty} a_n$ has bounded partial sums, therefore, the sequence $<S_n>$ is bounded.

$\Rightarrow$ $\exists$ a real number $M > 0$ such that $|S_n| \le M \ \forall n \in N$

$\therefore$ For $n > m$, we have $|a_{m+1} + a_{m+2} + \ldots + a_n| = |S_n - S_m|$
$\le |S_n| + |S_m|$
$\le M + M = 2M$ ...(1)

Since $<b_n>$ converges to 0, therefore, given $\varepsilon > 0$, $\exists$ a positive integer m_0 such that

$$b_n < \frac{\varepsilon}{2M} \ \forall n \ge m_0 \quad \ldots(2)$$

$\therefore$ From (1), by Abel's Lemma, for $n > m \ge m_0$, we have

$$|a_{m+1}b_{m+1} + a_{m+2}b_{m+2} + \ldots + a_nb_n| \le 2Mb_{m+1} < \varepsilon$$ [by (2)]

$\therefore$ By Cauchy's general principle of convergence, $\sum_{n=1}^{\infty} a_nb_n$ is convergent.

Cor. Leibnitz's Test as a particular case of Dirichlet's Test.

The series $\sum_{n=1}^{\infty} (-1)^{n-1}$ has bounded partial sums, since $S_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$

If $<a_n>$ is a monotonically decreasing sequence of positive numbers, converging to 0, *i.e.*, if (*i*) $a_n > 0$ (*ii*) $a_n \ge a_{n+1} \ \forall n \in N$ (*iii*) $a_n \to 0$ as $n \to \infty$

then by Dirichlet test, the series $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$, *i.e.*, the alternating series $a_1 - a_2 + a_3 - a_4 + \ldots$ is convergent.

ILLUSTRATIVE EXAMPLES

Example 1. *Test the convergence of the series :*

(*i*) $1-\frac{1}{3.2^2}+\frac{1}{5.3^2}-\frac{1}{7.4^2}+\ldots\ldots$ (*ii*) $1-\frac{1}{4.3}+\frac{1}{4^2.5}-\frac{1}{4^3.7}+\ldots\ldots$

(*iii*) $1-\frac{1}{5\sqrt{2}}+\frac{1}{9\sqrt{3}}-\frac{1}{13\sqrt{4}}+\ldots\ldots$ (*iv*) $\frac{1}{\log 2}-\frac{1}{2^3\log 3}+\frac{1}{3^3\log 4}-\frac{1}{4^3\log 5}+\ldots$

Sol. (*i*) The given series can be considered to have arisen as a result of multiplying the terms of the series $1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots$ with the terms of the sequence $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots\ldots$

Let $a_n = \frac{(-1)^{n-1}}{n^2}, b_n = \frac{1}{2n-1}$ then the given series can be written as $\sum_{n=1}^{\infty} a_n b_n$.

Now $|a_n| = \frac{1}{n^2}$, therefore Σa_n is convergent.

Since Σa_n is absolutely convergent, therefore, Σa_n is convergent.

Also $< b_n >$ is a monotonically decreasing sequence of positive terms.

$\therefore$ By Abel's test, the series $\sum_{n=1}^{\infty} a_n b_n$ is convergent.

(*ii*) The given series can be considered to have arisen as a result of multiplying the terms of the series

$$1-\frac{1}{4}+\frac{1}{4^2}-\frac{1}{4^3}+\ldots \text{ with the terms of the sequence } 1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots\ldots$$

Let $a_n = \left(-\frac{1}{4}\right)^{n-1}, b_n = \frac{1}{2n-1}$, then the given series can be written as $\sum_{n=1}^{\infty} a_n b_n$.

Now $|a_n| = \left(\frac{1}{4}\right)^{n-1}$, therefore, $\sum_{n=1}^{\infty} |a_n|$ is a geometric series with common ratio $\frac{1}{4}$.

$[|r| = \frac{1}{4} < 1]$

$\Rightarrow$ $\Sigma |a_n|$ is convergent.

$\Rightarrow$ Σa_n is convergent.

Also $< b_n >$ is a monotonically decreasing sequence of positive terms.

$\therefore$ By Abel's test, the series $\sum_{n=1}^{\infty} a_n b_n$ is convergent.

(*iii*) Let $a_n = \frac{(-1)^{n-1}}{\sqrt{n}}, b_n = \frac{1}{4n-3}$, then the given series can be written as $\sum_{n=1}^{\infty} a_n b_n$.

Now, by Leibnitz text, the series $\sum_{n=1}^{\infty} a_n$ is convergent.

Also $< b_n >$ is a monotonically decreasing sequence of positive terms.

$\therefore$ By Abel's test, the series $\sum_{n=1}^{\infty} a_n b_n$ is convergent.

(*iv*) Let $a_n = \frac{(-1)^{n-1}}{\log(n+1)}, b_n = \frac{1}{n^3}$, then the given series can be written as $\sum_{n=1}^{\infty} a_n b_n$.

Now, by Leibnitz test, the series $\sum_{n=1}^{\infty} a_n$ is convergent.

Also $< b_n >$ is a monotonically decreasing sequence of positive terms.

$\therefore$ By Abel's test, the series $\sum_{n=1}^{\infty} a_n b_n$ is convergent.

Note. Try part (*i*) by taking $a_n = \frac{(-1)^{n-1}}{2n-1}, b_n = \frac{1}{n^2}$

Try part (*ii*) by taking $a_n = \frac{(-1)^{n-1}}{2n-1}, b_n = \left(\frac{1}{4}\right)^{n-1}$

Try part (*iii*) by taking $a_n = \frac{(-1)^{n-1}}{4n-3}, b_n = \frac{1}{\sqrt{n}}$

Try part (*iv*) by taking $a_n = \frac{(-1)^{n-1}}{n^3}, b_n = \frac{1}{\log(n+1)}$

Example 2. *Show that the series* $\sum_{n=2}^{\infty} \frac{(n^3+1)^{1/3}-n}{\log n}$ *is convergent.*

Sol. Let $a_n = (n^3+1)^{1/3} - n, b_n = \frac{1}{\log n}$, then the given series can be written as $\sum_{n=1}^{\infty} a_n b_n$.

Now
$$a_n = (n^3+1)^{1/3} - n = n\left(1+\frac{1}{n^3}\right)^{1/3} - n$$
$$= n\left[1+\frac{1}{3}\cdot\frac{1}{n^3}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\cdot\frac{1}{n^6}+\ldots\ldots-1\right] = \frac{1}{n^2}\left[\frac{1}{3}-\frac{1}{9n^3}+\ldots\right]$$

Take $c_n = \frac{1}{n^2}$, then $\frac{a_n}{c_n} = \frac{1}{3} - \frac{1}{9n^3} + \ldots\ldots$

$$\lim_{n\to\infty} \frac{a_n}{c_n} = \frac{1}{3} \text{ which is finite and non-zero.}$$

$\therefore$ By comparison test, Σa_n and Σc_n converge or diverge together.

But $\sum c_n = \sum \frac{1}{n^2}$ is convergent.

$\therefore$ Σa_n is convergent.

Also $< b_n >$ is a monotonically decreasing sequence of positive terms.

$\therefore$ By Abel's test, the series $\sum_{n=2}^{\infty} a_n b_n$ is convergent.

Example 3. *Show that the convergence of $\sum_{n=1}^{\infty} a_n$ implies the convergence of each of the following series :*

(i) $\sum_{n=1}^{\infty} \frac{1}{n} a_n$ (ii) $\sum_{n=1}^{\infty} \frac{n+1}{n} a_n$

(iii) $\sum_{n=1}^{\infty} \frac{1}{n^p} a_n, p \geq 0$ (iv) $\sum_{n=1}^{\infty} n^{1/n} a^n$

(v) $\sum_{n=2}^{\infty} \frac{1}{\log n} a_n$ (vi) $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n a_n$.

Sol. (i) Let $b_n = \frac{1}{n}$, then the given series can be written as $\sum_{n=1}^{\infty} a_n b_n$.

Now $\sum_{n=1}^{\infty} a_n$ is given to be convergent.

Also $< b_n >$ is a monotonically decreasing sequence of positive terms.

$\therefore$ By Abel's test, the series $\sum_{n=1}^{\infty} a_n b_n$ is convergent.

(ii) *Please try yourself.*

(iii) Let $b_n = \frac{1}{n^p}$, then the given series can be written as $\sum_{n=1}^{\infty} a_n b_n$.

Now $\sum_{n=1}^{\infty} a_n$ is given to be convergent.

Also $< b_n >$ is a monotonically decreasing sequence of positive terms.

$\therefore$ By Abel's test, the series $\sum_{n=1}^{\infty} a_n b_n$ is convergent.

(iv) Let $b_n = n^{1/n}$, then the given series can be written as $\sum_{n=1}^{\infty} a_n b_n$.

Now $\sum_{n=1}^{\infty} a_n$ is given to be convergent.

Also
$$\left(1+\frac{1}{n}\right)^n = 1 + n \cdot \frac{1}{n} + \frac{n(n-1)}{2!} \cdot \frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!} \cdot \frac{1}{n^3} + \ldots + \frac{n(n-1)(n-2) \ldots 2.1}{n!} \cdot \frac{1}{n^n}$$
$$= 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \ldots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{n-1}{n}\right)$$

$$\leq 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!} < e < 3$$

$\therefore$ For $n \geq 3$, we have $n > \left(1+\frac{1}{n}\right)^n = \frac{(n+1)^n}{n^n} \Rightarrow n^{n+1} > (n+1)^n$

$\Rightarrow \quad n^{1/n} > (n+1)^{1/n+1} \Rightarrow b_n > b_{n+1}$

$\therefore$ $< b_n >$ is monotonically increasing sequence.

Also $\lim_{n \to \infty} b_n = \lim_{n \to \infty} n^{1/n} = 1$

$\Rightarrow$ $< b_n >$ is convergent. $\Rightarrow$ $< b_n >$ is bounded.

Since $< b_n >$ is monotonic and bounded, therefore, by Abel's test, the series $\sum_{n=1}^{\infty}$ is convergent.

(v) Let $b_n = \frac{1}{\log n}$, $n \geq 2$, then $< b_n >$ is a monotonically decreasing sequence of positive terms.

$\therefore$ By Abel's test, the series $\sum_{n=2}^{\infty} a_n b_n$ is convergent.

(vi) *Please try yourself.*

Example 4. *Show that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\left(1+\frac{1}{n}\right)^{-n}$ is convergent.*

Sol. Let $a_n = \frac{(-1)^{n-1}}{n}, b_n = \left(1+\frac{1}{n}\right)^{-n}$

By Leibnitz test, the series $\sum_{n=1}^{\infty} a_n$ is convergent.

Also
$$\left(1+\frac{1}{n}\right)^n = 1 + n \cdot \frac{1}{n} + \frac{n(n-1)}{2!} \cdot \frac{1}{n^2} + \ldots + \frac{n(n-1) \ldots 2.1}{n!} \cdot \frac{1}{n^n}$$
$$= 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \ldots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{n-1}{n}\right)$$

and
$$\left(1+\frac{1}{n+1}\right)^{n+1} = 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n+1}\right) + \frac{1}{3!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right) + \ldots + \frac{1}{(n+1)!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\ldots\left(1-\frac{n}{n+1}\right)$$

Since $1 - \frac{k}{n} < 1 - \frac{k}{n+1}$ for $k = 1, 2, \ldots, n$

$\therefore$ $\left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{n+1}\right)^{n+1}$

$$\Rightarrow \quad \left(1+\frac{1}{n}\right)^{-n} > \left(1+\frac{1}{n+1}\right)^{-(n+1)} \quad \Rightarrow \quad b_n > b_{n+1}$$

$\therefore$ $< b_n >$ is a monotonically decreasing sequence.

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{-n} = e^{-1} = \frac{1}{e}$$

$\Rightarrow$ $< b_n >$ is convergent. $\Rightarrow$ $< b_n >$ is bounded.

Since $\sum_{n=1}^{\infty} a_n$ is convergent and $< b_n >$ is monotonic and bounded, therefore, by Abel's test, the series

$$\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\left(1+\frac{1}{n}\right)^{-n} \text{ is convergent.}$$

Example 5. *Show that the series* $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}\left(1+\frac{1}{n}\right)^{n}$ *is convergent.*

Sol. *Please try yourself.*

Example 6. *Is the series* $\sum_{n=2}^{\infty} \frac{\sin nx}{n^2 \log n}$ *convergent ?*

Sol. Let $a_n = \frac{\sin nx}{n^2}$, $b_n = \frac{1}{\log n}$, then the given series can be written as $\sum_{n=2}^{\infty} a_n b_n$.

Now $|a_n| = \left|\frac{\sin nx}{n^2}\right| \le \frac{1}{n^2} \;\forall\, n$ and $\sum \frac{1}{n^2}$ converges.

$\therefore$ By comparison test, the series $\sum_{n=1}^{\infty} |a_n|$ is convergent.

$\Rightarrow$ $\sum_{n=1}^{\infty} a_n$ is convergent.

Also $< b_n >$ is a monotonically decreasing sequence of positive terms.

$\therefore$ By Abel's test, the series $\sum_{n=2}^{\infty} a_n b_n$ is convergent.

Example 7. *Show that the following series are convergent :*

(*i*) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n-\alpha}$, *α not an integer* ≥ 1. (*ii*) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n+\alpha}}$, *α not an integer* ≤ -1.

Sol. (*i*) Let $a_n = (-1)^{n-1}$, $b_n = \frac{1}{n-\alpha}$, then the given series can be written as $\sum_{n=1}^{\infty} a_n b_n$.

Now $\sum_{n=1}^{\infty} a_n$ has bounded partial sums, since $S_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$

Also $< b_n >$ is a decreasing sequence with $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n-\alpha} = 0$

$\therefore$ By Dirichlet's test, $\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n-\alpha}$ is convergent.

(*ii*) *Please try yourself.*

Example 8. *Test the convergence of the series* $\sum_{n=1}^{\infty} \frac{\cos nx}{n^p}$, $(p > 0)$.

Sol. Let $a_n = \cos nx$, $b_n = \frac{1}{n^p}$, $p > 0$, then the given series can be written as $\sum_{n=1}^{\infty} a_n b_n$.

Let S_n be the nth partial sum of the series Σa_n, then

$$S_n = \cos x + \cos 2x + \cos 3x + \dots + \cos nx = \frac{\cos\left(x + \frac{n-1}{2}x\right)\sin\frac{nx}{2}}{\sin\frac{x}{2}}$$

$$\left[\because \cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \dots \text{ to } n \text{ terms} = \frac{\cos\left(\alpha + \frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\right]$$

$$= \frac{\cos\left(\frac{n+1}{2}\right)x \sin\frac{nx}{2}}{\sin\frac{x}{2}}$$

$$\Rightarrow \quad |S_n| = \left|\frac{\cos\left(\frac{n+1}{2}\right)x\sin\frac{nx}{2}}{\sin\frac{x}{2}}\right| = \frac{\left|\cos\left(\frac{n+1}{2}\right)x\right|\left|\sin\frac{nx}{2}\right|}{\left|\sin\frac{x}{2}\right|} \le \frac{1}{\left|\sin\frac{x}{2}\right|} = \left|\operatorname{cosec}\frac{x}{2}\right|$$

$\Rightarrow$ $< S_n >$ is bounded for all x for which $\sin\frac{x}{2} \ne 0$, *i.e.*, for which $\frac{x}{2} \ne r\pi$ or $x \ne 2r\pi$, $r \in Z$.

Also $< b_n >$ is a decreasing sequence with $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n^p} = 0$

$\therefore$ By Dirichlet's test, $\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} \frac{\cos nx}{n^p}$, $p > 0$ is convergent for all $x \ne 2r\pi$, $r \in$ **Z**.

When $x = 2r\pi$ $\cos nx = 1$

$\therefore$ $\sum_{n=1}^{\infty} \frac{\cos nx}{n^p} = \sum_{n=1}^{\infty} \frac{1}{n^p}$, $(p > 0)$ is convergent if $p > 1$ and divergent if $0 < p \le 1$.

Example 9. *If* $< b_n >$ *is a monotone sequence converging to* θ, *show that the series* $\sum_{n=1}^{\infty} b_n \cos n\theta$ *converges for all real values of* $\theta \ne 2k\pi$, $k \in Z$.

Sol. *Please try yourself.*

[**Hint.** Show that $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \cos n\theta$ has bounded partial sums.]

Example 10. (*a*) *Show that the series* $\sum_{n=1}^{\infty} \frac{\sin nx}{n^p}$, $(p > 0)$, *converges for all real x.*

(*b*) *Use Dirichlet's test to show that the series* $\sum_{n=1}^{\infty} \frac{\sin nx}{n}$ *is convergent where* $x \in R$.

Sol. (*a*) *Please try yourself.*

[**Hint.** $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots\ldots$ to n terms $= \dfrac{\sin\left(\alpha + \frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}$]

(*b*) *Please try yourself.*

Example 11. *Show that the series* $\sum_{n=1}^{\infty} \frac{(-1)^n \sin nx}{n^p}$, $p > 0$, *converges for all real x.*

Sol. $\sum_{n=1}^{\infty} \frac{(-1)^n \sin nx}{n^p} = \sum_{n=1}^{\infty} \frac{\sin(n\pi + nx)}{n^p} = \sum_{n=1}^{\infty} \frac{\sin n(\pi + x)}{n^p}$

Let $a_n = \sin n(\pi + x)$, $b_n = \frac{1}{n^p}$, $p > 0$, then the given series can be written as $\sum_{n=1}^{\infty} a_b b_n$.

Let S_n be the nth partial sum of the series Σa_n, then

$$S_n = \sin(\pi + x) + \sin 2(\pi + x) + \sin 3(\pi + x) + \ldots\ldots + \sin n(\pi + x)$$

$$= \frac{\sin\left[(\pi+x) + \frac{n-1}{2}(\pi+x)\right]\sin\frac{n(\pi+x)}{2}}{\sin\frac{\pi+x}{2}} = \frac{\sin(n+1)\frac{\pi+x}{2}\sin\frac{n(\pi+x)}{2}}{\cos\frac{x}{2}}$$

$$\Rightarrow \quad |S_n| = \left|\frac{\sin(n+1)\frac{\pi+x}{2}\sin\frac{n(\pi+x)}{2}}{\cos\frac{x}{2}}\right| \le \frac{1}{\left|\cos\frac{x}{2}\right|} = \left|\sec\frac{x}{2}\right|$$

$\Rightarrow$ $< S_n >$ is bounded for all x for which $\cos\frac{x}{2} \neq 0$, *i.e.*, for which $\frac{x}{2} \neq (2r+1)\frac{\pi}{2}$ or $x \neq (2r+1)\pi$, $r \in Z$.

Also $< b_n >$ is a decreasing sequence with $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n^p} = 0$

$\therefore$ By Dirichlet's test, $\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} \frac{(-1)^n \sin nx}{n^p}$, $(p > 0)$ is convergent for all $x \neq (2r+1)\pi$, $r \in Z$.

When $x = (2r+1)\pi$, $\sin nx = 0$.

$\therefore$ Each term of the series is 0 and so that series is convergent.

Hence the given series is convergent for all real x.

Example 12. *Show that the series* $\sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^p}$, $p > 0$ *converges provided x is not an odd multiple of* π.

Sol. *Please try yourself.* [**Hint.** $(-1)^n \cos nx = \cos(n\pi + nx) = \cos n(\pi + x)$.]

Example 13. *Discuss the convergence of the series* $\sum_{n=1}^{\infty} \frac{1}{n}\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots + \frac{1}{n}\right)$ $\sin(n\theta + \alpha)$.

Sol. Let $a_n = \sin(n\theta + \alpha)$ and $b_n = \frac{1}{n}\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots + \frac{1}{n}\right)$.

Let S_n be the nth partial sum of $\sum_{n=1}^{\infty} a_n$, then

$$S_n = \sin(\theta + \alpha) + \sin(2\theta + \alpha) + \ldots\ldots + \sin(n\theta + \alpha)$$

$$= \frac{\sin\left[(\theta+\alpha) + \frac{n-1}{2}\theta\right]\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}$$

$$\Rightarrow \quad |S_n| = \left|\frac{\sin\left(\alpha + \frac{n+1}{2}\theta\right)\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}\right| \le \frac{1}{\left|\sin\frac{\theta}{2}\right|} = \left|\operatorname{cosec}\frac{\theta}{2}\right|$$

$\Rightarrow$ $< S_n >$ is bounded for each θ except which $\sin\frac{\theta}{2} = 0$,

i.e., when $\frac{\theta}{2} = r\pi$ or $\theta = 2r\pi$, $r \in Z$.

Also $b_n - b_{n+1}$

$$= \frac{1}{n}\left(1 + \frac{1}{2} + \ldots\ldots + \frac{1}{n}\right) - \frac{1}{n+1}\left(1 + \frac{1}{2} + \ldots\ldots + \frac{1}{n} + \frac{1}{n+1}\right)$$

$$= \left(1 + \frac{1}{2} + \ldots\ldots + \frac{1}{n}\right)\left(\frac{1}{n} - \frac{1}{n+1}\right) - \frac{1}{(n+1)^2}$$

$$\ge \frac{1}{n} - \frac{1}{n+1} - \frac{1}{(n+1)^2} \qquad \left[\because 1 + \frac{1}{2} + \ldots\ldots + \frac{1}{n} \ge 1\right]$$

$$= \frac{1}{n} - \frac{n+2}{(n+1)^2} = \frac{(n+1)^2 - n(n+2)}{n(n+1)^2} = \frac{1}{n(n+1)^2} > 0$$

$\Rightarrow$ $b_n > b_{n+1}$ $\forall$ n $\Rightarrow$ $< b_n >$ is a decreasing sequence.

Since $\lim_{n\to\infty} \frac{1}{n} = 0$

$\therefore$ **By Cauchy's first theorem on limits,** $\lim_{n\to\infty} \dfrac{1 + \frac{1}{2} + \ldots\ldots + \frac{1}{n}}{n} = 0 \Rightarrow \lim_{n\to\infty} b_n = 0$

$\therefore$ By Dirichlet's test, the series $\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} \frac{1}{n}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots+\frac{1}{n}\right) \sin(n\theta+\alpha)$ is convergent for $\theta \neq 2r\pi$, $r \in Z$.

When $\theta = 2r\pi$, $r \in Z$, $\sin(n\theta+\alpha) = \sin(2r\pi+\alpha) = \sin\alpha$.

The series reduces to $\sum_{n=1}^{\infty} \frac{1}{n}\left(1+\frac{1}{2}+\ldots\ldots+\frac{1}{n}\right)\sin\alpha$ which is convergent if α is a multiple of π (because, then each term is 0).

If α is not a multiple of π, the series reduces to $\sin\alpha \sum_{n=1}^{\infty} \frac{1}{n}\left(1+\frac{1}{2}+\ldots\ldots+\frac{1}{n}\right)$

Since $\frac{1}{n}\left(1+\frac{1}{2}+\ldots\ldots+\frac{1}{n}\right) \geq \frac{1}{n}$ and $\sum \frac{1}{n}$ is divergent, therefore, the given series is divergent.

Example 14. *Show that the following series are convergent :*

(*i*) $\sum_{n=1}^{\infty} \frac{(n\,!)^2}{2n\,!} \cos nx$ (*ii*) $\sum_{n=1}^{\infty} \frac{(n\,!)^2}{2n\,!} \sin nx.$

Sol. (*i*) Let $a_n = \cos nx$, $b_n = \frac{(n\,!)^2}{2n\,!}$.

The series $\sum_{n=1}^{\infty} a_n$ has bounded partial sums. (See Example 8)

Now $b_n = \frac{(n\,!)^2}{2n\,!}$, $b_{n+1} = \frac{[(n+1)\,!]^2}{(2n+2)\,!}$

$$\therefore \quad \frac{b_n}{b_{n+1}} = \frac{(2n+2)\,!}{2n\,!} \cdot \frac{(n\,!)^2}{[(n+1)\,!]^2} = \frac{(2n+2)(2n+1)\,2n\,!}{2n\,!} \cdot \left[\frac{n\,!}{(n+1)n\,!}\right]^2$$

$$= \frac{2(n+1)(2n+1)}{(n+1)^2} = \frac{2(2n+1)}{n+1} = \frac{(n+1)+(3n+1)}{n+1} = 1 + \frac{3n+1}{n+1} > 1 \;\forall\, n$$

$\Rightarrow$ $b_n > b_{n+1} \;\forall\, n \;\Rightarrow\; \langle b_n \rangle$ is monotonically decreasing.

Also $$\frac{b_{n+1}}{b_n} = \frac{n+1}{2(2n+1)} = \frac{1+\frac{1}{n}}{2\left(2+\frac{1}{n}\right)} \;\Rightarrow\; \lim_{n\to\infty} \frac{b_{n+1}}{b_n} = \frac{1}{4} \text{ and } \left|\frac{1}{4}\right| < 1$$

$\therefore$ $\langle b_n \rangle$ converges to 0.

$\left[\because \text{ If } \langle x_n \rangle \text{ is a sequence such that } \frac{x_{n+1}}{x_n} \to l \text{ where } |l| < 1, \text{ then } x_n \to 0\right]$

Since $\sum_{n=1}^{\infty} a_n$ has bounded partial sums and $\langle b_n \rangle$ is a decreasing sequence converging to zero, therefore, by Dirichlet's test, the series $\sum_{n=1}^{\infty} a_n b_n$ is convergent.

(*ii*) *Please try yourself.*

8.5. INSERTION AND REMOVAL OF PARENTHESES

Definition. Let $\sum_{n=1}^{\infty} a_n$ be an infinite series and let $\langle m_n \rangle$ be a strictly increasing sequence of positive integers so that $m_1 < m_2 < m_3 < \ldots\ldots$ Suppose

$$b_1 = a_1 + a_2 + \ldots\ldots + a_{m_1}$$
$$b_2 = a_{m_1+1} + a_{m_1+2} + \ldots\ldots + a_{m_2}$$
$$b_3 = a_{m_2+1} + a_{m_2+2} + \ldots\ldots + a_{m_3}$$
$$\vdots$$

Then the series $\sum_{n=1}^{\infty} b_n$ is said to have been obtained from $\sum_{n=1}^{\infty} a_n$ by a grouping of its terms (or by inserting parentheses). Obviously, $\sum_{n=1}^{\infty} a_n$ is obtained from $\sum_{n=1}^{\infty} b_n$ by removal of parentheses.

For example, $(a_1 + a_2) + (a_3 + a_4 + a_5 + a_6) + (a_7) + \ldots\ldots$ is a series obtained by a grouping of terms of the series $a_1 + a_2 + a_3 + \ldots\ldots$

In an infinite series, parentheses cannot be inserted or removed at will because it can affect the convergence of the series. However, under certain conditions, parentheses can be inserted or removed without affecting the convergence of the series.

For example, when we group the terms of the oscillatory series

$$\sum_{n=1}^{\infty} (-1)^{n-1} = 1 - 1 + 1 - 1 + \ldots\ldots \text{ as } (1-1) + (1-1) + (1-1) + \ldots\ldots$$

we get a convergent series.

8.6. INSERTION OF PARENTHESES

Theorem. *Any series obtained from a convergent series by a grouping of its terms converges and has the same sum as the original series.*

Proof. Let $\sum_{n=1}^{\infty} b_n$ be a series obtained from the series $\sum_{n=1}^{\infty} a_n$ by a grouping of its terms.

Let $\langle S_n \rangle$ be the sequence of partial sums of $\sum_{n=1}^{\infty} a_n$ and $\langle T_n \rangle$ be the sequence of partial sums of $\sum_{n=1}^{\infty} b_n$. Clearly, $\langle T_n \rangle$ is a subsequence of $\langle S_n \rangle$. Hence, if $\langle S_n \rangle$ converges to S, then $\langle T_n \rangle$ also converges to S.

$\Rightarrow$ If $\sum_{n=1}^{\infty} a_n$ converges and has sum S, then $\sum_{n=1}^{\infty} b_n$ also converges and has the same sum S.

8.7. REMOVAL OF PARENTHESES

Theorem I. *If the series $\sum_{n=1}^{\infty} b_n$ with parentheses converges to S and if $\sum_{n=1}^{\infty} a_n$ obtained from it by the removal of parentheses also converges, then $\sum_{n=1}^{\infty} a_n$ also converges to S.*

Proof. Suppose $\sum_{n=1}^{\infty} a_n$ converges to S′.

Let $< S_n >$ be the sequence of partial sums of Σa_n and $< T_n >$ be the sequence of partial sums of Σb_n.

Since Σa_n is obtained from Σb_n by the removal of parentheses, therefore, Σb_n is obtained from Σa_n by a grouping of its terms.

Clearly, $< T_n >$ is a subsequence of $< S_n >$. Hence, if $< S_n >$ converges to S′, then $< T_n >$ must also converge to S′. But $< T_n >$ converges to S.

$\therefore$ $S' = S$.

Note. From theorem with 8.6 and Theorem 1 with 8.7, we conclude that if Σa_n without parentheses and Σb_n with parentheses are both convergent, then they have same sum.

Theorem II. (*Without Proof*)

If the series $\sum_{n=1}^{\infty} b_n$ with parentheses converges to S and if the sum of the absolute values of the terms in b_n tends to zero as n tends to infinity, then the series $\sum_{n=1}^{\infty} a_n$ obtained on removing the parentheses, will also converge to S.

Note. If $b_n = a_{m_{n-1}+1} + a_{m_{n-1}+2} + \ldots\ldots + a_{m_n}$

then the sum of the absolute values of the terms in $b_n = |a_{m_{n-1}+1}| + |a_{m_{n-1}+2}| + \ldots\ldots + |a_{m_n}|$.

Example 1. *Show that the series* $\left(1-\frac{1}{2}\right)+\left(1-\frac{3}{4}\right)+\left(1-\frac{7}{8}\right)+\ldots\ldots$ *is convergent, but when the parentheses are removed, it oscillates.*

Sol. The given series with parentheses is

$$\sum_{n=1}^{\infty} b_n = \left(1-\frac{1}{2}\right)+\left(1-\frac{3}{4}\right)+\left(1-\frac{7}{8}\right)+\ldots\ldots = \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots\ldots$$

which is convergent, being a geometric series with common ratio $\frac{1}{2} < 1$.

Σb_n converges to the value $\dfrac{\frac{1}{2}}{1-\frac{1}{2}} = 1$.

Now $\quad b_n = 1 - \dfrac{2^n - 1}{2^n}$.

Sum of the absolute values of the terms in

$$b_n = |1| + \left|-\frac{2^n-1}{2^n}\right| = 1 + \frac{2^n-1}{2^n} = 2 - \frac{1}{2^n}$$

which tends to 2 (and not 0) as $n \to \infty$.

$\therefore$ The series $\sum_{n=1}^{\infty} a_n = 1 - \frac{1}{2} + 1 - \frac{3}{4} + 1 - \frac{7}{8} + \ldots\ldots$

obtained on removing the parentheses in $\sum_{n=1}^{\infty} b_n$, is not convergent.

Now, let S_n denote the nth partial sum of the series Σa_n, then

$$S_{2n} = 1 - \frac{1}{2} + 1 - \frac{3}{4} + 1 - \frac{7}{8} + \ldots\ldots \text{ to } 2n \text{ terms}$$

$$= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\ldots \text{ to } n \text{ terms} \qquad \text{(by adding in pairs)}$$

$$= \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$$

$$S_{2n+1} = S_{2n} + a_{2n+1} = 1 + 1 = 2$$

so that $\quad \lim_{n\to\infty} S_{2n} = 1$ and $\lim_{n\to\infty} S_{2n+1} = 2$.

$\therefore$ The series Σa_n is oscillatory.

Note. In the above example, the removal of parentheses affects the convergence of the given series.

Example 2. *Can the brackets be removed from the series*

$\left(1+\frac{1}{3}-\frac{1}{2}\right)+\left(\frac{1}{5}+\frac{1}{7}-\frac{1}{4}\right)+\left(\frac{1}{9}+\frac{1}{11}-\frac{1}{6}\right)+\ldots\ldots$ *without affecting its convergence?*

Sol. The given series with parentheses is

$$\sum_{n=1}^{\infty} b_n = \left(1+\frac{1}{3}-\frac{1}{2}\right)+\left(\frac{1}{5}+\frac{1}{7}-\frac{1}{4}\right)+\left(\frac{1}{9}+\frac{1}{11}-\frac{1}{6}\right)+\ldots$$

Here $\quad b_n = \dfrac{1}{4n-3} + \dfrac{1}{4n-1} - \dfrac{1}{2n}$

so that $\quad b_{n+1} = \dfrac{1}{4n+1} + \dfrac{1}{4n+3} - \dfrac{1}{2n+2}$

$$< \frac{1}{4n} + \frac{1}{4n} - \frac{1}{2(n+1)} < \frac{1}{2n} - \frac{1}{2(n+1)} = \frac{1}{2n(n+1)} < \frac{1}{2n^2}$$

Since the series $\frac{1}{2}\sum \frac{1}{n^2}$ is convergence.

$\therefore$ The series Σb_n is convergent.

Now $\quad b_{n+1} = \dfrac{1}{4n+1} + \dfrac{1}{4n+3} - \dfrac{1}{2n+2}$

Sum of absolute values of the terms in b_{n+1}

$$=\left|\frac{1}{4n+1}\right|+\left|\frac{1}{4n+3}\right|+\left|-\frac{1}{2n+2}\right|=\frac{1}{4n+1}+\frac{1}{4n+3}+\frac{1}{2n+2}$$

which tends to zero as $n\to\infty$.

$\therefore$ The series $\sum_{n=1}^{\infty} a_n = 1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\ldots$ obtained on removing the parentheses in $\sum_{n=1}^{\infty} b_n$ is also convergent.

Hence the brackets can be removed from the given series without affecting its convergence.

Example 3. *Show that the series* $\left(3-2\frac{1}{2}\right)+\left(2\frac{1}{3}-2\frac{1}{4}\right)+\left(2\frac{1}{5}-2\frac{1}{6}\right)+\ldots\ldots$ *is convergent but when the parentheses are removed, it does not converge.*

Sol. The given series with parentheses is

$$\sum_{n=1}^{\infty} b_n=\left(3-2\frac{1}{2}\right)+\left(2\frac{1}{3}-2\frac{1}{4}\right)+\left(2\frac{1}{5}-2\frac{1}{6}\right)+\ldots\ldots$$

$$=\left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)+\ldots\ldots=\frac{1}{1.2}+\frac{1}{3.4}+\frac{1}{5.6}+\ldots$$

Here $$b_n=\frac{1}{(2n-1)(2n)}=\frac{1}{4n^2\left(1-\frac{1}{2n}\right)}$$

Taking $c_n=\frac{1}{n^2}$, $\frac{b_n}{c_n}\to\frac{1}{4}$ as $n\to\infty$.

$\therefore$ By comparison test, Σb_n is convergent since Σc_n is convergent.

When the parentheses are removed, we get

$$3-2\frac{1}{2}+2\frac{1}{3}-2\frac{1}{4}+\ldots\ldots=\sum_{n=1}^{\infty}(-1)^{n-1}\left(2+\frac{1}{n}\right)=\sum_{n=1}^{\infty}(-1)^{n-1}a_n$$

which is an alternating series.

Since $a_n=2+\frac{1}{n}\to 2$ (and not 0) as $n\to\infty$, therefore, this series $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ is not convergent.

8.8. REARRANGEMENT OF TERMS

A series $\sum_{n=1}^{\infty} b_n$ is said to arise from a series $\sum_{n=1}^{\infty} a_n$ by a rearrangement of terms if there exists a one-to-one correspondence between the terms of the two series so that every a_n is some b_m and conversely.

For example, the series $1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\ldots\ldots$

is a rearrangement of the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\ldots\ldots$

If we add finitely many numbers, their sum has the same value, no matter how the terms of the sum are arranged. But this is not so when infinite series are involved. An *arrangement* (or equally well *derangement*) or change in the order of the terms in an infinite series may not only alter the sum but may change its nature all together.

The following theorem gives the condition under which we may rearrange the terms of the series without altering its sum.

Theorem 1. (Dirichlet's Theorem)

A series obtained from an absolutely convergent series by a rearrangement of terms converges absolutely and has the same sum as the original series.

Proof. We shall prove the theorem in two parts, first for series of positive terms and then for series of arbitrary terms.

Case 1. Let $\sum_{n=1}^{\infty} a_n$ be a series of positive terms converging to S.

Let $\sum_{n=1}^{\infty} b_n$ be any rearrangement of $\sum_{n=1}^{\infty} a_n$. Let $<S_n>$ and $<T_n>$ be the sequences of partial sums of $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ respectively.

Let $$b_1=a_{n_1}, b_2=a_{n_2}, \ldots\ldots, b_m=a_{n_m}$$

If $M=\max\{n_1, n_2, \ldots\ldots, n_m\}$, then S_M contains all the first m terms (and possibly some more) of $\sum_{n=1}^{\infty} b_n$.

$$\therefore \qquad T_m \le S_M \le S \quad \text{where} \quad \lim_{n\to\infty} S_n = S.$$

$\Rightarrow$ The sequence $<T_n>$ is bounded above by S.

$\Rightarrow$ The sequence $<T_n>$ and hence the series $\sum_{n=1}^{\infty} b_n$ converges.

If $\sum_{n=1}^{\infty} b_n$ converges to σ, then $\sigma\le S$.

Since $\sum_{n=1}^{\infty} a_n$ can also be thought of as a rearrangement of $\sum_{n=1}^{\infty} b_n$, we can similarly show that $S\le\sigma$.

Hence $\sigma = S$ and the two series converge to the same sum.

Case 2. Let $\sum_{n=1}^{\infty} a_n$ be an absolutely convergent series of arbitrary terms.

Let $\sum_{n=1}^{\infty} b_n$ be any rearrangement of $\sum_{n=1}^{\infty} a_n$.

Let
$$u_n = \begin{cases} a_n, & \text{if } a_n > 0 \\ 0, & \text{if } a_n \le 0 \end{cases}$$
$$v_n = \begin{cases} -a_n, & \text{if } a_n \le 0 \\ 0, & \text{if } a_n > 0 \end{cases}$$
$$u_n' = \begin{cases} b_n, & \text{if } b_n > 0 \\ 0, & \text{if } b_n \le 0 \end{cases}$$
$$v_n' = \begin{cases} -b_n, & \text{if } b_n \le 0 \\ 0, & \text{if } b_n > 0 \end{cases}$$

Then cleary u_n, v_n, u_n', v_n' are non-negative and

$$\left.\begin{aligned} a_n &= u_n - v_n, \quad | a_n | = u_n + v_n \\ b_n &= u_n' - v_n', \quad | b_n | = u_n' + v_n' \end{aligned}\right\} \quad ...(1)$$

$$\Rightarrow \quad \left.\begin{aligned} u_n &= \tfrac{1}{2}(| a_n | + a_n) \\ v_n &= \tfrac{1}{2}(| a_n | - a_n) \end{aligned}\right\} \quad ...(2)$$

and
$$\left.\begin{aligned} u_n' &= \tfrac{1}{2}(| b_n | + b_n) \\ v_n' &= \tfrac{1}{2}(| b_n | - b_n) \end{aligned}\right\} \quad ...(3)$$

Now $\sum_{n=1}^{\infty} a_n$ is absolutely convergent. $\Rightarrow \sum_{n=1}^{\infty} | a_n |$ and $\sum_{n=1}^{\infty} a_n$ are convergent.

$\therefore$ From (2), $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ are convergent series of positive terms and $\sum_{n=1}^{\infty} u_n'$ and $\sum_{n=1}^{\infty} v_n'$ are rearrangements of these series respectively.

$\therefore$ By Case 1, $\sum_{n=1}^{\infty} u_n'$ and $\sum_{n=1}^{\infty} v_n'$ converge and have same sum as $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ respectively.

If u, v, u', v' denote the sums of the series $\sum_{n=1}^{\infty} u_n, \sum_{n=1}^{\infty} v_n, \sum_{n=1}^{\infty} u_n'$ and $\sum_{n=1}^{\infty} v_n'$ respectively, then $u = u'$ and $v = v'$.

$\therefore$ From (1), it follows that $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} | b_n |$ are convergent and

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} u_n' - \sum_{n=1}^{\infty} v_n' = u' - v' = u - v = \sum_{n=1}^{\infty} u_n - \sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} a_n$$

$$\sum_{n=1}^{\infty} | b_n | = \sum_{n=1}^{\infty} u_n' + \sum_{n=1}^{\infty} v_n' = u' + v' = u + v = \sum_{n=1}^{\infty} u_n + \sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} | a_n |$$

Hence $\sum_{n=1}^{\infty} b_n$ converges absolutely and has the same sum as the original series $\sum_{n=1}^{\infty} a_n$.

Remarks 1. A series of positive terms, if convergent, has a sum independent of the order of its terms, but if divergent, it remains divergent however its terms may be rearranged.

2. Parentheses may be inserted or removed or terms rearranged in any order without changing the behaviour of a positive term series or an absolutely convergent series.

3. Since an absolutely convergent series remains convergent with unaltered sum, under any rearrangement of terms, it is also called unconditionally convergent.

Theorem 2. (Riemann's Theorem)

By a suitable rearrangement of the terms, a conditionally convergent series $\sum_{n=1}^{\infty} a_n$ *can be made*

(i) *to converge to any pre-assigned under* α, *or*

(ii) *to diverge to* ∞ *or* $-\infty$, *or*

(iii) *to oscillate finitely or infinitely.*

Proof. Let $u_1, u_2, u_3, ...$ be the positive terms and $-v_1, -v_2, -v_3, ...$ be the negative terms of $\sum_{n=1}^{\infty} a_n$.

For $n \in$ N, we define $\quad p_n = \frac{1}{2}(a_n + | a_n |) \quad$ and $\quad q_n = \frac{1}{2}(a_n - | a_n |)$

so that $\quad a_n = p_n$ if $a_n > 0 \quad$ and $\quad a_n = q_n$ if $a_n < 0$

$\Rightarrow p_n$ is the nth positive term and q_n is the nth negative term of $\sum_{n=1}^{\infty} a_n \Rightarrow p_n = u_n$ and $q_n = -v_n$.

Since $\sum_{n=1}^{\infty} a_n$ converges conditionally.

$\therefore \sum_{n=1}^{\infty} a_n$ is convergent and $\sum_{n=1}^{\infty} | a_n |$ is divergent.

$\Rightarrow \sum_{n=1}^{\infty} p_n$ and $\sum_{n=1}^{\infty} q_n$ are both divergent, $\sum_{n=1}^{\infty} p_n$ diverges to ∞ and $\sum_{n=1}^{\infty} q_n$ diverges to $-\infty$.

Also $\sum_{n=1}^{\infty} a_n$ converges $\Rightarrow \lim_{n\to\infty} a_n = 0 \Rightarrow \lim_{n\to\infty} | a_n = 0$ so that $\lim_{n\to\infty} p_n = 0$

and $$\lim_{n\to\infty} q_n = 0.$$

Let S_n and S_n' denote the nth partial sums of $\sum_{n=1}^{\infty} p_n$ and $\sum_{n=1}^{\infty} q_n$.

(i) We shall now construct strictly increasing sequences $< m_n >$ and $< k_n >$ of positive integers such that

$$p_1 + p_2 + ... + p_{m_1} + q_1 + q_2 + ... + q_{k_1} + p_{m_1+1} + p_{m_1+2} +$$
$$...... + p_{m_2} + q_{k_1+1} + q_{k_1+2} + ... + q_{k_2} + ... \quad ...(1)$$

is a rearrangement of $\sum_{n=1}^{\infty} a_n$, converging to α.

Since $\sum_{n=1}^{\infty} p_n$ diverges to ∞, it is always possible to find a partial sum of $\sum_{n=1}^{\infty} p_n$ which exceeds any pre-assigned number.

Also $\sum_{n=1}^{\infty} q_n$ diverges to $-\infty$, therefore, it is possible to find a partial sum of $\sum_{n=1}^{\infty} q_n$ which falls short of any pre-assigned number.

Let m_1 be the smallest positive integer such that the sum of first m_1 terms of $\sum_{n=1}^{\infty} p_n$ exceeds α.

Then $\quad p_1 + p_2 + \ldots + p_{m_1} > \alpha$

but $\quad p_1 + p_2 + \ldots + p_{m_1 - 1} < \alpha$ so that $S_{m_1 - 1} < \alpha < S_{m_1}$

Let k_1 be the smallest positive integer such that

$q_1 + q_2 + \ldots + q_{k_1}$ falls short of $\alpha - p_1 - p_2 \ldots - p_{m_1} = \alpha - S_{m_1}$

Then $\quad q_1 + q_2 + \ldots + q_{k_1} < \alpha - S_{m_1}$ and $q_1 + q_2 + \ldots + q_{k_1 - 1} > \alpha - S_{m_1}$

$\Rightarrow \quad S_{m_1} + S'_{k_1} < \alpha < S_{m_1} + S'_{k_1 - 1}$.

Let m_2 be the smallest positive integer such that $m_2 > m_1$ and the sum

$$p_1 + p_2 + \ldots + p_{m_1} + p_{m_1 + 1} + \ldots + p_{m_2} + q_1 + q_2 + \ldots + q_{k_1} \text{ exceeds } \alpha.$$

Then $\quad S_{m_2 - 1} + S'_{k_1} < \alpha < S_{m_2} + S'_{k_1}$

Let k_2 be the smallest positive integer such that $k_2 > k_1$ and the sum

$$q_1 + q_2 + \ldots + q_{k_1} + q_{k_1 + 1} + \ldots + q_{k_2} < \alpha - S_{m_2}$$

then $\quad S'_{k_2} < \alpha - S_{m_2} < S'_{m_{2-1}} \Rightarrow S_{m_2} + S'_{k_2} < \alpha < S_{m_2} + S'_{k_{2-1}}$

and so on. The process can be continued indefinitely because of the divergence of the two series

$$\sum_{n=1}^{\infty} p_n \quad \text{and} \quad \sum_{n=1}^{\infty} q_n.$$

Let $\sum_{n=1}^{\infty} b_n$ be the new series so constructed and σ_n be its nth partial sum.

The last term in σ_n will be either p_{m_r} or q_{k_s}. If the last term is p_{m_r}, then $\sigma_n - p_{m_r} < \alpha$, *i.e.*, $\sigma_n - \alpha < p_{m_r}$ and if the last term is q_{k_s}, then $\sigma_n - q_{k_s} > \alpha$ *i.e.*, $\sigma_n - \alpha > q_{k_s}$.

Since p_{m_r} is some positive term of $\sum_{n=1}^{\infty} a_n$ and q_{k_s} is some negative term of $\sum_{n=1}^{\infty} a_n$, therefore, if the last term in σ_n is a_l, then $|\sigma_n - \alpha| < |a_l|$

Since $\sum_{n=1}^{\infty} a_n$ is convergent, $a_l \to 0$ as $l \to \infty$

$\Rightarrow \quad |a_l| \to 0$ as $l \to \infty \Rightarrow \sigma_n - \alpha \to 0$ as $n \to \infty \Rightarrow \lim_{n \to \infty} \sigma_n = \alpha$

Hence $\sum_{n=1}^{\infty} b_n$ converges to α.

(*ii*) Now, we show that a rearrangement of $\sum_{n=1}^{\infty} a_n$ can be found which diverges to ∞.

Choose a positive integer m_1 such that

$$p_1 + p_2 + \ldots + p_{m_1} > 1 - q_1 \quad i.e., \quad p_1 + p_2 + \ldots + p_{m_1} + q_1 > 1$$

Now choose a positive integer $m_2 > m_1$ such that

$$(p_1 + p_2 + \ldots + p_{m_1}) + (p_{m_1+1} + p_{m_1+2} + \ldots + p_{m_2}) > 2 - q_1 - q_2$$

i.e., $$(p_1 + p_2 + \ldots + p_{m_1} + q_1) + (p_{m_1+1} + p_{m_1+2} + \ldots + p_{m_2} + q_2) > 2$$

Proceeding in this manner, we have

$$(p_1 + p_2 + \ldots + p_{m_1} + q_1) + (p_{m_1+1} + p_{m_1+2} + \ldots + p_{m_2} + q_2) + \ldots + (p_{m_{k-1}+1} + p_{m_{k-1}+2} + \ldots + p_{m_k} + q_k) > k$$

where k is a positive integer, however large.

Let $\sum_{n=1}^{\infty} b_n$ be the new series so constructed. Then the sequence of its partial sums is unbounded above and diverges to ∞.

Hence $\sum_{n=1}^{\infty} b_n$ diverges to ∞.

If we want to make the series diverge to $-\infty$, we choose a positive integer m_1 such that

$$q_1 + q_2 + \ldots + q_{m_1} < -1 - p_1 \quad i.e., \quad q_1 + q_2 + \ldots + q_{m_1} + p_1 < -1$$

Now we choose a positive integer $m_2 > m_1$ such that

$$(q_1 + q_2 + \ldots + q_{m_1}) + (q_{m_1+1} + q_{m_1+2} + \ldots + q_{m_2}) < -2 - p_1 - p_2$$

i.e., $$(q_1 + q_2 + \ldots + q_{m_1} + p_1) + (q_{m_1+1} + q_{m_1+2} + \ldots + q_{m_2} + p_2) < -2$$

Proceeding in this manner, we have

$$(q_1 + q_2 + \ldots + q_{m_1} + p_1) + (q_{m_1+1} + q_{m_1+2} + \ldots + q_{m_2} + p_2) + \ldots + (q_{m_{k-1}+1} + q_{m_{k-1}+2} + \ldots + q_{m_k} + p_k) < -k$$

where k is a positive integer, however large.

Let $\sum_{n=1}^{\infty} b_n$ be the new series so constructed. Then the sequence of its partial sums is unbounded below and diverges to $-\infty$.

Hence $\sum_{n=1}^{\infty} b_n$ diverges to $-\infty$.

(*iii*) Now we show that a rearrangement of $\sum_{n=1}^{\infty} a_n$ can be found which oscillates between two number α and β.

Take just sufficient number of positive terms so that the sum is greater than α and then take just sufficient number of negative terms so that the sum is less than β. Repeat the process indefinitely. The new series so formed will oscillate between $\beta - 1$ and $\alpha + 1$ finitely or infinitely.

Example 1. *Explain the fallacy in the following :*

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\ldots\ldots=(2-1)-\frac{1}{2}+\left(\frac{2}{3}-\frac{1}{3}\right)-\frac{1}{4}+\left(\frac{2}{5}-\frac{1}{5}\right)-\frac{1}{6}+\ldots..$$

$$=2-1-\frac{1}{2}+\frac{2}{3}-\frac{1}{3}-\frac{1}{4}+\frac{2}{5}-\frac{1}{5}-\frac{1}{6}+\ldots..$$

$$=2-1+\frac{2}{3}-\frac{1}{2}+\frac{2}{5}-\frac{1}{3}+\frac{2}{7}-\frac{1}{4}+\ldots\ldots$$

$$=2\left[1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\ldots\ldots\right] \qquad \ldots(1)$$

Also we know that $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots\ldots$ *converges and that its sum S is log 2 which is different from zero. Therefore, from (1), we have* $S = 2S$ *or* $1 = 2$.

Sol. The series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots..$ is conditionally convergent and, therefore, by Riemann's theorem, rearrangement of the terms may alter the sum of the series.

Thus $\quad 2-1-\frac{1}{2}+\frac{2}{3}-\frac{1}{3}-\frac{1}{4}+\frac{2}{5}-\frac{1}{5}-\frac{1}{6}+\ldots\ldots$

may not be equal to $\quad 2-1+\frac{2}{3}-\frac{1}{2}+\frac{2}{5}-\frac{1}{3}+\frac{2}{7}-\frac{1}{4}+\ldots\ldots$

Example 2. *Criticise the following paradox.*

$$\log 2 = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\ldots\ldots$$

$$=\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots\ldots\right)-2\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{10}+\ldots..\right)$$

$$=\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots\ldots\right)-\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots\ldots\right)=0.$$

Sol. The given series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots\ldots=\sum_{n=1}^{\infty}(-1)^{n-1}\cdot\frac{1}{n}$ is conditionally convergent and hence can be made to converge to any limit by a rearrangement of terms (Riemann's Theorem). Hence we are not justified in rearranging the terms of a conditionally convergent series and expecting the same sum.

Example 3. *What is wrong with the following ?*

$$2+2+2+2+2+2+\ldots..=(2+2)+(2+2)+(2+2)+\ldots..$$

$$=4+4+4+\ldots..=2(2+2+2+\ldots..)$$

$$\therefore \quad 2+2+2+\ldots..=0.$$

Sol. The series $2 + 2 + 2 + \ldots\ldots$ is divergent and tends to infinity, so that

$$2(2+2+2+\ldots\ldots)-(2+2+2+\ldots..)$$

is of indeterminate form $\infty - \infty$.

On account of this fallacy, we get an absurd result.

☞[**Note.** Riemann's method is of theoretical importance only. For practical applications, the method given by Pringsheim is useful.

Pringsheim's Method (*Without Proof*)

Let $f(n)$ be a positive function decreasing to zero as $n \to \infty$. Then by Leibnitz's test, the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1} f(n)$ is convergent.

Let the terms of the series $\sum_{n=1}^{\infty}(-1)^{n-1} f(n)$ be rearranged by taking alternately α positive and β negative terms.

If $g = mf(m)$ and $k = \frac{\alpha}{\beta}$ then the alteration in the sum due to this rearrangement is $\frac{1}{2} g \log k$.

In particular, if $f(n) = \frac{1}{n}$ so that $\sum_{n=1}^{\infty}(-1)^{n-1} f(n) = \sum_{n=1}^{\infty}(-1)^{n-1}\cdot\frac{1}{n} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots..$

then we know that the series is conditionally convergent and its sum is log 2.

Also $\qquad g = mf(m) = m\cdot\frac{1}{m} = 1.$

$\therefore$ If the terms are rearranged by taking alternately α positive and β negative terms, then the sum of the new series is

$$\log 2 + \tfrac{1}{2}\log k = \tfrac{1}{2}(2\log 2 + \log k) = \tfrac{1}{2}\log(4k).]$$

Example 4. *Find the sum of the series :*

(*i*) $1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\frac{1}{5}-\ldots..$ $\qquad$ (*ii*) $1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\frac{1}{9}+\ldots..$

(*iii*) $1+\frac{1}{3}+\frac{1}{5}-\frac{1}{2}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\frac{1}{4}+\ldots..$ (*iv*) $1+\frac{1}{3}+\frac{1}{5}-\frac{1}{2}-\frac{1}{4}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\frac{1}{6}-\frac{1}{8}+\ldots..$

Sol. (*i*) The given series $1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\frac{1}{5}-\ldots..$ is a rearrangement of the terms of conditionally convergent series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\ldots..=\sum_{n=1}^{\infty}(-1)^{n-1}\cdot\frac{1}{n} \text{ whose sum is log 2.}$$

Here the rearranged given series is formed by taking alternately one positive and two negative terms so that $\qquad k = \frac{\alpha}{\beta} = \frac{1}{2}$.

$\therefore$ Sum of the rearranged given series

$$= \log 2 + \tfrac{1}{2}\log k = \log 2 + \tfrac{1}{2}\log\tfrac{1}{2} = \log 2 - \tfrac{1}{2}\log 2 = \tfrac{1}{2}\log 2.$$

(*ii*) The given series $1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\frac{1}{9}+\ldots$ is a rearrangement of the terms of conditionally convergent series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\ldots = \sum_{n=1}^{\infty}(-1)^{n-1}\cdot\frac{1}{n} \text{ whose sum is } \log 2.$$

Here the rearranged series is formed by taking alternately two positive and one negative terms so that

$$k=\frac{\alpha}{\beta}=\frac{2}{1}=2.$$

$\therefore$ Sum of the rearranged given series

$$=\log 2+\tfrac{1}{2}\log k=\log 2+\tfrac{1}{2}\log 2=\frac{3}{2}\log 2.$$

(*iii*) *Please try yourself.* **[Ans.** $\frac{1}{2}\log 12$**]**

(*iv*) *Please try yourself.* **[Ans.** $\log\sqrt{6}$**]**

Example 5. *Investigate what derangement of the series* $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}\ldots$ *will reduce its sum to zero.*

Sol. The given series is $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\ldots=\sum(-1)^{n-1}\cdot\frac{1}{n}$

It is conditionally convergent with sum $\log 2$.

Let it be deranged by taking alternately α positive and β negative terms so that $k=\frac{\alpha}{\beta}$.

Sum of the deranged series $=\log 2+\frac{1}{2}\log k$

But it is given to be zero.

$$\therefore \quad \log 2+\tfrac{1}{2}\log k=0 \;\Rightarrow\; \log k=-2\log 2=\log 2^{-2}=\log\tfrac{1}{4} \;\Rightarrow\; k=\frac{\alpha}{\beta}=\frac{1}{4}$$

Hence to get the sum zero, one positive term should be followed by four negative terms.

The deranged series is $1-\frac{1}{2}-\frac{1}{4}-\frac{1}{6}-\frac{1}{8}+\frac{1}{3}-\frac{1}{10}-\frac{1}{12}-\frac{1}{14}-\frac{1}{16}+\frac{1}{5}-\ldots$

Example 6. *What derangement of the series* $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\ldots$ *will reduce its sum to* $\frac{1}{2}\log 2$ *?*

Sol. Proceeding as in Example 5,

$$\log 2+\tfrac{1}{2}\log k=\tfrac{1}{2}\log 2 \quad\Rightarrow\quad \tfrac{1}{2}\log k=-\tfrac{1}{2}\log 2$$

$$\Rightarrow \quad \log k=\log 2^{-1}=\log\tfrac{1}{2} \;\Rightarrow\; k=\tfrac{1}{2}$$

$\therefore$ The deranged series is $1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\frac{1}{5}-\frac{1}{10}-\frac{1}{12}+\frac{1}{7}-\ldots$

Example 7. *Find how the series* $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\ldots$ *should be deranged so that the sum is doubled.*

Sol. *Please try yourself.*

$$\left[\textbf{Ans. } k=4, \text{ Deranged series is } 1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}-\frac{1}{2}+\frac{1}{9}+\frac{1}{11}+\frac{1}{13}+\frac{1}{15}-\frac{1}{4}+\frac{1}{17}-\ldots\right]$$

Example 8. *Assuming that* $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$, *find the value of* $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$.

Sol. The series $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots$

of positive terms is convergent and hence we can derange its terms in any order.

$$\text{Now} \quad \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots$$

$$=\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots\right)-\left(\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\ldots\right)$$

$$=\sum_{n=1}^{\infty}\frac{1}{n^2}-\frac{1}{2^2}\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\right)$$

$$=\sum_{n=1}^{\infty}\frac{1}{n^2}-\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{3}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{3}{4}\times\frac{\pi^2}{6}=\frac{\pi^2}{8}.$$

8.9. CAUCHY PRODUCT OF TWO INFINITE SERIES

If $\sum_{n=1}^{\infty}a_n$ and $\sum_{n=1}^{\infty}b_n$ are two infinite series, then their product, called the Cauchy product, is defined as $\sum_{n=1}^{\infty}c_n$, where $c_n=a_1b_n+a_2b_{n-1}+a_3b_{n-2}+\ldots+a_nb_1=\sum_{n=1}^{\infty}a_r\,b_{n-r+1}$ for each $n\in\mathbb{N}$.

$$\text{Thus} \quad \sum_{n=1}^{\infty}c_n=\left(\sum_{n=1}^{\infty}a_n\right)\left(\sum_{n=1}^{\infty}b_n\right)=(a_1+a_2+\ldots)(b_1+b_2+\ldots)$$

$$=a_1b_1+(a_1b_2+a_2b_1)+(a_1b_3+a_2b_2+a_3b_1)+\ldots=c_1+c_2+c_3+\ldots$$

The terms in the product are so arranged that all the terms which have the same sum of suffixes are bracketed together.

Remarks 1. The Cauchy product of $\sum_{n=0}^{\infty}a_n$ and $\sum_{n=0}^{\infty}b_n$ is defined as $\sum_{n=0}^{\infty}c_n$ where

$$c_n=a_0b_n+a_1b_{n-1}+a_2b_{n-2}+\ldots+a_nb_0=\sum_{r=0}^{\infty}a_rb_{n-r} \text{ for each } n\in\mathbb{N}.$$

2. $$c_n=\sum_{r=1}^{\infty}a_r\,b_{n-r+1}=\sum_{r=1}^{\infty}a_{n-r+1}\,b_r \quad\text{and}\quad c_n=\sum_{r=0}^{\infty}a_r\,b_{n-r}=\sum_{r=0}^{\infty}a_{n-r}b_r$$

3. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ convergent, then it is not necessary that $\sum_{n=1}^{\infty} c_n = \left(\sum_{n=1}^{\infty} a_n\right)\left(\sum_{n=1}^{\infty} b_n\right)$ must converge. $\sum_{n=1}^{\infty} c_n$ converges if

(*i*) $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series of non-negative terms, or

(*ii*) $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are absolutely convergent, or

(*iii*) $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent and one of them is absolutely convergent.

Now we prove these assertions.

8.10. THEOREM 1.

If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are two series of non-negative terms converging to A and B respectively, then their Cauchy product $\sum_{n=1}^{\infty} c_n$ converges to AB.

Proof. Let A_n, B_n, C_n denote the nth partial sums of the series —

$$\sum_{n=1}^{\infty} a_n, \sum_{n=1}^{\infty} b_n \quad \text{and} \quad \sum_{n=1}^{\infty} c_n = \left(\sum_{n=1}^{\infty} a_n\right)\left(\sum_{n=1}^{\infty} b_n\right) \text{ respectively.}$$

Since $\sum_{n=1}^{\infty} a_n$ converges to A and $\sum_{n=1}^{\infty} b_n$ converges to B.

$$\therefore \quad \lim_{n\to\infty} A_n = A \quad \text{and} \quad \lim_{n\to\infty} B_n = B$$

Now

$$\begin{aligned} C_n &= a_1b_1 \\ &\quad + a_1b_2 + a_2b_1 \\ &\quad + a_1b_3 + a_2b_2 + a_3b_1 \\ &\quad \vdots \\ &\quad + a_1b_n + a_2b_{n-1} + a_3b_{n-2} + \ldots + a_nb_1 \\ &= a_1(b_1 + b_2 + \ldots + b_n) \\ &\quad + a_2(b_1 + b_2 + \ldots + b_{n-1}) \\ &\quad + b_3(b_1 + b_2 + \ldots + b_{n-2}) \\ &\quad \vdots \\ &\quad + a_nb_1 \\ &= a_1B_n + a_2B_{n-1} + a_3B_{n-2} + \ldots + a_nB_1 \end{aligned}$$

$$\begin{aligned} A_nB_n &= (a_1 + a_2 + a_3 + \ldots + a_n)B_n \\ &= a_1B_n + a_2B_n + a_3B_n + \ldots + a_nB_n \end{aligned}$$

and

$$\begin{aligned} C_{2n} &= a_1b_1 \\ &\quad + a_1b_2 + a_2b_1 \\ &\quad + a_1b_3 + a_2b_2 + a_3b_1 \\ &\quad \vdots \\ &\quad + a_1b_{2n} + a_2b_{2n-1} + a_3b_{2n-2} + \ldots + a_{2n}b_1 \\ &= a_1(b_1 + b_2 + \ldots + b_{2n}) \\ &\quad + a_2(b_1 + b_2 + \ldots + b_{2n-1}) \\ &\quad + a_3(b_1 + b_2 + \ldots + b_{2n-2}) \\ &\quad \vdots \\ &\quad + a_{2n}b_1 \\ &= a_1B_{2n} + a_2B_{2n-1} + a_3B_{2n-2} + \ldots a_{2n}B_1 \end{aligned}$$

Since $b_n \geq 0 \ \forall \ n \in N$, therefore, $i > j \Rightarrow B_i \geq B_j$

Clearly $\quad 0 \leq C_n \leq A_nB_n \leq C_{2n}$ ...(1)

$\Rightarrow \ < C_n >$ is a bounded sequence of non-negative numbers.

Also $< C_n >$ is monotonically increasing.

$\therefore \ < C_n >$ is convergent. Let it converge to C.

Then $\lim_{n\to\infty} C_n = C$ and the series $\sum_{n=1}^{\infty} C_n$ also converges to C.

Now, from (1), we have $\lim_{n\to\infty} C_n \leq \lim_{n\to\infty} (A_nB_n) \leq \lim_{n\to\infty} C_{2n}$

$\Rightarrow \quad C \leq (\lim_{n\to\infty} A_n)(\lim_{n\to\infty} B_n) \leq C$

$\Rightarrow \quad C \leq AB \leq C \ \Rightarrow \ C = AB$

Hence the Cauchy product of $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converges to AB.

Remark. Note that in Theorem 1, $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series of **non-negative terms.**

Theorem 2. (Cauchy's Theorem)

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two absolutely convergent series such that $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$. Then their Cauchy product $\sum_{n=1}^{\infty} c_n$ is also absolutely convergent and $\sum_{n=1}^{\infty} c_n = AB$.

Proof. Let A_n, B_n be the nth partial sums of the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ respectively.

Since $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$

$\therefore$ $\lim_{n\to\infty} A_n = A$ and $\lim_{n\to\infty} B_n = B$

Let A_n', B_n' denote the nth partial sums of the series $\sum_{n=1}^{\infty} |a_n|$ and $\sum_{n=1}^{\infty} |b_n|$ respectively.

Since $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are absolutely convergent, therefore, $\sum_{n=1}^{\infty} |a_n|$ and $\sum_{n=1}^{\infty} |b_n|$ are convergent. Suppose they converge to A' and B' respectively.

Then $\lim_{n\to\infty} A_n' = A'$ and $\lim_{n\to\infty} B_n' = B'$.

Let the Cauchy product of $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be $\sum_{n=1}^{\infty} c_n$, then $c_n = a_1 b_n + a_2 b_{n-1} + \ldots + a_n b_1$

Let the Cauchy product of $\sum_{n=1}^{\infty} |a_n|$ and $\sum_{n=1}^{\infty} |b_n|$ be $\sum_{n=1}^{\infty} d_n$, then

$$d_n = |a_1||b_n| + |a_2||b_{n-1}| + \ldots + |a_n||b_1|.$$

Since $\sum_{n=1}^{\infty} |a_n|$ and $\sum_{n=1}^{\infty} |b_n|$ are series of **non-negative terms**, converging to A' and B' respectively, therefore, their Cauchy product $\sum_{n=1}^{\infty} d_n$ converges to $A'B'$.

Now
$$|c_n| = |a_1 b_n + a_2 b_{n-1} + \ldots + a_n b_1|$$
$$\le |a_1||b_n| + |a_2||b_{n-1}| + \ldots + |a_n||b_1| = d_n \ \forall\, n$$

and $\sum_{n=1}^{\infty} d_n$ is convergent.

$\Rightarrow \sum_{n=1}^{\infty} |c_n|$ is convergent (by comparison test) $\Rightarrow \sum_{n=1}^{\infty} c_n$ is absolutely convergent.

Now we shall prove that $\sum_{n=1}^{\infty} c_n = AB$.

Let C_n and D_n denote the nth partial sums of the series $\sum_{n=1}^{\infty} c_n$ and $\sum_{n=1}^{\infty} d_n$ respectively.

$$\begin{aligned}
C_n - A_n B_n &= (a_1 b_1 \\
&\quad + a_1 b_2 + a_2 b_1 \\
&\quad + a_1 b_3 + a_2 b_2 + a_3 b_1 \\
&\quad \vdots \\
&\quad + a_1 b_n + a_2 b_{n-1} + a_3 b_{n-2} + \ldots + a_n b_1) \\
&\quad - (a_1 + a_2 + \ldots + a_n)(b_1 + b_2 + \ldots + b_n) \\
&= a_1(b_1 + b_2 + \ldots + b_n) \\
&\quad + a_2(b_1 + b_2 + \ldots + b_{n-1}) \\
&\quad + a_3(b_1 + b_2 + \ldots + b_{n-2}) \\
&\quad \vdots \\
&\quad + a_n b_1 \\
&\quad - a_1(b_1 + b_2 + \ldots + b_n) \\
&\quad - a_2(b_1 + b_2 + \ldots + b_n) \\
&\quad - a_3(b_1 + b_2 + \ldots + b_n) \\
&\quad \vdots \\
&\quad - a_n(b_1 + b_2 + \ldots + b_n) \\
&= - a_2 b_n - a_3(b_{n-1} + b_n) - \ldots \\
&\quad - a_n(b_2 + b_3 + \ldots + b_n)
\end{aligned}$$

$$\Rightarrow \quad |C_n - A_n B_n| \le |a_2||b_n| + |a_3|(|b_{n-1}| + |b_n|) + \ldots + |a_n|(|b_2| + |b_3| + \ldots + |b_n|) = A_n'B_n' - D_n \qquad \ldots(1)$$

Now $\lim_{n\to\infty} A_n' = A'$, $\lim_{n\to\infty} B_n' = B'$ and $\lim_{n\to\infty} D_n = A'B'$

so that $A_n'B_n' - D_n \to 0$ as $n \to \infty$.

$\therefore$ From (1), $|C_n - A_n B_n| \to 0$ as $n \to \infty$

Also $|A_n B_n - AB| \to 0$ as $n \to \infty$

$\therefore$ $|C_n - AB| = |(C_n - A_n B_n) + (A_n B_n - AB)| \le |C_n - A_n B_n| + |A_n B_n - AB|$

$\Rightarrow$ $|C_n - AB| \to 0$ as $n \to \infty$ $\Rightarrow$ $C_n - AB \to 0$ as $n \to \infty$ $\Rightarrow$ $\lim_{n\to\infty} C_n = AB$

Hence $\sum_{n=1}^{\infty} c_n = AB$.

Remark. Note that in Cauchy's theorem, $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are both absolutely convergent.

Theorem 3. (Merten's Theorem)

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two convergent series and let $\sum_{n=1}^{\infty} a_n$ converge absolutely. If $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$, then their Cauchy product $\sum_{n=1}^{\infty} c_n$ converges to AB.

Proof. Let A_n, B_n, C_n denote the nth partial sums of the series $\sum_{n=1}^{\infty} a_n, \sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} c_n$ respectively.

Then $\lim_{n\to\infty} A_n = A$ and $\lim_{n\to\infty} B_n = B$

Let $\beta_n = B_n - B \ \forall\, n$ so that $\lim_{n\to\infty} \beta_n = \lim_{n\to\infty} B_n - B = B - B = 0$

Now

$$\begin{aligned}
C_n &= a_1 b_1 \\
&\quad + a_1 b_2 + a_2 b_1 \\
&\quad + a_1 b_3 + a_2 b_2 + a_3 b_1 \\
&\quad \vdots \\
&\quad \vdots \\
&\quad + a_1 b_n + a_2 b_{n-1} + a_3 b_{n-2} + \ldots\ldots + a_n b_1 \\
&= a_1(b_1 + b_2 + \ldots\ldots + b_n) \\
&\quad + a_2(b_1 + b_2 + \ldots\ldots + b_{n-1}) \\
&\quad \vdots \\
&\quad \vdots \\
&\quad + a_n b_1 \\
&= a_1 B_n + a_2 B_{n-1} + \ldots.. + a_n B_1 \\
&= a_1(B + \beta_n) + a_2(B + \beta_{n-1}) + \ldots.. + a_n(B + \beta_1) \quad [\because \ B_n = B + \beta_n \ \forall\, n] \\
&= B(a_1 + a_2 + \ldots\ldots + a_n) + (a_1\beta_n + a_2\beta_{n-1} + \ldots.. + a_n\beta_1) \\
&= BA_n + \gamma_n \text{ where } \gamma_n = a_1\beta_n + a_2\beta_{n-1} + \ldots. + a_n\beta_1
\end{aligned}$$

$\therefore$ To prove that $\lim_{n\to\infty} C_n = AB$, it is sufficient to prove that $\lim_{n\to\infty} \gamma_n = 0$.

Now $\sum_{n=1}^{\infty} a_n$ converges absolutely $\Rightarrow$ $\sum_{n=1}^{\infty} |a_n|$ converges.

Let $$\sum_{n=1}^{\infty} |a_n| = \alpha.$$

Since $\lim_{n\to\infty} \beta_n = 0$, $<\beta_n>$ converges.

$\Rightarrow$ $<\beta_n>$ is bounded. $\Rightarrow$ There exists a real number $K > 0$ such that $|\beta_n| < K \ \forall\, n$ ...(1)

Also $\lim_{n\to\infty} \beta_n = 0$ $\Rightarrow$ given $\varepsilon > 0$, there exists a positive integer p such that

$$|\beta_n| < \frac{\varepsilon}{2\alpha + 1} \ \forall\, n > p.$$

Since $\sum_{n=1}^{\infty} |a_n|$ converges, by Cauchy's general principle of convergence, there exists a positive integer q such that

$$\big||a_{q+1}| + |a_{q+2}| + \ldots.. + |a_n|\big| < \frac{\varepsilon}{2K+1} \ \forall\, n > q$$

or $$|a_{q+1}| + |a_{q+2}| + \ldots.. + |a_n| < \frac{\varepsilon}{2K+1} \ \forall\, n > q$$

If $m = \max\{p, q\}$, then for $n > m$, we have $|\beta_n| < \dfrac{\varepsilon}{2\alpha+1}$ ...(2)

and $$|a_{m+1}| + |a_{m+2}| + \ldots.. + |a_n| < \frac{\varepsilon}{2K+1} \quad \ldots(3)$$

$\therefore$ For $n > 2m$, we have $n - m > m$ and

$$\begin{aligned}
|\gamma_n| &= |a_1\beta_n + a_2\beta_{n-1} + \ldots.. + a_n\beta_1| \\
&= |\beta_1 a_n + \beta_n a_{n-1} + \ldots.. + \beta_{m+1} a_{n-m} + \beta_{m+2} a_{n-m-1} + \ldots.. + \beta_n a_1| \\
&\le (|\beta_1||a_n| + |\beta_2||a_{n-1}| + \ldots.. + |\beta_{m+1}||a_{n-m}|) \\
&\qquad + (|\beta_{m+2}||a_{n-m-1}| + \ldots.. + |\beta_n||a_1|) \\
&< K(|a_{n-m}| + \ldots\ldots + |a_{n-1}| + |a_n|) \\
&\qquad + \frac{\varepsilon}{2\alpha+1}(|a_1| + |a_2| + \ldots.. + |a_{n-m+1}|) \quad \text{[using (1) and (2)]} \\
&< K \cdot \frac{\varepsilon}{2K+1} + \frac{\varepsilon}{2\alpha+1} \cdot \alpha \quad \text{[using (3)]} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

$$\left[\begin{array}{l} \because \sum_{n=1}^{\infty} |a_n| = \alpha \\ \therefore \sum_{n=1}^{n-m+1} |a_n| < \alpha \end{array}\right]$$

$\Rightarrow$ Given $\varepsilon > 0$, there exists a positive integer $2m$ such that

$$|\gamma_n| < \varepsilon \ \forall\, n > 2m \Rightarrow \lim_{n\to\infty} \gamma_n = 0$$

Now $$C_n = BA_n + \gamma_n$$

$\therefore$ $$\lim_{n\to\infty} C_n = \lim_{n\to\infty} (BA_n + \gamma_n) = B \lim_{n\to\infty} A_n + \lim_{n\to\infty} \gamma_n = BA + 0 = AB.$$

Hence $\sum_{n=1}^{\infty} C_n$ converges to AB.

Remark. Note the in Merten's theorem, $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge and one of them converges absolutely.

Theorem 4. (Cesaro's Theorem)

If two sequences $<a_n>$ and $<b_n>$ converge to a and b respectively, then the sequence $<x_n>$ where $x_n = \dfrac{a_1 b_n + a_2 b_{n-1} + \ldots\ldots + a_n b_1}{n}$ converges to ab.

Proof. Let $a_n = a + \alpha_n \ \forall\, n \in N$

Then $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} a_n - a = a - a = 0 \Rightarrow \lim_{n\to\infty} |\alpha_n| = 0$

Now
$$\begin{aligned}
x_n &= \frac{a_1 b_n + a_2 b_{n-1} + \ldots\ldots + a_n b_1}{n} = \frac{(a+\alpha_1)b_n + (a+\alpha_2)b_{n-1} + \ldots\ldots + (a+\alpha_n)b_1}{n} \\
&= \frac{a(b_n + b_{n-1} + \ldots\ldots + b_1) + (\alpha_1 b_n + \alpha_2 b_{n-1} + \ldots\ldots + \alpha_n b_1)}{n}
\end{aligned}$$

$$= a\left(\frac{b_1 + b_2 + \ldots\ldots + b_n}{n}\right) + \left(\frac{b_n\alpha_1 + b_{n-1}\alpha_2 + \ldots\ldots + b_1\alpha_n}{n}\right) \quad \ldots(1)$$

By Cauchy's first theorem on limits

$$\lim_{n\to\infty} b_n = b \Rightarrow \lim_{n\to\infty} \frac{b_1 + b_2 + \ldots\ldots + b_n}{n} = b \quad \ldots(2)$$

and $$\lim_{n\to\infty} |\alpha_n| = 0 \Rightarrow \lim_{n\to\infty} \frac{|\alpha_1| + |\alpha_2| + \ldots\ldots + |\alpha_n|}{n} = 0$$

Since $< b_n >$ converges, $< b_n >$ is bounded.

$\therefore$ $\exists$ a real number $K > 0$ such that $|b_n| < K \; \forall \; n \in N$ ...(3)

Now $$0 \le \left|\frac{b_n\alpha_1 + b_{n-1}\alpha_2 + \ldots\ldots + b_1\alpha_n}{n}\right|$$

$$\le \frac{|b_n||\alpha_1| + |b_{n-1}||\alpha_2| + \ldots\ldots + |b_1||\alpha_n|}{n} \quad \text{[Using (3)]}$$

$$\le K\left(\frac{|\alpha_1| + |\alpha_2| + \ldots\ldots + |\alpha_n|}{n}\right) \to 0 \text{ as } n \to \infty \quad \text{[Using (2)]}$$

$\therefore$ By Squeeze principle, $\lim_{n\to\infty} \frac{b_n\alpha_1 + b_{n-1}\alpha_2 + \ldots\ldots + b_1\alpha_n}{n} = 0$

From (1), $$\lim_{n\to\infty} x_n = a \lim_{n\to\infty} \left(\frac{b_1 + b_2 + \ldots\ldots + b_n}{n}\right) + \lim_{n\to\infty} \frac{b_n\alpha_1 + b_{n-1}\alpha_2 + \ldots\ldots + b_1\alpha_n}{n}$$

$$= a(b) + 0 = ab$$

Hence the sequence $< x_n >$ converges to ab.

Theorem 5. (Abel's Test)

Let $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *be two convergent series such that* $\sum_{n=1}^{\infty} a_n = A$ *and* $\sum_{n=1}^{\infty} b_n = B$. *If their Cauchy product* $\sum_{n=1}^{\infty} c_n$ *converges, then* $\sum_{n=1}^{\infty} c_n = AB$.

Proof. Let A_n, B_n, C_n denote the nth partial sums of the series $\sum_{n=1}^{\infty} a_n, \sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} c_n$ respectively.

Now
$$\begin{aligned}
C_n &= a_1b_1 \\
&+ a_1b_2 + a_2b_1 \\
&+ a_1b_3 + a_2b_2 + a_3b_1 \\
&\vdots \\
&\vdots \\
&+ a_1b_n + a_2b_{n-1} + a_3b_{n-2} + \ldots\ldots + a_nb_1 \\
&= a_1(a_1 + b_2 + \ldots. + b_n) \\
&+ a_2(b_1 + b_2 + \ldots.. + b_{n-1}) \\
&+ a_3(b_1 + b_2 + \ldots.. + b_{n-2}) \\
&\vdots \\
&\vdots \\
&+ a_nb_1
\end{aligned}$$

$$= a_1B_n + a_2B_{n-1} + \ldots\ldots + a_nB_1$$

$$\Rightarrow \quad C_n = a_1B_n + a_2B_{n-1} + a_3B_{n-2} + \ldots\ldots + a_nB_1$$

Changing n to $n-1, n-2, \ldots\ldots, 1$ successively, we get

$$C_{n-1} = a_1B_{n-1} + a_2B_{n-2} + \ldots\ldots + a_{n-1}B_1$$
$$C_{n-2} = a_1B_{n-2} + a_2B_{n-3} + \ldots\ldots + a_{n-2}B_1$$
$$\vdots$$
$$C_1 = a_1B_1$$

Adding, we have

$$C_1 + C_2 + \ldots\ldots + C_n = a_1B_n + (a_1 + a_2)B_{n-1} + (a_1 + a_2 + a_3)B_{n-2} + \ldots\ldots + (a_1 + a_2 + \ldots\ldots + a_n)B_1$$
$$= A_1B_n + A_2B_{n-1} + A_3B_{n-2} + \ldots\ldots + A_nB_1 \quad (\because \; a_1 = A_1)$$

$$\Rightarrow \quad \frac{C_1 + C_2 + \ldots\ldots + C_n}{n} = \frac{A_1B_n + A_2B_{n-1} + \ldots\ldots + A_nB_1}{n} \quad \ldots(1)$$

Suppose $\sum_{n=1}^{\infty} c_n$ converges and $\sum_{n=1}^{\infty} c_n = C$, then $\lim_{n\to\infty} C_n = C$

$$\Rightarrow \quad \lim_{n\to\infty} \frac{C_1 + C_2 + \ldots\ldots + C_n}{n} = C \quad \text{(By Cauchy's first theorem on limits)}$$

Also $< A_n >$ converges to A and $< B_n >$ converges to B

$$\Rightarrow \quad \lim_{n\to\infty} \frac{A_1B_n + A_2B_{n-1} + \ldots\ldots + A_nB_1}{n} = AB \quad \text{(By Cesaro's theorem)}$$

$\therefore$ From (1), $C = AB$.

Remark. Note that in Abel's theorem, $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge to A and B respectively. Abel's theorem does not confirm (under the conditions) the convergence of $\sum_{n=1}^{\infty} c_n$. However, if $\sum_{n=1}^{\infty} c_n$ converges, then $\sum_{n=1}^{\infty} c_n = AB$.

ILLUSTRATIVE EXAMPLES

Example 1. *Show that the Cauchy product of the convergent series* $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ *with itself is not convergent.*

Sol. Let $$a_n = b_n = \frac{(-1)^{n-1}}{n}, \; n \in N.$$

By Leibnitz's test, the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are both convergent (but not absolutely).

The Cauchy product of the two series is $\sum_{n=1}^{\infty} c_n$, where

$$c_n = a_1b_n + a_2b_{n-1} + \dots + a_nb_1$$
$$= \frac{(-1)^0}{1}\cdot\frac{(-1)^{n-1}}{n} + \frac{(-1)^1}{2}\cdot\frac{(-1)^{n-2}}{n-1} + \dots\dots + \frac{(-1)^{n-1}}{n}\cdot\frac{(-1)^0}{1}$$
$$= (-1)^{n-1}\left[\frac{1}{1.n} + \frac{1}{2(n-1)} + \dots\dots + \frac{1}{n.1}\right]$$
$$\geq (-1)^{n-1}\left[\frac{1}{n.n} + \frac{1}{n.n} + \dots\dots + \frac{1}{n.n}\right] \qquad \left[\because\ r \leq n \Rightarrow \frac{1}{r} \geq \frac{1}{n}\right]$$
$$= (-1)^{n-1}\cdot\frac{n}{n^2} = \frac{(-1)^{n-1}}{n} \quad\Rightarrow\quad |c_n| \geq \frac{1}{n}\ \forall\ n \in N$$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, $\sum_{n=1}^{\infty} |c_n|$ is divergent. $\Rightarrow \lim_{n\to 1} c_n \neq 0$.

Hence $\sum_{n=1}^{\infty} c_n$ cannot converge.

Note 1. The condition $\lim_{n\to\infty} a_n = 0$ is absolutely essential for convergence of any series $\sum_{n=1}^{\infty} a_n$.

Note 2. The above example illustrates that the Cauchy product of two conditionally convergent series need not be necessarily convergent.

Example 2. *Show that the Cauchy product of the convergent series* $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ *with itself is not convergent.*

Sol. Let $\qquad a_n = b_n = \frac{(-1)^{n-1}}{\sqrt{n}}$, $n \in N$.

By Leibnitz's test, the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are both convergent (but not absolutely).

The Cauchy product of the two series is $\sum_{n=1}^{\infty} c_n$, where

$$c_n = a_1b_n + a_2b_{n-1} + \dots + a_nb_1$$
$$= \frac{(-1)^0}{\sqrt{1}}\cdot\frac{(-1)^{n-1}}{\sqrt{n}} + \frac{(-1)^1}{\sqrt{2}}\cdot\frac{(-1)^{n-2}}{\sqrt{n-1}} + \dots\dots + \frac{(-1)^{n-1}}{\sqrt{n}}\cdot\frac{(-1)^0}{1}$$
$$= (-1)^{n-1}\left[\frac{1}{\sqrt{1.n}} + \frac{1}{\sqrt{2(n-1)}} + \dots\dots + \frac{1}{\sqrt{n.1}}\right]$$
$$\geq (-1)^{n-1}\left[\frac{1}{\sqrt{n.n}} + \frac{1}{\sqrt{n.n}} + \dots\dots + \frac{1}{\sqrt{n.n}}\right]$$
$$= (-1)^{n-1}\left[\frac{1}{n} + \frac{1}{n} + \dots + \frac{1}{n}\right] = (-1)^{n-1}\cdot\frac{n}{n} = (-1)^{n-1}$$

$\Rightarrow \qquad |c_n| \geq 1\ \forall\ n \in N \Rightarrow \lim_{n\to\infty} c_n \neq 0$

Hence $\sum_{n=1}^{\infty} c_n$ cannot converge.

Example 3. *Show that the Cauchy product of the convergent series* $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ *with itself is not convergent.*

Sol. Let $\qquad a_n = b_n = \frac{(-1)^n}{\sqrt{n+1}}$, $n \in N$

By Leibnitz's test, the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are both convergent (but not absolutely).

The Cauchy product of the two series is $\sum_{n=1}^{\infty} c_n$, where

$$c_n = a_1b_n + a_2b_{n-1} + \dots + a_nb_1$$
$$= \frac{(-1)^1}{\sqrt{2}}\cdot\frac{(-1)^n}{\sqrt{n+1}} + \frac{(-1)^2}{\sqrt{3}}\cdot\frac{(-1)^{n-1}}{\sqrt{n}} + \dots\dots + \frac{(-1)^n}{\sqrt{n+1}}\cdot\frac{(-1)^1}{\sqrt{2}}$$
$$= (-1)^{n-1}\left[\frac{1}{\sqrt{2(n+1)}} + \frac{1}{\sqrt{3n}} + \dots\dots + \frac{1}{\sqrt{(n+1).2}}\right]$$
$$\geq (-1)^{n-1}\cdot\left[\frac{1}{\sqrt{(n+1)(n+1)}} + \frac{1}{\sqrt{(n+1)(n+1)}} + \dots\dots + \frac{1}{\sqrt{(n+1)(n+1)}}\right]$$
$$= (-1)^{n-1}\frac{n}{n+1} \Rightarrow |c_n| \geq \frac{n}{n+1}\ \forall\ n \in N$$

Since $\sum_{n=1}^{\infty} \frac{n}{n+1}$ is divergent. $\qquad \left(\because\ \lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0\right)$

$\Sigma\,|c_n|$ is divergent. $\Rightarrow \lim_{n\to\infty} c_n \neq 0$

Hence $\sum_{n=1}^{\infty} c_n$ cannot converge.

Example 4. *Show that the Cauchy product of the series* $\sum_{n=1}^{\infty} \frac{1}{n^2}$ *with itself converges to* $\frac{\pi^4}{36}$

Sol. Let $\qquad a_n = b_n = \frac{1}{n^2}$, $n \in N$ then $A_n = \sum_{n=1}^{\infty} a_r = \frac{1}{1^2} + \frac{1}{2^2} + \dots\dots + \frac{1}{n^2} = \frac{\pi^2}{6}$

and $\qquad B_n = A_n = \frac{\pi^2}{6} \quad\Rightarrow\quad \lim_{n\to\infty} A_n = \frac{\pi^2}{6} = \lim_{n\to\infty} B_n$

$\Rightarrow \qquad \sum_{n=1}^{\infty} a_n = \frac{\pi^2}{6}$ and $\sum_{n=1}^{\infty} b_n = \frac{\pi^2}{6}$

Since $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series of positive terms, their Cauchy product $\sum_{n=1}^{\infty} c_n$ is also convergent and $\sum_{n=1}^{\infty} c_n = \left(\sum_{n=1}^{\infty} a_n\right)\left(\sum_{n=1}^{\infty} b_n\right) = \frac{\pi^2}{6} \times \frac{\pi^2}{6} = \frac{\pi^4}{36}$

Hence the Cauchy product of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ with itself converges to $\frac{\pi^4}{36}$.

Example 5. *Show that the Cauchy product of two divergent series*

$$\sum_{n=1}^{\infty} a_n = 2 + 2 + 2^2 + 2^3 + \ldots\ \textit{and}\ \sum_{n=1}^{\infty} b_n = -1 + 1 + 1 + 1 + \ldots\ \textit{is convergent.}$$

Sol. For $n \geq 2$, Σa_n and Σb_n are geometric series with common ratios 2 and 1 respectively.

Since the geometric series Σr^n is divergent for $r \geq 1$, the series Σa_n and Σb_n are both divergent.

The Cauchy product of the two given series is $\sum_{n=1}^{\infty} c_n$ where

$$\begin{aligned} c_n &= a_1 b_n + a_2 b_{n-1} + a_3 b_{n-2} + a_4 b_{n-3} + \ldots + a_{n-1} b_2 + a_n b_1 \\ &= 2 \,.\, 1 + 2 \,.\, 1 + 2^2 \,.\, 1 + 2^3 \,.\, 1 + \ldots + 2^{n-2} \,.\, 1 + 2^{n-1} \,.\, (-1) \\ &= 2 + (2 + 2^2 + 2^3 + \ldots + 2^{n-2}) - 2^{n-1} \\ &= 2 + \frac{2(2^{n-2} - 1)}{2 - 1} - 2^{n-1} = 2 + n^{n-1} - 2 - 2^{n-1} = 0 \ \forall\, n \geq 2 \end{aligned}$$

and $\quad c_1 = a_1 b_1 = 2(-1) = -2$

Thus $\sum_{n=1}^{\infty} c_n = -2 + 0 + 0 + 0 + \ldots$ which converges to -2.

Example 6. *Show that the Cauchy product of two divergent series*

$$\sum_{n=0}^{\infty} a_n = 1 - \frac{3}{2} - \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^3 - \ldots\ \textit{and}\ \sum_{n=0}^{\infty} b_n$$

$$= 1 + \left(2 + \frac{1}{2^2}\right) + \frac{3}{2}\left(2^2 + \frac{1}{2^3}\right) + \left(\frac{3}{2}\right)^2\left(2^3 + \frac{1}{2^4}\right) + \ldots$$

is convergent.

Sol. For $n \geq 2$, Σa_n is a geometric series with common ratio $\frac{2}{2}$ (> 1).

$\Rightarrow$ Σa_n is divergent.

Also Σb_n is a series of positive terms and $b_n \geq 1 \ \forall\, n \in \mathbb{N}$.

Since $\lim_{n \to \infty} b_n \neq 0$, Σb_n is divergent.

The Cauchy product of the two given series is $\sum_{n=1}^{\infty} c_n$ where $c_0 = a_0 b_0 = 1 \times 1 = 1$ and for $n \geq 1$,

$$\begin{aligned} c_n &= a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \ldots a_{n-1} b_1 + a_n b_0 \\ &= 1 \,.\, \left(\frac{3}{2}\right)^{n-1}\left(2^n + \frac{1}{2^{n+1}}\right) - \left(\frac{3}{2}\right)\cdot\left(\frac{3}{2}\right)^{n-2}\left(2^{n-1} + \frac{1}{2^n}\right) \\ &\quad - \left(\frac{3}{2}\right)^2 \cdot \left(\frac{3}{2}\right)^{n-3}\left(2^{n-2} + \frac{1}{2^{n-1}}\right) - \ldots - \left(\frac{3}{2}\right)^{n-1}\left(2 + \frac{1}{2^2}\right) - \left(\frac{3}{2}\right)^n \,.\, 1 \\ &= \left(\frac{3}{2}\right)^{n-1}\left[\left(2^n + \frac{1}{2^{n+1}}\right) - \left(2^{n-1} + \frac{1}{2^n}\right) - \left(2^{n-2} + \frac{1}{2^{n-1}}\right) - \ldots - \left(2 + \frac{1}{2^2}\right)\right] - \left(\frac{3}{2}\right)^n \\ &= \left(\frac{3}{2}\right)^{n-1}\left[\left(2^n + \frac{1}{2^{n+1}}\right) - (2^{n-1} + 2^{n-2} + \ldots + 2) - \left(\frac{1}{2^n} + \frac{1}{2^{n-1}} + \ldots + \frac{1}{2^n}\right)\right] - \left(\frac{3}{2}\right)^n \\ &= \left(\frac{3}{2}\right)^{n-1}\left[2^n + \frac{1}{2^{n+1}} - \frac{2(2^{n-1} - 1)}{2 - 1} - \frac{\frac{1}{2^2}\left(1 - \frac{1}{2^{n-1}}\right)}{1 - \frac{1}{2}}\right] - \left(\frac{3}{2}\right)^n \\ &= \left(\frac{3}{2}\right)^{n-1}\left[2^n + \frac{1}{2^{n+1}} - 2^n + 2 - \frac{1}{2} + \frac{1}{2^n}\right] - \left(\frac{3}{2}\right)^n = \left(\frac{3}{2}\right)^{n-1}\left[\frac{3}{2} + \frac{1}{2^{n+1}} + \frac{1}{2^n}\right] - \left(\frac{3}{2}\right)^n \\ &= \left(\frac{3}{2}\right)^{n-1}\left[\frac{3}{2} + \frac{3}{2^{n+1}} - \frac{3}{2}\right] = \frac{3^n}{2^{2n}} = \left(\frac{3}{4}\right)^n \end{aligned}$$

$$\Rightarrow \qquad \sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n$$

which being a geometric series of positive terms with common ratio $\frac{3}{4}$ (< 1), is absolutely convergent.

Example 7. *Show that the Cauchy product of the two divergent series*

$$\sum_{n=1}^{\infty} a_n = 1 - \frac{3}{2} - \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^3 - \ldots$$

and $$\sum_{n=1}^{\infty} b_n = 1 + \left(2 + \frac{1}{2^2}\right) + \frac{3}{2}\left(2^2 + \frac{1}{2^3}\right) + \left(\frac{3}{2}\right)^2\left(2^3 + \frac{1}{2^4}\right) + \ldots\ \textit{is convergent.}$$

Sol. *Please try yourself.*

[**Note.** In example 6, a_n is the $(n + 1)$th term of $\sum_{n=0}^{\infty} a_n$ whereas in example 7, a_n is the nth terms of $\sum_{n=1}^{\infty} a_n$.]

Example 8. *Prove that the Cauchy product of the two series* $3 + \sum_{n=1}^{\infty} 3^n$ *and* $-2 + \sum_{n=1}^{\infty} 2^n$ *is absolutely convergent, although both the series are divergent.*

Sol. Let $\sum_{n=0}^{\infty} a_n = 3 + 3 + 3^2 + 3^3 + \ldots$ and $\sum_{n=0}^{\infty} b_n = -2 + 2 + 2^2 + 2^3 + \ldots$

For $n \geq 1$, $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are geometric series with common ratios 3 and 2 respectively.

Since the geometric series $\sum_{n=1}^{\infty} r^n$ is divergent for $r \geq 1$, the series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are both divergent.

The Cauchy product of the two given series is $\sum_{n=0}^{\infty} c_n$ where $c_0 = a_0 b_0 = (3)(-2) = -6$

and for $n \geq 1$,

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \ldots\ldots + a_{n-1} b_1 + a_n b_0$$
$$= 3 . 2^n + 3 . 2^{n-1} + 3^2 . 2^{n-2} + \ldots\ldots + 3^{n-1} . 2 + 3^n . (-2)$$
$$= 2^n \left[3 + \frac{3}{2} + \left(\frac{3}{2}\right)^2 + \ldots\ldots + \left(\frac{3}{2}\right)^{n-1}\right] - 2 . 3^n$$
$$= 2^n \left[3 + \frac{\frac{3}{2}\left\{\left(\frac{3}{2}\right)^{n-1} - 1\right\}}{\frac{3}{2} - 1}\right] - 2 . 3^n = 2^n \left[3 + 3 . \left(\frac{3}{2}\right)^{n-1} - 3\right] - 2 . 3^n$$
$$= 2^n . \frac{3^n}{2^{n-1}} - 2 . 3^n = 2 . 3^n - 2 . 3^n = 0$$

Thus $\sum_{n=0}^{\infty} c_n = -6 + 0 + 0 + 0 + \ldots\ldots$ which converges absolutely.

Example 9. *Show that* $\sum_{n=0}^{\infty} (-1)^n \left[\frac{1}{(n+1).1} + \frac{1}{n.2} + \ldots\ldots + \frac{1}{1.(n+1)}\right] = (\log 2)^2$.

Sol. We know that $\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots\ldots$

$\therefore$ The series $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\ldots$ converges (conditionally) to log 2.

By Abel's test, if the product of $\sum_{n=0}^{\infty} a_n$ with itself converges, it will converge to $(\log 2)^2$. ...(1)

Let $\sum_{n=0}^{\infty} c_n$ denote the product of $\sum_{n=0}^{\infty} a_n$ with $\sum_{n=0}^{\infty} b_n$ where $b_n = a_n \ \forall\ n \geq 0$, then $c_0 = a_0 b_0 = 1.1 = 1$

and for $n \geq 1$,

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \ldots\ldots + a_{n-1} b_1 + a_n b_0$$
$$= 1 . \frac{(-1)^n}{n+1} - \frac{1}{2} . \frac{(-1)^{n-1}}{n} + \frac{1}{3} . \frac{(-1)^{n-2}}{n-1} - \ldots\ldots + \frac{(-1)^{n-1}}{n} . \left(-\frac{1}{2}\right) + \frac{(-1)^n}{n+1} . 1$$

$$= (-1)^n \left[\frac{1}{1.(n+1)} + \frac{1}{2n} + \frac{1}{3(n-1)} + \ldots\ldots + \frac{1}{n.2} + \frac{1}{(n+1).1}\right] \quad \ldots(2)$$
$$= \frac{(-1)^n}{n+2}\left[\frac{n+2}{1.(n+1)} + \frac{n+2}{2n} + \ldots\ldots + \frac{n+2}{n.2} + \frac{n+2}{(n+1).1}\right]$$
$$= \frac{(-1)^n}{n+2}\left[\left(1 + \frac{1}{n+1}\right) + \left(\frac{1}{2} + \frac{1}{n}\right) + \ldots\ldots + \left(\frac{1}{n} + \frac{1}{2}\right) + \left(\frac{1}{n+1} + 1\right)\right]$$
$$= \frac{(-1)^n}{n+2}\left[2 + \frac{2}{2} + \ldots\ldots + \frac{2}{n} + \frac{2}{n+1}\right] = (-1)^n . \frac{2}{n+2}\left[1 + \frac{1}{2} + \ldots\ldots + \frac{1}{n} + \frac{1}{n+1}\right]$$

Now $$| c_n | = \frac{2}{n+2}\left(1 + \frac{1}{2} + \ldots\ldots + \frac{1}{n+1}\right) = 2\left(\frac{n+1}{n+2}\right)\left(\frac{1 + \frac{1}{2} + \ldots\ldots + \frac{1}{n+1}}{n+1}\right)$$
$$= 2\left(1 - \frac{1}{n+2}\right)\left(\frac{1 + \frac{1}{2} + \ldots\ldots + \frac{1}{n+1}}{n+1}\right) \to 0 \text{ as } n \to \infty$$

(by Cauchy's first theorem on limits)

Also $$| c_{n+1} | - | c_n | = \frac{2}{n+3}\left(1 + \frac{1}{2} + \ldots\ldots + \frac{1}{n+1} + \frac{1}{n+2}\right) - \frac{2}{n+2}\left(1 + \frac{1}{2} + \ldots\ldots + \frac{1}{n+1}\right)$$
$$= \left(\frac{2}{n+3} - \frac{2}{n+2}\right)\left(1 + \frac{1}{2} + \ldots\ldots + \frac{1}{n+1}\right) + \frac{2}{(n+2)(n+3)}$$
$$= \frac{-2}{(n+2)(n+3)}\left(1 + \frac{1}{2} + \ldots\ldots + \frac{1}{n+1} - 1\right)$$
$$= \frac{-2}{(n+2)(n+3)}\left(\frac{1}{2} + \frac{1}{3} + \ldots\ldots + \frac{1}{n+1}\right) < 0$$

$\Rightarrow$ $| c_n | > | c_{n+1} |$

$\therefore$ By Leibnitz's test, the alternating series

$$\sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty} (-1)^n \left[\frac{1}{(n+1).1} + \frac{1}{n.2} + \ldots\ldots + \frac{1}{1.(n+1)}\right] \text{ [by (2)] converges.}$$

Hence, from (1), we have $\sum_{n=0}^{\infty} (-1)^n \left[\frac{1}{(n+1).1} + \frac{1}{n.2} + \ldots\ldots + \frac{1}{1.(n+1)}\right] = (\log 2)^2$.

Example 10. *Resolving* $\frac{(a+1)(a+3)\ldots\ldots(a+2n-1)}{a(a+2)\ldots\ldots(a+2n)}$ *into partial fractions, show that if* $a > 0$ *and* $| x | < 1$ *then,*

$$\left(1 + \frac{1}{2} . \frac{a}{a+2} x + \frac{1.3}{2.4} . \frac{a}{a+4} x^2 + \ldots\ldots\right)\left(1 + \frac{1}{2} x + \frac{1.3}{2.4} x^2 + \ldots\ldots\right)$$
$$= 1 + \frac{a+1}{a+2} x + \frac{(a+1)(a+3)}{(a+2)(a+4)} x^2 + \ldots\ldots$$

Sol. Resolving into partial fractions, we have

$$\frac{(a+1)(a+3)\ldots\ldots(a+2n-1)}{a(a+2)(a+4)\ldots\ldots(a+2n)} = \frac{1.3\ldots\ldots(2n-1)}{2.4\ldots\ldots(2n)}\cdot\frac{1}{a} + \frac{1}{2}\cdot\frac{1.3\ldots\ldots(2n-3)}{2.4\ldots\ldots(2n-2)}\cdot\frac{1}{a+2} + \ldots + \frac{1.3\ldots\ldots(2n-1)}{2.4\ldots\ldots(2n)}\cdot\frac{1}{a+2n} \quad \ldots(1)$$

(by putting $a = 0, -2, -4, \ldots\ldots, -2n$)

Let $$\sum_{n=0}^{\infty} u_n = 1 + \frac{1}{2}\cdot\frac{a}{a+2}x + \frac{1.3}{2.4}\cdot\frac{a}{a+4}x^2 + \ldots\ldots$$

and $$\sum_{n=0}^{\infty} v_n = 1 + \frac{1}{2}x + \frac{1.3}{2.4}x^2 + \ldots$$

$$|u_n| = \frac{1.3\ldots\ldots(2n-1)}{2.4\ldots\ldots(2n)}\cdot\frac{a}{a+2n}\cdot|x|^n$$

$$|u_{n+1}| = \frac{1.3\ldots\ldots(2n-1)(2n+1)}{2.4\ldots\ldots(2n)(2n+2)}\cdot\frac{a}{a+2(n+1)}\cdot|x|^{n+1}$$

$$\therefore \quad \frac{|u_n|}{|u_{n+1}|} = \frac{2n+2}{2n+1}\cdot\frac{a+2+2n}{a+2n}\cdot\frac{1}{|x|} = \frac{1+\frac{1}{n}}{1+\frac{1}{2n}}\cdot\frac{1+\frac{a+2}{2n}}{1+\frac{a}{2n}}\cdot\frac{1}{|x|} \to \frac{1}{|x|} \text{ as } n\to\infty$$

If $|x| < 1$, then $\lim\limits_{n\to\infty}\dfrac{|u_n|}{|u_{n+1}|} > 1$.

$\Rightarrow \sum\limits_{n=0}^{\infty} |u_n|$ converges by D'Alembert's ratio test.

$\Rightarrow \sum\limits_{n=0}^{\infty} u_n$ converges absolutely for $|x| < 1$.

Similarly, $$\frac{|v_n|}{|v_{n+1}|} = \frac{1+\frac{1}{n}}{1+\frac{1}{2n}}\cdot\frac{1}{|x|} \to \frac{1}{|x|} \text{ as } n\to\infty.$$

$\Rightarrow \sum\limits_{n=0}^{\infty} v_n$ converges absolutely for $|x| < 1$.

Let $\sum\limits_{n=0}^{\infty} c_n$ denote the Cauchy product of the series $\sum\limits_{n=0}^{\infty} u_n$ and $\sum\limits_{n=0}^{\infty} v_n$. Then $c_0 = u_0v_0 = 1\times 1 = 1$ and for $n \ge 1$,

$$c_n = u_0v_n + u_1v_{n-1} + \ldots + u_nv_0$$

$$= 1\cdot\frac{1.3\ldots\ldots(2n-1)}{2.4\ldots\ldots(2n)}x^n + \frac{1}{2}\cdot\frac{a}{a+2}x\cdot\frac{1.3\ldots\ldots(2n-3)}{2.4\ldots\ldots(2n-2)}\cdot x^{n-1} + \ldots\ldots + \frac{1.3\ldots\ldots(2n-1)}{2.4\ldots\ldots(2n)}\cdot\frac{a}{a+2n}x^n\cdot 1$$

$$= \left[\frac{1.3\ldots\ldots(2n-1)}{2.4\ldots\ldots(2n)} + \frac{1}{2}\cdot\frac{1.3\ldots\ldots(2n-3)}{2.4\ldots\ldots(2n-2)}\cdot\frac{a}{a+2} + \ldots\ldots + \frac{1.3\ldots\ldots(2n-1)}{2.4\ldots\ldots(2n)}\cdot\frac{a}{a+2n}\right]x^n$$

$$= a\left[\frac{1.3\ldots\ldots(2n-1)}{2.4\ldots\ldots(2n)}\cdot\frac{1}{a} + \frac{1}{2}\cdot\frac{1.3\ldots\ldots(2n-3)}{2.4\ldots\ldots(2n-2)}\cdot\frac{1}{a+2} + \ldots\ldots + \frac{1.3\ldots\ldots(2n-1)}{2.4\ldots\ldots(2n)}\cdot\frac{1}{a+2n}\right]x^n$$

$$= \frac{(a+1)(a+3)\ldots\ldots(a+2n-1)}{(a+2)(a+4)\ldots\ldots(a+2n)}x^n \qquad \text{[using (1)]}$$

$$\Rightarrow \quad \sum_{n=0}^{\infty} c_n = 1 + \frac{a+1}{a+2}x + \frac{(a+1)(a+3)}{(a+2)(a+4)}x^2 + \ldots\ldots$$

Since $\sum\limits_{n=0}^{\infty} u_n$ and $\sum\limits_{n=0}^{\infty} v_n$ converge absolutely for $|x| < 1$, therefore, by Cauchy's theorem, $\sum\limits_{n=0}^{\infty} c_n$ converges absolutely for $|x| < 1$ and $\left(\sum\limits_{n=0}^{\infty} u_n\right)\left(\sum\limits_{n=0}^{\infty} v_n\right) = \sum\limits_{n=0}^{\infty} c_n$

$$\Rightarrow \left(1 + \frac{1}{2}\cdot\frac{a}{a+2}x + \frac{1.3}{2.4}\cdot\frac{a}{a+4}x^2 + \ldots\ldots\right)\times\left(1 + \frac{1}{2}x + \frac{1.3}{2.4}x^2 + \ldots\ldots\right)$$

$$= 1 + \frac{a+1}{a+2}x + \frac{(a+1)(a+3)}{(a+2)(a+4)}x^2 + \ldots\ldots$$

Example 11. *Show that* $\left(1 - \frac{1}{2} + \frac{1}{3} - \ldots\ldots\right)^2 = \sum\limits_{n=1}^{\infty}(-1)^{n-1}\left[\frac{1}{1.n} + \frac{1}{2(n-1)} + \ldots\ldots + \frac{1}{n.1}\right].$

Sol. Let $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\ldots = \sum\limits_{n=1}^{\infty}\frac{(-1)^{n-1}}{n} = \sum\limits_{n=1}^{\infty} a_n$, then $\sum\limits_{n=1}^{\infty} a_n$ converges (conditionally).

By Abel's test, if the Cauchy product $\sum\limits_{n=1}^{\infty} c_n$ of $\sum\limits_{n=1}^{\infty} a_n$ with itself converges, then

$$\left(\sum_{n=1}^{\infty} a_n\right)^2 = \sum_{n=1}^{\infty} c_n \quad \ldots(1)$$

Now, $$c_n = 1\cdot\frac{(-1)^{n-1}}{n} - \frac{1}{2}\cdot\frac{(-1)^{n-2}}{n-1} + \ldots\ldots + \frac{(-1)^{n-2}}{n-1}\left(-\frac{1}{2}\right) + \frac{(-1)^{n-1}}{n}\cdot 1$$

$$= (-1)^{n-1}\left[\frac{1}{1.n} + \frac{1}{2(n-1)} + \ldots\ldots + \frac{1}{(n-1).2} + \frac{1}{n.1}\right] \quad \ldots(2)$$

$$= \frac{(-1)^{n-1}}{n+1}\left[\frac{n+1}{1.n} + \frac{n+1}{2(n-1)} + \ldots\ldots + \frac{n+1}{(n-1).2} + \frac{n+1}{n.1}\right]$$

$$= \frac{(-1)^{n-1}}{n+1}\left[\left(1 + \frac{1}{n}\right) + \left(\frac{1}{2} + \frac{1}{n-1}\right) + \ldots\ldots + \left(\frac{1}{n-1} + \frac{1}{2}\right) + \left(\frac{1}{n} + 1\right)\right]$$

$$= \frac{(-1)^{n-1}}{n+1}\cdot\left[2+\frac{2}{2}+\ldots\ldots+\frac{2}{n-1}+\frac{2}{n}\right]$$

$$= (-1)^{n-1}\cdot\frac{2}{n+1}\left[1+\frac{1}{2}+\ldots\ldots+\frac{1}{n-1}+\frac{1}{n}\right]$$

$$\therefore \quad |c_n| = \frac{2}{n+1}\left(1+\frac{1}{2}+\ldots\ldots+\frac{1}{n}\right) = \frac{2n}{n+1}\left[\frac{1+\frac{1}{2}+\ldots\ldots+\frac{1}{n}}{n}\right]$$

$$= \frac{2}{1+\frac{1}{n}}\left[\frac{1+\frac{1}{2}+\ldots\ldots+\frac{1}{n}}{n}\right] \to 0 \text{ as } n \to \infty$$

(by Cauchy's first theorem on limits)

Also $$|c_{n+1}| - |c_n| = \frac{2}{n+2}\left(1+\frac{1}{2}+\ldots\ldots+\frac{1}{n}+\frac{1}{n+1}\right) - \frac{2}{n+1}\left(1+\frac{1}{2}+\ldots\ldots+\frac{1}{n}\right)$$

$$= \left(\frac{2}{n+2}-\frac{2}{n+1}\right)\left(1+\frac{1}{2}+\ldots\ldots+\frac{1}{n}\right)+\frac{2}{(n+2)(n+1)}$$

$$= \frac{-2}{(n+2)(n+1)}\left(1+\frac{1}{2}+\ldots\ldots+\frac{1}{n}-1\right)$$

$$= \frac{-2}{(n+2)(n+1)}\left(\frac{1}{2}+\frac{1}{3}+\ldots\ldots+\frac{1}{n}\right) \ngtr 0$$

$\Rightarrow \quad |c_n| > |c_{n+1}|$

$\therefore$ By Leibnitz's test, the alternating series

$$\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} (-1)^{n-1}\left[\frac{1}{1.n}+\frac{1}{2(n-1)}+\ldots\ldots+\frac{1}{n.1}\right] \text{ [by (2)] converges.}$$

Hence, from (1), we have

$$\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots\ldots\right)^2 = \sum_{n=1}^{\infty} (-1)^{n-1}\left[\frac{1}{1.n}+\frac{1}{2(n-1)}+\ldots\ldots+\frac{1}{n.1}\right].$$

Example 12. *Show that*

$$\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots\ldots\right)^2 = 2\left[\frac{1}{2}-\frac{1}{3}\left(1+\frac{1}{2}\right)+\frac{1}{4}\left(1+\frac{1}{2}+\frac{1}{3}\right)-\frac{1}{5}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)+\ldots\ldots\right].$$

Sol. Proceeding as in example 11, we have $\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots\ldots\right)^2$

$$= \sum_{n=1}^{\infty} (-1)^{n-1}\left[\frac{1}{1.n}+\frac{1}{2(n-1)}+\ldots\ldots+\frac{1}{n.1}\right]$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1}\cdot\frac{2}{n+1}\left[1+\frac{1}{2}+\ldots\ldots+\frac{1}{n}\right]$$

$$= 2\left[\frac{1}{2}-\frac{1}{3}\left(1+\frac{1}{2}\right)+\frac{1}{4}\left(1+\frac{1}{2}+\frac{1}{3}\right)-\frac{1}{5}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)+\ldots\ldots\right].$$

Example 13. *Show that*

$$\frac{1}{2}\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots\ldots\right)^2 = \frac{1}{2}-\frac{1}{4}\left(1+\frac{1}{3}\right)+\frac{1}{6}\left(1+\frac{1}{3}+\frac{1}{5}\right)-\ldots\ldots$$

Sol. Let $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots\ldots = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1} = \sum_{n=1}^{\infty} a_n,$

then $\sum_{n=1}^{\infty} a_n$ converges (conditionally).

By Abel's test, if the Cauchy product $\sum_{n=1}^{\infty} c_n$ of $\sum_{n=1}^{\infty} a_n$ with itself converges, then

$$\left(\sum_{n=1}^{\infty} a_n\right)^2 = \sum_{n=1}^{\infty} c_n \qquad \ldots(1)$$

Now $$c_n = 1\cdot\frac{(-1)^{n-1}}{2n-1}-\frac{1}{3}\cdot\frac{(-1)^{n-2}}{2n-3}+\ldots\ldots+\frac{(-1)^{n-2}}{2n-3}\left(-\frac{1}{3}\right)+\frac{(-1)^{n-1}}{2n-1}\cdot 1$$

$$= (-1)^{n-1}\left[\frac{1}{1.(2n-1)}+\frac{1}{3(2n-3)}+\ldots\ldots+\frac{1}{(2n-3).3}+\frac{1}{(2n-1).1}\right]$$

$$= \frac{(-1)^{n-1}}{2n}\left[\frac{(2n-1)+1}{1.(2n-1)}+\frac{(2n-3)+3}{3(2n-3)}+\ldots\ldots+\frac{(2n-3)+3}{(2n-3).3}+\frac{(2n-1)+1}{(2n-1).1}\right]$$

$$= \frac{(-1)^{n-1}}{2n}\left[\left(1+\frac{1}{2n-1}\right)+\left(\frac{1}{3}+\frac{1}{2n-3}\right)+\ldots\ldots+\left(\frac{1}{2n-3}+\frac{1}{3}\right)+\left(\frac{1}{2n-1}+1\right)\right]$$

$$= \frac{(-1)^{n-1}}{2n}\left[2+\frac{2}{3}+\ldots\ldots+\frac{2}{2n-3}+\frac{2}{2n-1}\right]$$

$$= \frac{(-1)^{n-1}}{n}\left[1+\frac{1}{3}+\ldots\ldots+\frac{1}{2n-1}\right] \qquad \ldots(2)$$

$$\therefore \quad |c_n| = \frac{1+\frac{1}{3}+\ldots\ldots+\frac{1}{2n-1}}{n} \to 0 \text{ as } n \to \infty \quad \text{(by Cauchy's first theorem on limits)}$$

Also $$|c_{n+1}| - |c_n| = \frac{1}{n+1}\left(1+\frac{1}{3}+\ldots\ldots+\frac{1}{2n-1}+\frac{1}{2n+1}\right)-\frac{1}{n}\left(1+\frac{1}{3}+\ldots\ldots+\frac{1}{2n-1}\right)$$

$$= \left(\frac{1}{n+1}-\frac{1}{n}\right)\left(1+\frac{1}{3}+\ldots\ldots+\frac{1}{2n-1}\right)+\frac{1}{(n+1)(2n+1)}$$

$$= \frac{-1}{n(n+1)}\left(1+\frac{1}{3}+\ldots\ldots+\frac{1}{2n-1}\right)+\frac{1}{(n+1)(2n+1)}$$

$$< \frac{-1}{n(n+1)}+\frac{1}{(n+1)(2n+1)} \qquad \left[\because 1+\frac{1}{3}+\ldots\ldots+\frac{1}{2n-1} > 1\right]$$

$$= \frac{-(2n+1)+n}{n(n+1)(2n+1)} = \frac{-(n+1)}{n(n+1)(2n+1)} = \frac{-1}{n(2n+1)} < 0$$

$\Rightarrow \quad |c_n| > |c_{n+1}|$

By Leibnitz's test, the alternating series

$$\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\left[1+\frac{1}{3}+\dots\dots+\frac{1}{2n-1}\right] \text{ [by (2)] converges.}$$

Hence, from (1), we have

$$\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots\dots\right)^2 = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\left[1+\frac{1}{3}+\dots\dots+\frac{1}{2n-1}\right]$$

$$= \frac{1}{1}(1) - \frac{1}{2}\left(1+\frac{1}{3}\right)+\frac{1}{3}\left(1+\frac{1}{3}+\frac{1}{5}\right)-\dots\dots$$

$$\Rightarrow \frac{1}{2}\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots\dots\right)^2 = \frac{1}{2}-\frac{1}{4}\left(1+\frac{1}{3}\right)+\frac{1}{6}\left(1+\frac{1}{3}+\frac{1}{5}+\dots\dots\right).$$

Example 14. *Show that* $\frac{1}{2}\left(x-\frac{1}{2}x^2+\frac{1}{3}x^3-\dots\dots\right)^2$

$$= \sum_{n=2}^{\infty} (-1)^n \left(1+\frac{1}{2}+\frac{1}{3}+\dots\dots+\frac{1}{n-1}\right)\frac{x^n}{n}$$

when (i) $|x|<1$ *and (ii)* $x=1$.

Sol. (*i*) Let $x-\frac{1}{2}x^2+\frac{1}{3}x^3-\dots = \sum_{n=2}^{\infty} \frac{(-1)^n}{n-1}x^{n-1} = \sum_{n=2}^{\infty} a_n$

then $\quad |a_n| = \frac{|x|^{n+1}}{n+1}$

$$|a_{n+1}| = \frac{|x|^{n+2}}{n+2}$$

$$\therefore \quad \frac{|a_n|}{|a_{n+1}|} = \frac{n+2}{n+1}\cdot\frac{1}{|x|} = \frac{1+\frac{2}{n}}{1+\frac{1}{n}}\cdot\frac{1}{|x|} \to \frac{1}{|x|} \text{ as } n\to\infty$$

Since $\quad |x|<1, \frac{1}{|x|}>1$

$\therefore$ By ratio test, the series $\sum_{n=2}^{\infty} |a_n|$ converges.

$\Rightarrow$ the series $\sum_{n=2}^{\infty} a_n$ converges absolutely.

By Cauchy's theorem, the Cauchy product $\sum_{n=2}^{\infty} c_n$ of $\sum_{n=2}^{\infty} a_n$ with itself also converges absolutely and

$$\left(\sum_{n=2}^{\infty} a_n\right)^2 = \sum_{n=2}^{\infty} c_n \quad \dots(1)$$

Now $\quad c_n = x\cdot\frac{(-1)^n}{n-1}x^{n-1} - \frac{1}{2}x^2\cdot\frac{(-1)^{n-1}}{n-2}x^{n-2}+\dots\dots$

$$+\frac{(-1)^{n-1}}{n-2}x^{n-2}\left(-\frac{1}{2}x^2\right)+\frac{(-1)^n}{n-1}x^{n-1}\cdot x \text{ for } n\ge 2$$

$$= \frac{(-1)^n}{n}x^n\left[\frac{(n-1)+1}{1.(n-1)}+\frac{(n-2)+2}{2(n-2)}+\dots\dots+\frac{(n-2)+2}{(n-1).2}+\frac{(n-1)+1}{(n-1).1}\right]$$

$$= (-1)^n\cdot\frac{x^n}{n}\left[\left(1+\frac{1}{n-1}\right)+\left(\frac{1}{2}+\frac{1}{n-2}\right)+\dots\dots+\left(\frac{1}{n-2}+\frac{1}{2}\right)+\left(\frac{1}{n-1}+1\right)\right]$$

$$= (-1)^n\cdot\frac{x^n}{n}\left[2+\frac{2}{2}+\dots\dots+\frac{2}{n-2}+\frac{2}{n-1}\right] = 2(-1)^n\cdot\frac{x^n}{n}\left(1+\frac{1}{2}+\dots\dots+\frac{1}{n-1}\right)$$

$\therefore$ From (1), we have

$$\left(x-\frac{1}{2}x^2+\frac{1}{3}x^3-\dots\dots\right)^2 = \sum_{n=2}^{\infty} 2(-1)^n\cdot\frac{x^n}{n}\left(1+\frac{1}{2}+\dots\dots+\frac{1}{n-1}\right)$$

$$\Rightarrow \frac{1}{2}\left(x-\frac{1}{2}x^2+\frac{1}{3}x^3-\dots\dots\right)^2 = \sum_{n=2}^{\infty}(-1)^n\left(1+\frac{1}{2}+\frac{1}{3}+\dots\dots+\frac{1}{n-1}\right)\cdot\frac{x^n}{n}$$

(*ii*) Here $x = 1$

$\therefore$ We have to show that $\frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\dots\dots\right)^2 = \sum_{n=2}^{\infty}\frac{(-1)^n}{n}\left(1+\frac{1}{2}+\frac{1}{3}+\dots\dots+\frac{1}{n-1}\right)$

Let $\quad 1-\frac{1}{2}+\frac{1}{3}-\dots\dots = \sum_{n=2}^{\infty}\frac{(-1)^n}{n-1} = \sum_{n=2}^{\infty} a_n$ then $\sum_{n=2}^{\infty} a_n$ converges (conditionally).

By Abel's test, if the Cauchy product $\sum_{n=2}^{\infty} c_n$ of $\sum_{n=2}^{\infty} a_n$ with itself converges, then

$$\left(\sum_{n=2}^{\infty} a_n\right)^2 = \sum_{n=2}^{\infty} c_n \quad \dots(2)$$

Now $\quad c_n = 1\cdot\frac{(-1)^n}{n-1}-\frac{1}{2}\cdot\frac{(-1)^{n-1}}{n-2}+\dots\dots+\frac{(-1)^{n-1}}{n-2}\cdot\left(-\frac{1}{2}\right)+\frac{(-1)^n}{n-1}\cdot 1$

$$= \frac{(-1)^n}{n}\left[\frac{(n-1)+1}{1.(n-1)}+\frac{(n-2)+2}{2(n-2)}+\dots\dots+\frac{(n-2)+2}{(n-2).2}+\frac{(n-1)+1}{(n-1).1}\right]$$

$$= \frac{(-1)^n}{n}\left[\left(1+\frac{1}{n-1}\right)+\left(\frac{1}{2}+\frac{1}{n-2}\right)+\dots\dots+\left(\frac{1}{n-2}+\frac{1}{2}\right)+\left(\frac{1}{n-1}+1\right)\right]$$

$$= \frac{(-1)^n}{n}\left(2+\frac{2}{2}+\dots\dots+\frac{2}{n-2}+\frac{2}{n-1}\right] = (-1)^n\cdot\frac{2}{n}\left(1+\frac{1}{2}+\dots\dots+\frac{1}{n-1}\right) \quad \dots(3)$$

$$\therefore \quad |c_n| = \frac{2}{n}\left(1+\frac{1}{2}+\dots\dots+\frac{1}{n-1}\right) = \frac{2(n-1)}{n}\cdot\left(\frac{1+\frac{1}{2}+\dots\dots+\frac{1}{n-1}}{n-1}\right)$$

$$= 2\left(1-\frac{1}{n}\right)\left(\frac{1+\frac{1}{2}+\ldots\ldots+\frac{1}{n-1}}{n-1}\right) \to 0 \text{ as } n \to \infty$$

(By Cauchy's first theorem on limits)

Also $\quad |c_{n+1}| - |c_n| = \frac{2}{n+1}\left(1+\frac{1}{2}+\ldots\ldots+\frac{1}{n-1}+\frac{1}{n}\right) - \frac{2}{n}\left(1+\frac{1}{2}+\ldots\ldots+\frac{1}{n-1}\right)$

$$= \left(\frac{2}{n+1}-\frac{2}{n}\right)\left(1+\frac{1}{2}+\ldots\ldots+\frac{1}{n-1}\right)+\frac{2}{n(n+1)}$$

$$= \frac{-2}{n(n+1)}\left(1+\frac{1}{2}+\ldots\ldots+\frac{1}{n-1}-1\right)$$

$$= \frac{-2}{n(n+1)}\left(\frac{1}{2}+\frac{1}{3}+\ldots\ldots+\frac{1}{n-1}\right) < 0$$

$\Rightarrow \quad |c_n| > |c_{n+1}|$

$\therefore$ By Leibnitz's test, the alternating series

$$\sum_{n=2}^{\infty} c_n = \sum_{n=2}^{\infty} (-1)^n \cdot \frac{2}{n}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots+\frac{1}{n-1}\right) \text{ [by (3)] converges.}$$

Hence, from (2), we have

$$\left(1-\frac{1}{2}+\frac{1}{3}-\ldots\ldots\right)^2 = \sum_{n=2}^{\infty} (-1)^n \cdot \frac{2}{n}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots+\frac{1}{n-1}\right)$$

$$\Rightarrow \quad \frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\ldots\ldots\right)^2 = \sum_{n=2}^{\infty} \frac{(-1)^n}{n}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots+\frac{1}{n-1}\right).$$

Example 15. *Show that*

$$\frac{1}{2}\left(x-\frac{1}{2}x^2+\frac{1}{3}x^3-\ldots\ldots\right)^2 = \sum_{n=1}^{\infty} (-1)^{n+1}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots+\frac{1}{n}\right)\cdot\frac{x^{n+1}}{n+1}.$$

when (i) $|x| < 1$ *and (ii)* $x = 1$.

Sol. *Please try yourself.* $\qquad \left[\text{Here } a_n = \frac{(-1)^{n+1}}{n} x^n\right]$

Example 16. *Prove that*

$$\left(1+x+\frac{x^2}{2!}+\ldots\ldots\right)\left(1+y+\frac{y^2}{2!}+\ldots\ldots\right) = 1+\frac{x+y}{1!}+\frac{(x+y)^2}{2!}+\frac{(x+y)^3}{3!}+\ldots\ldots$$

for all values of x and y.

Sol. We know that $\quad e^x = 1+x+\frac{x^2}{2!}+\ldots\ldots \quad$ for all $x \in \mathbf{R}$

and $\quad e^y = 1+y+\frac{y^2}{2!}+\ldots\ldots \quad$ for all $y \in \mathbf{R}$

$\therefore$ The series $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1+x+\frac{x^2}{2!}+\ldots\ldots$

converges to e^x, and the series $\sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} \frac{y^n}{n!} = 1+y+\frac{y^2}{2!}+\ldots\ldots$ converges to e^y.

Let $\sum_{n=0}^{\infty} c_n$ denote the Cauchy product of the series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$, then $c_0 = a_0 b_0$ $= 1 \times 1 = 1$

and for $n \geq 1$,

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \ldots\ldots + a_n b_0$$

$$= 1 . \frac{y^n}{n!} + x \cdot \frac{y^{n-1}}{(n-1)!} + \frac{x^2}{2!}\cdot\frac{y^{n-2}}{(n-2)!} + \ldots\ldots + \frac{x^n}{n!} . 1$$

$$= \frac{1}{n!}\left[y^n + ny^{n-1} + \frac{n(n-1)}{2!} y^{n-2} x^2 + \ldots\ldots + x^n\right] = \frac{(y+x)^n}{n!}$$

$\therefore \quad \sum_{n=0}^{\infty} c_n = 1 + \frac{x+y}{1!} + \frac{(x+y)^2}{2!} + \frac{(x+y)^3}{3!} + \ldots\ldots$ which converges to e^{x+y}.

$\therefore$ By Abel's test, $\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) = \sum_{n=0}^{\infty} c_n$

$$\Rightarrow \quad \left(1+x+\frac{x^2}{2!}+\ldots\ldots\right)\left(1+y+\frac{y^2}{2!}+\ldots\ldots\right) = 1+\frac{x+y}{1!}+\frac{(y+y)^2}{2!}+\frac{(x+y)^3}{3!}+\ldots\ldots$$

Example 17. *Show that the Cauchy product of the series* $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$, $(p > 0)$ *with itself*

(i) converges if $p > \frac{1}{2}$ $\qquad$ *(ii) does not converge if* $p \leq \frac{1}{2}$.

Sol. Let $a_n = b_n = \frac{(-1)^{n-1}}{n^p}$.

The Cauchy product of the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ is $\sum_{n=1}^{\infty} c_n$

where $\quad c_n = a_1 b_n + a_2 b_{n-1} + a_3 b_{n-2} + \ldots\ldots + a_n b_1$

$$= \frac{1}{1^p}\cdot\frac{(-1)^{n-1}}{n^p} - \frac{1}{2^p}\cdot\frac{(-1)^{n-2}}{(n-1)^p} + \frac{1}{3^p}\cdot\frac{(-1)^{n-3}}{(n-2)^p} - \ldots\ldots + \frac{(-1)^{n-1}}{n^p}\cdot\frac{1}{1^p}$$

$$= (-1)^{n-1}\left[\frac{1}{1^p . n^p} + \frac{1}{2^p . (n-1)^p} + \frac{1}{3^p (n-2)^p} + \ldots\ldots + \frac{1}{n^p . 1^p}\right]$$

$$= (-1)^{n-1} . d_n$$

where $\quad d_n = \frac{1}{1^p . n^p} + \frac{1}{2^p . (n-1)^p} + \frac{1}{3^p (n-2)^p} + \ldots\ldots + \frac{1}{n^p . 1^p}$

$$= \sum_{r=1}^{n} \frac{1}{r^p (n-r+1)^p} = \sum_{r=1}^{n} \frac{1}{[r(n-r+1)]^p}$$

Now $r(n-r+1) = -r^2 + (n+1)r$

$$= -\left[r^2 - (n+1)\,r + \left(\frac{n+1}{2}\right)^2\right] + \left(\frac{n+1}{2}\right)^2 = \left(\frac{n+1}{2}\right)^2 - \left(r - \frac{n+1}{2}\right)^2 \le \left(\frac{n+1}{2}\right)^2$$

$\Rightarrow$ $r(n-r+1)$ is maximum when $r - \frac{n+1}{2} = 0$ *i.e.,* when $r = \frac{n+1}{2}$ and the max. value is $\left(\frac{n+1}{2}\right)^2$.

$$\Rightarrow \quad r(n-r+1) \le \left(\frac{n+1}{2}\right)^2 \Rightarrow r^p (n-r+1)^p \le \left(\frac{n+1}{2}\right)^{2p}$$

$$\Rightarrow \frac{1}{[r(n-r+1)]^p} \ge \left(\frac{2}{n+1}\right)^{2p}$$

$$\Rightarrow \quad d_n \ge \sum_{r=1}^{n} \left(\frac{2}{n+1}\right)^{2p} = \left(\frac{2}{n+1}\right)^{2p} . n = \left(\frac{2n}{n+1}\right)^{2p} . n^{1-2p} = \left(\frac{2}{1+\frac{1}{n}}\right)^{2p} . n^{1-2p} \ge n^{1-2p}$$

$$\left[\because \forall n \in \mathbb{N}, \frac{1}{n} \le 1 \Rightarrow 1 + \frac{1}{n} \le 2 \Rightarrow \frac{2}{1+\frac{1}{n}} \ge 1\right]$$

If $1 - 2p < 0$ *i.e.,* $p > \frac{1}{2}$, $d_n \to 0$ as $n \to \infty$ and by Leibnitz's test, the series $\sum_{n=1}^{\infty} c_n$ converges.

If $1 - 2p \ge 0$ *i.e.,* $p \le \frac{1}{2}$, $d_n \ge 1$ and the series $\sum_{n=1}^{\infty} c_n$ does not converge.

Example 18. *Assuming* $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$, *prove that*

$$(\tan^{-1} x)^2 = 2 \sum_{n=0}^{\infty} (-1)^n . \frac{x^{2n+2}}{2n+2}\left(1 + \frac{1}{3} + \ldots + \frac{1}{2n+1}\right)$$

Also, show that the series is absolutely convergent if $|x| < 1$ *and convergent if* $x = 1$.

Or

For $-1 < x \le 1$, *show that* $\frac{1}{2}\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots\right)^2$

$$= \sum_{n=0}^{\infty} (-1)^n . \frac{x^{2n+2}}{2n+2}\left(1 + \frac{1}{3} + \ldots + \frac{1}{2n+1}\right).$$

Sol. Let $x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots = \sum_{n=0}^{\infty} (-1)^n . \frac{x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} a_n$

then $$|a_n| = \frac{|x|^{2n+1}}{2n+1} \text{ and } |a_{n+1}| = \frac{|x|^{2n+3}}{2n+3}$$

$$\therefore \quad \frac{|a_n|}{|a_{n+1}|} = \frac{2n+3}{2n+1} . \frac{1}{|x|^2} = \frac{1+\frac{3}{2n}}{1+\frac{1}{2n}} . \frac{1}{x^2}$$

$$\lim_{n \to \infty} \frac{|a_n|}{|a_{n+1}|} = \frac{1}{x^2}.$$

$\therefore$ by ratio test, the series $\sum_{n=0}^{\infty} |a_n|$ is convergent if $\frac{1}{x^2} > 1$ *i.e.,* if $x^2 < 1$ *i.e.,* if $|x| < 1$

$\Rightarrow$ the series $\sum_{n=0}^{\infty} a_n$ converges absolutely for $|x| < 1$.

By Cauchy's theorem, the Cauchy product $\sum_{n=0}^{\infty} c_n$ of $\sum_{n=0}^{\infty} a_n$ with itself converges absolutely for $|x| < 1$ and $\left(\sum_{n=0}^{\infty} a_n\right)^2 = \sum_{n=0}^{\infty} c_n$.

Now $c_0 = x . x = x^2$ and for $n \ge 1$,

$$c_n = x . (-1)^n \frac{x^{2n+1}}{2n+1} - \frac{x^3}{3} . (-1)^{n-1} \frac{x^{2n-1}}{2n-1} + \frac{x^5}{5} . (-1)^{n-2} \frac{x^{2n-3}}{2n-3} - \ldots + (-1)^n \frac{x^{2n+1}}{2n+1} . x$$

$$= (-1)^n x^{2n+2}\left[\frac{1}{1.(2n+1)} + \frac{1}{3(n-1)} + \frac{1}{5(2n-3)} + \ldots + \frac{1}{(2n+1).1}\right]$$

$$= (-1)^n . \frac{x^{2n+2}}{2n+2}\left[\frac{(2n+1)+1}{1.(2n+1)} + \frac{(2n-1)+3}{3(2n-1)} + \frac{(2n-3)+5}{5(2n-3)} + \ldots + \frac{(2n+1)+1}{(2n+1).1}\right]$$

$$= (-1)^n . \frac{x^{2n+2}}{2n+2}\left[\left(1 + \frac{1}{2n+1}\right) + \left(\frac{1}{3} + \frac{1}{2n-1}\right) + \left(\frac{1}{5} + \frac{1}{2n-3}\right) + \ldots + \left(\frac{1}{2n+1} + 1\right)\right]$$

$$= (-1)^n . \frac{x^{2n+2}}{2n+2}\left[2 + \frac{2}{3} + \frac{2}{5} + \ldots + \frac{2}{2n+1}\right] = 2(-1)^n . \frac{x^{2n+2}}{2n+2}\left(1 + \frac{1}{3} + \ldots + \frac{1}{2n+1}\right)$$

For $|x| < 1$, $$\left(\sum_{n=0}^{\infty} a_n\right)^2 = \sum_{n=0}^{\infty} c_n.$$

$$\Rightarrow \quad \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots\right)^2 = 2\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{2n+2}\left(1 + \frac{1}{3} + \ldots + \frac{1}{2n+1}\right)$$

But $$x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots = \tan^{-1} x.$$

$$\therefore \quad (\tan^{-1} x)^2 = 2 \sum_{n=0}^{\infty} (-1)^n . \frac{x^{2n+2}}{2n+2}\left(1+\frac{1}{3}+\ldots\ldots+\frac{1}{2n+1}\right)$$

The result holds good for $x = 1$ also. (See example 13)

Example 19. *Assuming that* $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots\ldots$, *prove that*

$$(\tan^{-1} x)^2 = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}\left(1+\frac{1}{3}+\ldots\ldots+\frac{1}{2n-1}\right).$$

Sol. *Please try yourself.*

Example 20. *For all* $x \in R$, *show that*

$$\left(1+\frac{x}{1^2}+\frac{x^2}{(2!)^2}+\frac{x^3}{(3!)^2}+\ldots\ldots\right)\left(1-\frac{x}{1^2}+\frac{x^2}{(2!)^2}-\frac{x^3}{(3!)^2}+\ldots\ldots\right)$$

$$= 1 - \frac{x^2}{1^2 . 2!} + \frac{x^4}{(2!)^2 4!} - \frac{x^6}{(3!)^2 6!} + \ldots\ldots$$

Sol. Let $\quad 1+\frac{x}{1^2}+\frac{x^2}{(2!)^2}+\frac{x^3}{(3!)^2}+\ldots\ldots = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2} = \sum_{n=0}^{\infty} a_n$

and $\quad 1-\frac{x}{1^2}+\frac{x^2}{(2!)^2}-\frac{x^3}{(3!)^2}+\ldots\ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{(n!)^2} = \sum_{n=0}^{\infty} b_n$

Then $\quad | a_n | = \frac{|x|^n}{(n!)^2}$ and $| a_{n+1} | = \frac{|x|^{n+1}}{(x+1!)^2}$

$$\therefore \quad \frac{|a_n|}{|a_{n+1}|} = \frac{(n+1)^2}{|x|} \to \infty \text{ as } n \to \infty \text{ for all } x \neq 0.$$

$\Rightarrow$ The series $\sum_{n=0}^{\infty} | a_n |$ converges for all $x \neq 0$

For $x = 0$, the series becomes $1 + 0 + 0 + 0 + \ldots$

$\therefore$ the series $\sum_{n=0}^{\infty} a_n$ converges absolutely for all $x \in R$.

Similarly, the series $\sum_{n=0}^{\infty} b_n$ converges absolutely for all $x \in R$.

$\therefore$ By Cauchy's theorem, the Cauchy product $\sum_{n=0}^{\infty} c_n$ of $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ also converges absolutely and $\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) = \sum_{n=0}^{\infty} c_n$.

Now $c_0 = 1 \times 1 = 1$ and for $n \geq 1$,

$$c_n = 1 . (-1)^n \frac{x^n}{(n!)^2} + \frac{x}{1^2} . (-1)^{n-1} \frac{x^{n-1}}{[(n-1)!]^2} + \frac{x^2}{(2!)^2} . (-1)^{n-2} \frac{x^{n-2}}{[(n-2)!]^2} + \ldots\ldots + \frac{x^n}{(n!)^2} . 1$$

$$= \frac{(-1)^n x^n}{(n!)^2}\left[1 - \frac{(n!)^2}{[1^2 . (n-1)!]^2} + \frac{(n!)^2}{(2!)^2 (n-2!)^2} - \ldots\ldots + (-1)^n \frac{(n!)^2}{(n!)^2}\right]$$

$$= \frac{(-1)^n x^n}{(n!)^2}\left[1 - \left(\frac{n}{1}\right)^2 + \left(\frac{n(n-1)^2}{2!}\right)^2 + \ldots\ldots + (-1)^n (1)^2\right]$$

$$= \frac{(-1)^n x^n}{(n!)^2}\left[1 - ({}^nC_1)^2 + ({}^nC_2)^2 - \ldots\ldots + (-1)^n . ({}^nC_n)^2\right]$$

$$= \frac{(-1)^n x^n}{(n!)^2} \times \begin{cases} 0 & , \text{ if } n \text{ is odd} \\ (-1)^{n/2} . {}^nC_{n/2} , & \text{if } n \text{ is even} \end{cases}$$

$$= \begin{cases} 0 & , \text{ if } n \text{ is odd} \\ \frac{(-1)^{3n/2}}{(n!)^2} . {}^nC_{n/2} x^n , & \text{if } n \text{ is even} \end{cases}$$

$$= \begin{cases} 0 & , \text{ if } n \text{ is odd} \\ \frac{(-1)^{3n/2}}{n!\left(\frac{n}{2}!\right)^2} . x^n , & \text{if } n \text{ is even} \end{cases} \qquad \left[\because \ {}^nC_{n/2} = \frac{n!}{\left(\frac{n}{2}\right)^2}\right]$$

$\therefore$ For all $x \in R$, $\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) = \sum_{n=0}^{\infty} c_n$.

$$\Rightarrow \left(1+\frac{x}{1^2}+\frac{x^2}{(2!)^2}+\frac{x^3}{(3!)^2}+\ldots\ldots\right)\left(1-\frac{x}{1^2}+\frac{x^2}{(2!)^2}-\frac{x^3}{(3!)^2}+\ldots\ldots\right)$$

$$= 1 - \frac{x^2}{1^2 . 2!} + \frac{x^4}{(2!)^2 4!} - \frac{x^6}{(3!)^2 6!} + \ldots\ldots$$

Note. $\quad (1-x^2)^n = (1+x)^n (1-x)^n = (x+1)^n (1-x)^n$

$$= [C_0 x^n + C_1 x^{n-1} + C_2 x^{n-2} + \ldots\ldots + C_n] [C_0 - C_1 x + C_2 x^2 - \ldots\ldots + (-1)^n C_n x^n]$$

$\therefore \quad C_0^2 - C_1^2 + C_2^2 - \ldots\ldots + (-1)^n C_n^2$ is the co-efficient of x^n on R.H.S.

Also $x^n = (x^2)^{n/2}$, so that the term containing x^n in $(1-x^2)^n$ will be $\left(\frac{n}{2}+1\right)$ th

$$T_{\frac{n}{2}+1} \text{ of } (1-x^2)^n = {}^nC_{n/2} . (-x^2)^{n/2} = (-1)^{n/2} . {}^nC_{n/2} . x^n$$

$\therefore \quad C_0^2 - C_1^2 + C_2^2 - \ldots\ldots + (-1)^n C_n^2 = (-1)^{n/2} . {}^nC_{n/2}$.

Moreover, the term containing x^n occurs in the expansion of $(1-x^2)^n$ only when $n/2 \in N$ *i.e.*, only when n is even.

Example 21. *If for* $| x | < 1$, *the series* $a_0 + a_1 x + a_2 x^2 + \ldots\ldots + a_n x^n + \ldots\ldots$ *is absolutely convergent to* $A(x)$, *then show that*

$$(1-x)^{-1} A(x) = \sum_{n=0}^{\infty} S_n x^n, \text{ where } S_n = a_0 + a_1 + \ldots\ldots + a_n.$$

Hence, show that $\sum_{n=0}^{\infty} (n+1) x^n = (1-x)^{-2}$.

Sol. The geometric series $\sum_{n=0}^{\infty} x^n$ converges absolutely for $|x| < 1$ and has sum $= \frac{1}{1-x}$ $= (1-x)^{-1}$.

Thus $$\sum_{n=0}^{\infty} x^n = (1-x)^{-1}$$

Also, for $|x| < 1$, the series $a_0 = a_1 x + a_2 x^2 + \ldots = \sum_{n=0}^{\infty} a_n x^n$ converges absolutely and has sum $A(x)$.

$$\therefore \quad \sum_{n=0}^{\infty} a_n x^n = A(x).$$

By Cauchy's theorem, the Cauchy product $\sum_{n=0}^{\infty} c_n$ of $\sum_{n=0}^{\infty} x^n$ and $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $|x| < 1$ and

$$= \left(\sum_{n=0}^{\infty} x^n\right)\left(\sum_{n=0}^{\infty} a_n x^n\right) = \sum_{n=0}^{\infty} c_n \quad \textit{i.e.,} \quad (1-x)^{-1} . A(x) = \sum_{n=0}^{\infty} c_n \qquad \ldots(1)$$

Now $c_0 = 1 \times a_0 = a_0$ and for $n \geq 1$,

$$c_n = 1 . a_n x^n + x . a_{n-1} x^{n-1} + x^2 . a_{n-2} x^{n-2} + \ldots + x^n . a_0$$
$$= (a_0 + a_1 + \ldots + a_n) x^n = S_n x^n$$

$$\therefore \text{ From (1), for } |x| < 1, (1-x)^{-1} . A(x) = \sum_{n=0}^{\infty} S_n x^n \qquad \ldots(2)$$

If $a_n = 1 \ \forall \ n \geq 0$, then $S_n = 1 + 1 + \ldots + 1 = n + 1$

and $$A(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} = (1-x)^{-1}$$

$$\therefore \text{ From (2), we have } \sum_{n=0}^{\infty} (n+1)x^n = (1-x)^{-1}(1-x)^{-1} = (1-x)^{-2}.$$

Example 22. *If $|x| < 1$, show that* $\frac{1}{1-x} \log \frac{1}{1-x} = \sum_{n=1}^{\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right) x^n$.

Sol. For $|x| < 1$, we know that

$$x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots = \sum_{n=1}^{\infty} \frac{x^n}{n} \quad \text{and} \quad 1 + x + x^2 + \ldots = \sum_{n=1}^{\infty} x^{n-1}$$

converge absolutely to $-\log(1-x) = \log \frac{1}{1-x}$ and $\frac{1}{1-x}$ respectively.

$\therefore$ By Cauchy's theorem, the Cauchy product $\sum_{n=1}^{\infty} c_n$ of $\sum_{n=1}^{\infty} \frac{x^n}{n}$ and $\sum_{n=1}^{\infty} x^{n-1}$ converges absolutely for $|x| < 1$ and $\left(\sum_{n=1}^{\infty} \frac{x^n}{n}\right)\left(\sum_{n=1}^{\infty} x^n\right) = \sum_{n=1}^{\infty} c_n$. $\qquad \ldots(1)$

Now $$c_n = x . x^{n-1} + \frac{x^2}{2} . x^{n-2} + \frac{x^3}{3} . x^{n-3} + \ldots + \frac{x^n}{n} . 1 = \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right) x^n$$

$$\therefore \text{ From (1), we have } \frac{1}{1-x} \log \frac{1}{1-x} = \sum_{n=1}^{\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right) x^n.$$

8.11. INFINITE PRODUCTS

Definitions. If $< a_n >$ is a sequence, then the product $a_1 a_2 a_3 \ldots a_n \ldots$ is called an **infinite product** and is denoted by $\prod_{n=1}^{\infty} a_n$, or simply by $\prod a_n$.

a_n is called the **nth factor** of the product. The product of first n terms of the sequence is called the nth partial product and is denoted by P_n.

Thus $$P_n = a_1 a_2 a_3 \ldots a_n = \prod_{r=1}^{\infty} a_r.$$

The sequence $< P_n >$ is called the sequence of partial products of the sequence $< a_n >$.

8.12. CONVERGENCE OF INFINITE PRODUCTS

Let $P_n = \prod_{r=1}^{\infty} a_r$ be the nth partial product of the infinite product $\prod_{n=1}^{\infty} a_n$.

(*i*) If no factor a_n is zero, then the product $\prod_{n=1}^{\infty} a_n$ converges if the sequence $< P_n >$ converges to a **non-zero** finite number P (say), *i.e.,* if $\lim_{n \to \infty} P_n = P$

P is called the value of the product and we write $\prod_{n=1}^{\infty} a_n = P$.

If $\lim_{n \to \infty} P_n = \infty$, then the product $\prod_{n=1}^{\infty} a_n$ is said to diverge to ∞.

If $\lim_{n \to \infty} P_n = 0$, then the product $\prod_{n=1}^{\infty} a_n$ is said to diverge to 0.

(*ii*) If infinitely many factors a_n are zero, then the product $\prod_{n=1}^{\infty} a_n$ is said to diverge to 0.

(*iii*) If finitely many factors a_n are zero, then the product $\prod_{n=1}^{\infty} a_n$ is said to converge if it converges when the zero factors are removed.

(*iv*) If a finite number of factors are negative, then there exists a positive integer m such that $a_n > 0 \ \forall \ n > m$ and the product $\prod_{n=1}^{\infty} a_n$ is said to converge if the product $\prod_{n=m+1}^{\infty} a_n$ converges, since

$$\prod_{n=1}^{\infty} a_n = a_1 a_2 \ldots\ldots a_m = \prod_{n=m+1}^{\infty} a_n.$$

(v) If the sequence $< P_n >$ oscillates, then the product $\prod_{n=1}^{\infty} a_n$ is said to oscillate.

Note 1. It is usually convenient to write the factors of the infinite product as $1 + a_n$ instead of a_n.

Thus an infinite product is usually written as $\prod_{n=1}^{\infty} (1 + a_n)$ and $P_n = \prod_{r=1}^{\infty} (1 + a_r)$.

Note 2. We shall assume throughout the chapter that $a_n > -1$ *i.e.*, $1 + a_n > 0 \ \forall \ n$ so that log $(1 + a_n)$ is defined for all n.

Note 3. For $a_n > -1$, let P_n denote the nth partial product of $\prod_{n=1}^{\infty} (1 + a_n)$, then

$$P_n = (1 + a_1)(1 + a_2) \ldots\ldots (1 + a_n)$$

$$\Rightarrow \quad \log P_n = \log (1 + a_1) + \log (1 + a_2) + \ldots\ldots + \log (1 + a_n) = S_n$$

where $S_n = \sum_{r=1}^{n} \log (1 + a_r)$ is the nth partial sum of the series $\sum_{n=1}^{\infty} \log (1 + a_n)$.

$$\Rightarrow \quad P_n = e^{S_n}$$

If $\quad \lim_{n\to\infty} S_n = S$, then $\lim_{n\to\infty} P_n = e^S$

Thus, to say that the product $\prod_{n=1}^{\infty} (1 + a_n)$ diverges to 0, *i.e.*, $\lim_{n\to\infty} P_n = 0$ is equivalent to saying that the series $\sum_{n=1}^{\infty} \log (1 + a_n)$ diverges to $-\infty$, *i.e.*, $\lim_{n\to\infty} S_n = -\infty$.

ILLUSTRATIVE EXAMPLES

Example 1. *Show that the infinite product* $\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right) \ldots$ *converges to* $\frac{1}{2}$.

Sol. The given infinite product is

$$\prod^{\infty}\left(1 - \frac{1}{(n+1)^2}\right) = \prod_{n=1}^{\infty} \frac{(n+1)^2 - 1}{(n+1)^2} = \prod_{n=1}^{\infty} \frac{n(n+2)}{(n+1)^2} = \prod_{n=1}^{\infty}\left(\frac{n}{n+1} \cdot \frac{n+2}{n+1}\right)$$

$$\therefore \quad P_n = \left(\frac{1}{2} \cdot \frac{3}{2}\right)\left(\frac{2}{3} \cdot \frac{4}{3}\right)\left(\frac{3}{4} \cdot \frac{5}{4}\right) \ldots\ldots \left(\frac{n}{n+1} \cdot \frac{n+2}{n+1}\right)$$

$$= \left(\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \ldots\ldots \frac{n}{n+1}\right)\left(\frac{3}{2} \cdot \frac{4}{3} \cdot \frac{5}{4} \ldots\ldots \frac{n+2}{n+1}\right)$$

$$= \frac{1}{n+1} \cdot \frac{n+2}{2} = \frac{1}{2}\left(1 + \frac{1}{n+1}\right)$$

$$\Rightarrow \quad \lim_{n\to\infty} P_n = \tfrac{1}{2}$$

Hence the given infinite product converges to $\frac{1}{2}$ *i.e.*, $\prod_{n=1}^{\infty}\left(1 - \frac{1}{(n+1)^2}\right) = \frac{1}{2}$.

Example 2. *Show that the infinite product* $\left(1 - \frac{2}{2.3}\right)\left(1 - \frac{2}{3.4}\right)\left(1 - \frac{2}{4.5}\right) \ldots$ *converges to* $\frac{1}{3}$.

Sol. The given infinite product is

$$\prod_{n=1}^{\infty}\left[1 - \frac{2}{(n+1)(n+2)}\right] = \prod_{n=1}^{\infty} \frac{(n^2 + 3n + 2) - 2}{(n+1)(n+2)} = \prod_{n=1}^{\infty} \frac{n(n+3)}{(n+1)(n+2)}$$

$$\therefore \quad P_n = \left(\frac{1.4}{2.3}\right)\left(\frac{2.5}{3.4}\right)\left(\frac{3.6}{4.5}\right) \ldots\ldots \left(\frac{n}{n+1} \cdot \frac{n+3}{n+2}\right)$$

$$= \left(\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \ldots\ldots \frac{n}{n+1}\right)\left(\frac{4}{3} \cdot \frac{5}{4} \cdot \frac{6}{5} \ldots\ldots \frac{n+3}{n+2}\right) = \frac{1}{n+1} \cdot \frac{n+3}{3} = \frac{1}{3}\left(1 + \frac{2}{n+1}\right)$$

$$\Rightarrow \quad \lim_{n\to\infty} P_n = \tfrac{1}{3}$$

Hence the given infinite product converges to $\frac{1}{3}$ *i.e.*, $\prod_{n=1}^{\infty}\left[1 - \frac{2}{(n+1)(n+2)}\right] = \frac{1}{3}$.

Example 3. *Show that the infinite products*

(i) $\prod_{n=1}^{\infty}\left(1 + \frac{1}{n}\right)$ *and*   (ii) $\prod_{n=2}^{\infty}\left(1 - \frac{1}{n}\right)$ *are both divergent.*

Sol. (i) The given infinite product is $\prod_{n=1}^{\infty}\left(1 + \frac{1}{n}\right) = \prod_{n=1}^{\infty}\left(\frac{n+1}{n}\right)$

$$\therefore \quad P_n = \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \ldots\ldots \frac{n+1}{n} = n + 1 \ \Rightarrow \ \lim_{n\to\infty} P_n = \infty$$

$\therefore$ The given infinite product diverges to ∞.

(ii) The given infinite product is $\prod_{n=2}^{\infty}\left(1 - \frac{1}{n}\right) = \prod_{n=2}^{\infty}\left(\frac{n-1}{n}\right)$

$$\therefore \quad P_n = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \ldots\ldots \frac{n}{n+1} = \frac{1}{n+1} \ \Rightarrow \ \lim_{n\to\infty} P_n = 0$$

$\therefore$ The given infinite product diverges to 0.

Example 4. *Show that the infinite product* $\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right) \ldots\ldots$ *is convergent.*

Sol. Let $\quad P = \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right) \ldots\ldots$ to ∞

then $\quad \log P = \log\left(1 - \frac{1}{2^2}\right) + \log\left(1 - \frac{1}{3^2}\right) + \log\left(1 - \frac{1}{4^2}\right) + \ldots\ldots$ to ∞

$$= \sum_{n=2}^{\infty} \log\left(1-\frac{1}{n^2}\right) = \sum_{n=2}^{\infty} a_n \text{ (say).} \qquad \ldots(1)$$

Now $\quad a_n = \log\left(1-\frac{1}{n^2}\right) = -\left[\frac{1}{n^2}+\frac{1}{2n^4}+\frac{1}{3n^6}+\ldots\ldots\right]$

since $\log(1-x) = -\left(x+\frac{x^2}{2}+\frac{x^3}{3}+\ldots\ldots\right)$

$$= -\frac{1}{n^2}\left[1+\frac{1}{2n^2}+\frac{1}{3n^4}+\ldots\ldots\right]$$

Comparing with the convergent series $\sum_{n=2}^{\infty} \frac{1}{n^2}$, we find that $\sum_{n=2}^{\infty} a_n$ is convergent, *i.e.*,

$\sum_{n=2}^{\infty} a_n$ = a finite number S (say)

$\therefore$ From (1), log P = a finite number S when $n \to \infty$

$\Rightarrow$ P = *a* finite number e^S when $n \to \infty$

$\therefore$ The given infinite product is convergent.

Example 5. *Show that the infinite product* $\left(1+\frac{1}{1^3}\right)\left(1+\frac{1}{2^3}\right)\left(1+\frac{1}{3^3}\right)$ ...... *is convergent.*

Sol. Please try yourself.

Example 6. *Show that the infinite product* $\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{3}{4}\cdot\frac{5}{4}\cdot\frac{5}{6}\cdot\frac{7}{6}\ldots\ldots\frac{2n-1}{2n}\cdot\frac{2n+1}{2n}\ldots\ldots$ *tends to a finite limit as* $n \to \infty$.

Sol. Let $\quad P = \left(\frac{1}{2}\cdot\frac{3}{2}\right)\cdot\left(\frac{3}{4}\cdot\frac{5}{4}\right)\left(\frac{5}{6}\cdot\frac{7}{6}\right)\ldots\left(\frac{2n-1}{2n}\cdot\frac{2n+1}{2n}\right)\ldots$ to ∞

then $\quad \log P = \log\left(\frac{1}{2}\cdot\frac{3}{2}\right)+\log\left(\frac{3}{4}\cdot\frac{5}{4}\right)+\ldots\ldots+\log\left(\frac{2n-1}{2n}\cdot\frac{2n+1}{2n}\right)+\ldots\ldots$ to ∞

$$= \sum_{n=1}^{\infty} \log\left(\frac{2n-1}{2n}\cdot\frac{2n+1}{2n}\right) = \sum_{n=1}^{\infty} \log\left(\frac{4n^2-1}{4n^2}\right)$$

$$= \sum_{n=1}^{\infty} \log\left(1-\frac{1}{4n^2}\right) = \sum_{n=1}^{\infty} a_n \text{ (say)} \qquad \ldots(1)$$

Now $\quad a_n = \log\left(1-\frac{1}{4n^2}\right)$

$$= -\left[\frac{1}{4n^2}+\frac{1}{2}\left(\frac{1}{4n^2}\right)^2+\frac{1}{3}\left(\frac{1}{4n^2}\right)^3+\ldots\right] = -\frac{1}{n^2}\left[\frac{1}{4}+\frac{1}{32n^2}+\ldots\ldots\right]$$

Comparing with the convergent series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ we find that $\sum_{n=1}^{\infty} a_n$ is convergent, *i.e.*,

$\sum_{n=1}^{\infty} a_n$ = a finite number S, (say).

$\therefore$ From (1), log P = a finite number S when $n \to \infty$

$\Rightarrow$ P = a finite number e^S when $n \to \infty$.

Example 7. *Show that the infinite product* $\left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\left(1-\frac{1}{5}\right)$ ......*converges to 1.*

Sol. Let $\quad P = \left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\left(1-\frac{1}{5}\right)\ldots\ldots\left(1+\frac{1}{2n}\right)\left(1-\frac{1}{2n+1}\right)\ldots\ldots$ to ∞

then $\quad \log P = \log\left\{\left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\right\}+\log\left\{\left(1+\frac{1}{4}\right)\left(1-\frac{1}{5}\right)\right\}$

$$+\ldots+\log\left\{\left(1+\frac{1}{2n}\right)\left(1-\frac{1}{2n+1}\right)\right\}+\ldots\ldots \text{ to } \infty$$

$$= \sum_{n=1}^{\infty} \log\left\{\left(1+\frac{1}{2n}\right)\left(1-\frac{1}{2n+1}\right)\right\} = \sum_{n=1}^{\infty} \log\left\{1+\left(\frac{1}{2n}-\frac{1}{2n+1}\right)-\frac{1}{2n(2n+1)}\right\}$$

$$= \sum_{n=1}^{\infty} \log\left\{1+\frac{1}{2n(2n+1)}-\frac{1}{2n(2n+1)}\right\} = \sum_{n=1}^{\infty} \log 1 = 0 \Rightarrow \quad P = e^0 = 1.$$

Example 8. *Prove that the infinite product* $\prod_{n=1}^{\infty} \cos\frac{x}{2^n}$ *converges to* $\frac{\sin x}{x}$. *(x is an arbitrary fixed non-zero number).*

Sol. Here $\quad P_n = \cos\frac{x}{2}\cos\frac{x}{2^2}\ldots\ldots\cos\frac{x}{2^{n-1}}\cos\frac{x}{2^n}$

Multiplying both sides by $\sin\frac{x}{2^n}$ and applying sucessively the formula $\sin 2\theta = 2\sin\theta\cos\theta$, we have

$$P_n \cdot \sin\frac{x}{2^n} = \sin\frac{x}{2^n}\cos\frac{x}{2^n}\cos\frac{x}{2^{n-1}}\ldots\ldots\frac{x}{2^2}\cos\frac{x}{2}$$

$$= \frac{1}{2}\left(2\sin\frac{x}{2^n}\cos\frac{x}{2^n}\right)\cos\frac{x}{2^{n-1}}\ldots\ldots\cos\frac{x}{2}$$

$$= \frac{1}{2}\sin\frac{x}{2^{n-1}}\cos\frac{x}{2^{n-1}}\cos\frac{x}{2^{n-2}}\ldots\ldots\cos\frac{x}{2}$$

$$= \frac{1}{2^2}\left(2\sin\frac{x}{2^{n-1}}\cos\frac{x}{2^{n-1}}\right)\cos\frac{x}{2^{n-2}}\ldots\ldots\cos\frac{x}{2}$$

$$= \frac{1}{2^2}\sin\frac{x}{2^{n-2}}\cos\frac{x}{2^{n-2}}\ldots\ldots\cos\frac{x}{2}$$

....

....

$$= \frac{1}{2^n}\sin x$$

$\Rightarrow \quad P_n = \frac{\sin x}{x}\left(\frac{\frac{x}{2^n}}{\sin\frac{x}{2^n}}\right) \Rightarrow \lim_{n\to\infty} P_n = \frac{\sin x}{x}$ since $\left(\frac{\frac{x}{2^n}}{\sin\frac{x}{2^2}}\right) \to$ as $n \to \infty$

Hence $\prod_{n=1}^{\infty} \cos \frac{x}{2^n}$ converges to $\frac{\sin x}{x}$.

(If $x = 0$, then each factor is 1 and the product converges to 1).

☞ For simpler and better solutions to examples 4 and 5, see the next 'Illustrative Examples'.

8.13. THEOREM. (A necessary condition for convergence)

If the product $\prod_{n=1}^{\infty} (1 + a_n)$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

Proof. Suppose the product $\prod_{n=1}^{\infty} (1 + a_n)$ converges to P, then

$$P \neq 0 \quad \text{and} \quad \lim_{n\to\infty} P_n = P. \text{ Also } \lim_{n\to\infty} P_{n-1} = P$$

Now
$$\frac{P_n}{P_{n-1}} = \frac{(1+a_1)(1+a_2)\ldots\ldots(1+a_{n-1})(1+a_n)}{(1+a_1)(1+a_2)\ldots\ldots(1+a_{n-1})} = 1 + a_n$$

$$\therefore \quad \lim_{n\to\infty} (1 + a_n) = \lim_{n\to\infty} \frac{P_n}{P_{n-1}} = \frac{\lim_{n\to\infty} P_n}{\lim_{n\to\infty} P_{n-1}} = \frac{P}{P} = 1$$

Hence $\lim_{n\to\infty} a_n = 0$.

Note. The converse of above theorem need not be true.

Consider the infinite product $\prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)$.

Here $a_n = \frac{1}{n} \to 0$ as $n \to \infty$ but the product is divergent. [See example 3 (*i*)].

8.14. GENERAL PRINCIPLE OF CONVERGENCE OF AN INFINITE PRODUCT

Statement. *A necessary and sufficient condition for the convergence of the infinite product $\prod_{n=1}^{\infty} (1 + a_n)$ is that for every $\varepsilon > 0$, there exists a positive integer m such that*

$$\left|\frac{P_{n+p}}{P_n} - 1\right| < \varepsilon \quad \forall\, n \geq m,\, p \geq 1.$$

Proof. The condition is necessary.

Let $$P_n = \prod_{r=1}^{n} (1 + a_r)$$

Since the product $\prod_{n=1}^{\infty} (1 + a_n)$ is convergent, the sequence $< P_n >$ converges to a non-zero finite limit.

$\therefore$ There exists a positive number k such that $| P_n | > k \quad \forall\, n$ ...(1)

Also, since $< P_n >$ is convergent, by Cauchy's general principle of convergence, for every $\varepsilon > 0$, there exists a positive integer m such that

$$| P_{n+p} - P_n | < k\varepsilon \quad \forall\, n \geq m, p \geq 1 \qquad \ldots(2)$$

Now $$\left|\frac{P_{n+p}}{P_n} - 1\right| = \left|\frac{P_{n+p} - P_n}{P_n}\right| = \frac{|P_{n+p} - P_n|}{|P_n|} < \frac{k\varepsilon}{k} = \varepsilon \qquad \text{[by (1) and (2)]}$$

The condition is sufficient

For every $\varepsilon > 0$, let $\left|\frac{P_{n+p}}{P_n} - 1\right| < \varepsilon \quad \forall\, n \geq m, p \geq 1$

then $$1 - \varepsilon < \frac{P_{n+p}}{P_n} < 1 + \varepsilon \quad \forall\, n \geq m, p \geq 1$$

In particular, for $n = m$, we have $1 - \varepsilon < \frac{P_{m+p}}{P_m} < 1 + \varepsilon$

Since $\varepsilon > 0$ can be taken arbitrarily small.

$\therefore$ $| P_{m+p} | > (1 - \varepsilon) \, | P_m | , p \geq 1$

$\Rightarrow$ $| P_n |$ > some fixed positive number when $n \geq m$

$\Rightarrow$ **P_n cannot tend to zero.**

Also $(1 - \varepsilon) P_m < P_{m+p} < (1 + \varepsilon) P_m \; \forall\, p \geq 1$

But P_m is the product of a finite number of factors and is, therefore, finite.

$\Rightarrow$ $P_n \to$ *a* non-zero finite number as $n \to \infty$

$\Rightarrow$ The infinite product $\prod_{n=1}^{\infty} (1 + a_n)$ is convergent.

8.15. WEIERSTRASS'S INEQUALITIES

For any infinite product $\prod_{n=1}^{\infty} (1 + a_n)$, prove that

(*i*) $$\frac{1}{1 - \sum_1^n a_n} > \prod_1^n (1 + a_n) > 1 + \sum_1^n a_n$$

(*ii*) $$\frac{1}{1 + \sum_1^n a_n} > \prod_1^n (1 - a_n) > 1 - \sum_1^n a_n \text{ where } 0 < a_n < 1 \; \forall\, n.$$

Proof. Since $0 < a_n < 1$, we have

$$(1 + a_1)(1 + a_2) = 1 + (a_1 + a_2) + a_1 a_2 > 1 + (a_1 + a_2)$$
$$(1 + a_1)(1 + a_2)(1 + a_3) > [1 + (a_1 + a_2)](1 + a_3) > 1 + (a_1 + a_2 + a_3)$$

Proceeding in this manner, we shall have

$$(1 + a_1)(1 + a_2) \ldots\ldots (1 + a_n) > 1 + (a_1 + a_2 + \ldots\ldots + a_n)$$

$$\therefore \quad \prod_1^n (1 + a_n) > 1 + \sum_1^n a_n \qquad \ldots(1)$$

Also
$$(1 - a_1)(1 - a_2) = 1 - (a_1 + a_2) + a_1 a_2 > 1 - (a_1 + a_2)$$
$$(1 - a_1)(1 - a_2)(1 - a_3) > [1 - (a_1 + a_2)](1 - a_3) > 1 - (a_1 + a_2 + a_3)$$

Proceeding in this manner, we shall have

$$(1-a_1)(1-a_2)\ldots\ldots(1-a_n) > 1-(a_1+a_2+\ldots\ldots+a_n)$$

$$\therefore \qquad \prod_1^n (1-a_n) > 1-\sum_1^n a_n \qquad \ldots(2)$$

Now $\qquad 1+a_1 = \dfrac{1-a_1^2}{1-a_1} < \dfrac{1}{1-a_1}$, since $1-a_1^2 < 1$

$$1+a_2 < \frac{1}{1-a_2} \text{ and so on.}$$

$$\therefore \quad (1+a_1)(1+a_2)\ldots\ldots(1+a_n) < \frac{1}{(1-a_1)(1-a_2)\ldots\ldots(1-a_n)} < \frac{1}{1-(a_1+a_2+\ldots\ldots a_n)}$$

or
$$\prod_1^n (1+a_n) < \frac{1}{1-\sum_1^n a_n} \qquad \ldots(3)$$

Similarly, $\qquad 1-a_1 = \dfrac{1-a_1^2}{1+a_1} < \dfrac{1}{1+a_1}$

$$1-a_2 < \frac{1}{1+a_2} \text{ and so on.}$$

$$\therefore \quad (1-a_1)(1-a_2)\ldots\ldots(1-a_n) < \frac{1}{(1+a_1)(1+a_2)\ldots\ldots(1+a_n)} < \frac{1}{1+(a_1+a_2+\ldots\ldots+a_n)}$$

or
$$\prod_1^n (1-a_n) < \frac{1}{1+\sum_1^n a_n} \qquad \ldots(4)$$

From (1), (2), (3) and (4), we have

$$\frac{1}{1-\sum_1^n a_n} > \prod_1^n (1+a_n) > 1+\sum_1^n a_n$$

and
$$\frac{1}{1+\sum_1^n a_n} > \prod_1^n (1-a_n) > 1-\sum_1^n a_n.$$

Note. In order to establish the convergence of an infinite product by means of an infinite series, we prove three theorems when

(*i*) $a_n \geq 0$ $\qquad$ (*ii*) $-1 < a_n \leq 0$ $\qquad$ (*iii*) a_n may be of either sign.

8.16. THEOREM I

If $a_n \geq 0$, then the series $\sum_{n=1}^{\infty} a_n$ and the product $\sum_{n=1}^{\infty} (1+a_n)$ converge or diverge together.

Proof. We know that $e^x = 1+\dfrac{x}{1!}+\dfrac{x^2}{2!}+\ldots\ldots \geq 1+x$ for $x \geq 0$

$$\therefore \qquad 1+x \leq e^x \quad \text{for } x \geq 0$$

Putting $x = a_1, a_2, \ldots\ldots, a_n$ in succession and multiplying, we have

$$(1+a_1)(1+a_2)\ldots\ldots(1+a_n) \leq e^{a_1}\cdot e^{a_2}\ldots\ldots e^{a_n} = e^{a_1+a_2+\ldots\ldots+a_n}$$

$$\therefore \qquad a_1+a_2+\ldots\ldots+a_n < (1+a_1)(1+a_2)\ldots\ldots(1+a_n) \leq e^{a_1+a_2+\ldots\ldots+a_n}$$

$$\Rightarrow \qquad S_n < P_n \leq e^{S_n}$$

where $\qquad S_n = a_1+a_2+\ldots\ldots+a_n = \sum_{r=1}^{n} a_r \qquad \ldots(1)$

and $\qquad P_n = (1+a_1)(1+a_2)\ldots\ldots(1+a_n) = \prod_{r=1}^{n} (1+a_r)$

Since $a_n \geq 0$, the sequences $<S_n>$ and $<P_n>$ are monotonic increasing and they have a limit :

finite, $\qquad$ if bounded above

and $\qquad$ infinite, $\qquad$ if unbounded above.

They cannot oscillate.

(*i*) If $\sum_{n=1}^{\infty} a_n$ is convergent, then the sequence $<S_n>$ is convergent *i.e.*, $S_n \to$ a finite number S (say).

$\therefore$ From (1), $P_n \leq e^{S_n}$ $\Rightarrow$ $<P_n>$ is bounded above.

$\Rightarrow$ $<P_n>$ and hence the product $\prod_{n=1}^{\infty} (1+a_n)$ converges.

(*ii*) If $\sum_{n=1}^{\infty} a_n$ is divergent, then the sequence $<S_n>$ is divergent *i.e.*, $S_n \to \infty$.

$\therefore$ From (1), $P_n \leq e^{S_n}$ $\Rightarrow$ $<P_n>$ is unbounded above.

$\Rightarrow$ $<P_n>$ and hence the product $\prod_{n=1}^{\infty} (1+a_n)$ diverges to ∞.

(*iii*) If $\prod_{n=1}^{\infty} (1+a_n)$ is convergent, then the sequence $<P_n>$ converges to a non-zero finite number P (say).

$\therefore$ From (1), $S_n < P_n$ $\Rightarrow$ $<S_n>$ is bounded above.

$\Rightarrow$ $<S_n>$ and hence the series $\sum_{n=1}^{\infty} a_n$ converges.

(*iv*) If $\prod_{n=1}^{\infty} (1+a_n)$ is divergent, then the sequence $<P_n>$ is divergent *i.e.*, $P_n \to \infty$

$\therefore$ From (1), $S_n < P_n$ $\Rightarrow$ $<S_n>$ is unbounded above.

$\Rightarrow$ $<S_n>$ and hence the series $\sum_{n=1}^{\infty} a_n$ diverges to ∞.

Hence the series $\sum_{n=1}^{\infty} a_n$ and the product $\prod_{n=1}^{\infty} (1+a_n)$ converge or diverge together.

Theorem II. *If $-1 < a_n \le 0$, then the series $\sum_{n=1}^{\infty} a_n$ and the product $\prod_{n=1}^{\infty} (1+a_n)$ converge or diverge together.*

Proof. Let $b_n = -a_n$ so that $0 \le b_n < 1$

Since $1 - x \le e^{-x}$ for $0 \le x < 1$

$\therefore$ Putting $x = b_1, b_2, \ldots\ldots, b_n$ in succession and mulitplying, we have

$$0 < (1-b_1)(1-b_2) \ldots\ldots (1-b_n) \le e^{-b_1} \cdot e^{-b_2} \ldots\ldots e^{-b_n}$$

$$\Rightarrow \quad 0 < (1+a_1)(1+a_2) \ldots\ldots (1+a_n) \le e^{-(b_1+b_2+\ldots\ldots+b_n)}$$

$$\Rightarrow \quad 0 < (1+a_1)(1+a_2) \ldots\ldots (1+a_n) \le e^{a_1+a_2+\ldots\ldots+a_n}$$

$$\Rightarrow \quad 0 < P_n \le e^{S_n} \qquad \ldots(1)$$

where $S_n = a_1 + a_2 + \ldots\ldots + a_n = \sum_{r=1}^{n} a_r$ and $P_n = (1+a_1)(1+a_2) \ldots\ldots (1+a_n) = \prod_{r=1}^{n} (1+a_r)$

Now, if $\sum_{n=1}^{\infty} a_n$ diverges, it will diverge to $-\infty$ so that $S_n \to -\infty$.

From (1), $P_n \to 0$. Hence the product $\prod_{n=1}^{\infty} (1+a_n)$ diverges to zero.

Next, let $\sum_{n=1}^{\infty} a_n$ be convergent, then $\sum_{n=1}^{\infty} b_n$ is also convergent.

$\therefore$ for each $\varepsilon > 0$, $\exists$ a positive integer m such that

$$0 \le |\, b_{m+1} + b_{m+2} + \ldots\ldots + b_n \,| < \varepsilon \ \forall\ n > m$$

or $\quad b_{m+1} + b_{m+2} + \ldots\ldots + b_n < \varepsilon \ \forall\ n > m \qquad \ldots(2)$

But
$$(1-b_{m+1})(1-b_{m+2}) = 1 - (b_{m+1} + b_{m+2}) + b_{m+1}\, b_{m+2}$$
$$\ge 1 - (b_{m+1} + b_{m+2})$$
$$(1-b_{m+1})(1-b_{m+2})(1-b_{m+3}) \ge [1-(b_{m+1}+b_{m+2})](1-b_{m+3})$$
$$\ge 1 - (b_{m+1} + b_{m+2} + b_{m+3})$$

....

$\therefore$ for $n > m$, $(1-b_{m+1})(1-b_{m+2}) \ldots\ldots (1-b_n) \ge 1 - (b_{m+1} + b_{m+2} + \ldots\ldots + b_n) > 1 - \varepsilon$ [using (2)]

$$\Rightarrow \quad \frac{P_n}{P_m} > 1 - \varepsilon \qquad \ldots(3)$$

Now each factor of $\frac{P_n}{P_m}$ is less than unity $\qquad (\because\ b_n < 1\ \forall\ n)$

$$\therefore \quad \frac{P_n}{P_m} < 1 \ \Rightarrow\ P_n < P_m \ \forall\ n > m$$

$\therefore$ the sequence $< \frac{P_n}{P_m} >$ is a monotonically decreasing sequence and is bounded below.

$\Rightarrow$ P_n tends to a non-zero finite limit and, hence, the product $\prod_{n=1}^{\infty} (1-b_n)$ or $\prod_{n=1}^{\infty} (1+a_n)$ is convergent.

Remark. The above theorem may be stated as :

If $0 \le b_n < 1$, then $\prod_{n=1}^{\infty} (1-b_n)$ converges to a non-zero finite limit if $\sum_{n=1}^{\infty} b_n$ converges and diverges to zero if $\sum_{n=1}^{\infty} b_n$ diverges.

Theorem III. *If the series $\sum_{n=1}^{\infty} a_n^2$ is convergent, then the product $\prod_{n=1}^{\infty} (1+a_n)$ and the series $\sum_{n=1}^{\infty} a_n$ converge or diverge together.*

Proof. Since $\sum_{n=1}^{\infty} a_n^2$ is convergent, $\therefore$ $a_n^2 \to 0$ as $n \to \infty$. $\Rightarrow$ $a_n \to 0$ as $n \to \infty$.

$\therefore$ Taking $\varepsilon = \frac{1}{2}$, there exists a positive integer m such that $|\, a_n \,| < \frac{1}{2}\ \forall\ n > m$

$\therefore$ for $n > m$,

$$|\, a_n - \log(1+a_n) \,| = \left| a_n - \left(a_n - \frac{a_n^2}{2} + \frac{a_n^3}{3} - \frac{a_n^4}{4} + \ldots\ldots \right) \right| = \left| \frac{a_n^2}{2} - \frac{a_n^3}{3} + \frac{a_n^4}{4} - \ldots\ldots \right|$$

$$\le \left| \frac{a_n^2}{2} \right| + \left| \frac{a_n^3}{3} \right| + \left| \frac{a_n^4}{4} \right| + \ldots\ldots \le \tfrac{1}{2} |a_n^2| + \tfrac{1}{2} |a_n^3| + \tfrac{1}{2} |a_n^4| + \ldots\ldots$$

$$= \tfrac{1}{2} |\, a_n \,|^2 (1 + |\, a_n \,| + |\, a_n \,|^2 + \ldots\ldots) = \frac{1}{2} a_n^2 \left(\frac{1}{1 - |\, a_n \,|} \right) < a_n^2$$

$$\left[\because\ |a_n| < \frac{1}{2} \ \Rightarrow\ 1 - |a_n| > \frac{1}{2} \ \Rightarrow\ \frac{1}{1 - |a_n|} < 2 \right]$$

Since the series $\sum_{n=1}^{\infty} a_n^2$ is convergent, therefore, by comparison test, the series $\sum_{n=1}^{\infty} |\, a_n - \log(1+a_n) \,|$ is also convergent.

$$\Rightarrow \quad \sum_{n=1}^{\infty} (a_n - \log(1+a_n)) \text{ is convergent.}$$

$\Rightarrow \sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} \log(1+a_n)$ converge or **diverge together.**

$\Rightarrow \sum_{n=1}^{\infty} a_n$ and $\prod_{n=1}^{\infty} (1+a_n)$ converge or diverge together.

Remark. The above theorem gives an easily applied test for the convergence of an infinite product in which a_n may be of either sign.

Note. If $\sum_{n=1}^{\infty} a_n^2$ is convergent, then we have proved that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} \log(1+a_n)$ converge or diverge together.

Also $\sum_{n=1}^{\infty} \log(1+a_n)$ and $\prod_{n=1}^{\infty} (1+a_n)$ converge or diverge together.

$\therefore$ We have

(*i*) if $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} \log(1+a_n)$ converges and therefore $\prod_{n=1}^{\infty} (1+a_n)$ converges.

(*ii*) if $\sum_{n=1}^{\infty} a_n$ diverges to ∞, then $\sum_{n=1}^{\infty} \log(1+a_n)$ diverges to ∞ and therefore $\prod_{n=1}^{\infty} (1+a_n)$ diverges to ∞.

(*iii*) if $\sum_{n=1}^{\infty} a_n$ diverges to $-\infty$, then $\sum_{n=1}^{\infty} \log(1+a_n)$ diverges to $-\infty$ and therefore $\prod_{n=1}^{\infty} (1+a_n)$ diverges to zero.

Also, if $\sum_{n=1}^{\infty} a_n^2$ diverges and $\sum_{n=1}^{\infty} a_n$ converges or oscillates finitely, then $\prod_{n=1}^{\infty} (1+a_n)$ diverges to zero.

8.17. ABSOLUTE CONVERGENCE OF INFINITE PRODUCTS

The product $\prod_{n=1}^{\infty} (1+a_n)$ is said to be absolutely convergent if the product $\prod_{n=1}^{\infty} (1+|a_n|)$ is convergent.

Theorem I. *The product $\prod_{n=1}^{\infty} (1+a_n)$ is absolutely convergent if and only if the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.*

Proof. The product $\prod_{n=1}^{\infty} (1+a_n)$ is absolutely convergent.

$\Leftrightarrow$ The product $\prod_{n=1}^{\infty} (1+|a_n|)$ is convergent.

$\Leftrightarrow$ The series $\sum_{n=1}^{\infty} |a_n|$ is convergent.

$\Leftrightarrow$ The series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

Theorem II. *The product $\prod_{n=1}^{\infty} (1+a_n)$ is absolutely convergent if and only if the series $\sum_{n=1}^{\infty} \log(1+a_n)$ is absolutely convergent.*

Proof. The condition is necessary. –

Let $\prod_{n=1}^{\infty} (1+a_n)$ be absolutely convergent, then $\prod_{n=1}^{\infty} (1+|a_n|)$ is convergent.

$\Rightarrow \sum_{n=1}^{\infty} |a_n|$ is convergent.

$\Rightarrow |a_n| \to 0$ as $n \to \infty \Rightarrow a_n \to 0$ as $n \to \infty$

$\therefore$ Taking $\varepsilon = \frac{1}{2}$, $\exists$ a positive integer m such that $|a_n| < \frac{1}{2}\ \forall\ n > m$.

Now, for $n > m$, we have $\log(1+a_n) = a_n - \frac{1}{2}a_n^2 + \frac{1}{3}a_n^3 - \frac{1}{4}a_n^4 + \ldots$

$$\Rightarrow \left|\frac{\log(1+a_n)}{a_n} - 1\right| = \left|-\frac{1}{2}a_n + \frac{1}{3}a_n^2 - \frac{1}{4}a_n^3 + \ldots\right|$$

$$\le \tfrac{1}{2}|a_n| + \tfrac{1}{3}|a_n|^2 + \tfrac{1}{4}|a_n|^3 + \ldots \le \tfrac{1}{2}|a_n| + \tfrac{1}{2}|a_n|^2 + \tfrac{1}{2}|a_n|^3 + \ldots$$

$$\le \tfrac{1}{2}\cdot\tfrac{1}{2} + \tfrac{1}{2}(\tfrac{1}{2})^2 + \tfrac{1}{2}(\tfrac{1}{2})^3 + \ldots = \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \ldots = \frac{\frac{1}{2^2}}{1-\frac{1}{2}} = \frac{1}{2}$$

$$\Rightarrow \left|\frac{\log(1+a_n)}{a_n} - 1\right| < \frac{1}{2} \Rightarrow 1 - \frac{1}{2} < \frac{\log(1+a_n)}{a_n} < 1 + \frac{1}{2}$$

$$\Rightarrow \frac{1}{2} < \frac{\log(1+a_n)}{a_n} < \frac{3}{2} \Rightarrow \frac{1}{2} < \left|\frac{\log(1+a_n)}{a_n}\right| < \frac{3}{2}$$

$[\because\ |x| = x \text{ if } x > 0]$

$$\Rightarrow \frac{1}{2}|a_n| < |\log(1+a_n)| < \frac{3}{2}|a_n|\ \forall\ n > m$$

Since $\sum_{n=1}^{\infty} |a_n|$ is convergent, it follows that $\sum_{n=1}^{\infty} |\log(1+a_n)|$ is convergent *i.e.*, $\sum_{n=1}^{\infty} \log(1+a_n)$ is absolutely convergent.

The condition is sufficient

Let $\sum_{n=1}^{\infty} \log(1+a_n)$ be absolutely convergent, then $\sum_{n=1}^{\infty} |\log(1+a_n)|$ is convergent.

$\Rightarrow \log(1+a_n) \to 0$ as $n \to \infty \Rightarrow 1 + a_n \to 1$ as $n \to \infty \Rightarrow a_n \to 0$ as $n \to \infty$

Proceeding as in the necessary part, we have $|a_n| < 2|\log(1+a_n)|\ \forall\, n > m$

Since $\sum_{n=1}^{\infty} |\log(1+a_n)|$ is convergent, it follows that $\sum_{n=1}^{\infty} |a_n|$ is convergent.

$\Rightarrow \prod_{n=1}^{\infty} (1+|a_n|)$ is convergent.

$\Rightarrow \prod_{n=1}^{\infty} (1+a_n)$ is absolutely convergent.

Theorem III. *Every absolutely convergent infinite product is convergent.*

Proof. Let $\prod_{n=1}^{\infty} (1+a_n)$ be absolutely convergent, then $\prod_{n=1}^{\infty} (1+|a_n|)$ is convergent.

$\therefore$ Given $\varepsilon > 0$, there exists a positive integer m such that

$$\left|\frac{P_{n+p}}{P_n} - 1\right| < \varepsilon \quad \forall\, n \ge m,\ p \ge 1$$

where P_n is the nth partial product of $\prod_{n=1}^{\infty} (1+|a_n|)$

$\Rightarrow \quad |(1+|a_{n+1}|)(1+|a_{n+2}|)\ \ldots\ (1+|a_{n+p}|) - 1| < \varepsilon \ \forall\, n \ge m,\ p \ge 1$

Since $|(1+a_{n+1})(1+a_{n+2})\ \ldots\ (1+a_{n+p}) - 1|$

$\le |(1+|a_{n+1}|)(1+|a_{n+2}|)\ \ldots\ (1+|a_{n+p}|) - 1| \quad [\because\ x \le |x|]$

$\therefore \quad |(1+a_{n+1})(1+a_{n+2})\ \ldots\ (1+a_{n+p}) - 1| < \varepsilon\ \forall\, n \ge m,\ p \ge 1$

$\Rightarrow \quad \left|\frac{P_{n+p}}{P_n} - 1\right| < \varepsilon\ \forall\, n \ge m,\ p \ge 1$ where P_n is the nth partial product of $\prod_{n=1}^{\infty}(1+a_n)$.

$\Rightarrow \prod_{n=1}^{\infty} (1+a_n)$ is convergent.

Note. The factors of an absolutely convergent infinite product may be rearranged in any order without affecting its convergence.

ILLUSTRATIVE EXAMPLES

Example 1. *Discuss the convergence of the infinite products :*

(i) $\prod_{n=1}^{\infty}\left(1+\frac{1}{n^2}\right)$ (ii) $\prod_{n=1}^{\infty}\left(1+\frac{1}{n^{3/2}}\right)$ (iii) $\prod_{n=1}^{\infty}\left(1+\frac{1}{n^{\alpha}}\right),\ \alpha > 1$

(iv) $\prod_{n=1}^{\infty}\left(1+\frac{1}{n^{\alpha}}\right),\ 0 < \alpha \le 1$ (v) $\prod_{n=1}^{\infty}\left(1+\frac{1}{\sqrt{n}}\right)$ (vi) $\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)$.

Sol. (i) The given product is $\prod_{n=1}^{\infty}\left(1+\frac{1}{n^2}\right) = \prod_{n=1}^{\infty}(1+a_n)$ where $a_n = \frac{1}{n^2} > 0$.

$\therefore$ The product $\prod_{n=1}^{\infty}(1+a_n)$ and the series $\sum_{n=1}^{\infty} a_n$ converge or diverge together.

But the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent. $(\because\ p = 2 > 1)$

$\therefore$ The given product is convergent.

(ii) *Please try yourself.* **[Ans.** Convergent]

(iii) The given product is $\prod_{n=1}^{\infty}\left(1+\frac{1}{n^{\alpha}}\right) = \prod_{n=1}^{\infty}(1+a_n)$ where $a_n = \frac{1}{n^{\alpha}} > 0$

$\therefore$ The product $\prod_{n=1}^{\infty}(1+a_n)$ and the series $\sum_{n=1}^{\infty} a_n$ converge or diverge together.

But the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}$ is convergent since $\alpha > 1$.

$\therefore$ The given product is convergent.

(iv) The given product is $\prod_{n=1}^{\infty}\left(1+\frac{1}{n^{\alpha}}\right) = \prod_{n=1}^{\infty}(1+a_n)$ where $a_n = \frac{1}{n^{\alpha}} > 0$.

$\therefore$ The product $\prod_{n=1}^{\infty}(1+a_n)$ and the series $\sum_{n=1}^{\infty} a_n$ converge or diverge together.

But the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}$ diverges to ∞ since $0 < \alpha \le 1$.

$\therefore$ the given product diverges to ∞.

(v) *Please try yourself.* **[Ans.** Divergent]

(vi) *Please try yourself.* **[Ans.** Divergent]

Example 2. *Discuss the convergence of the infinite products :*

(i) $\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right)$ (ii) $\prod_{n=2}^{\infty}\left(1-\frac{1}{n}\right)$

(iii) $\prod_{n=2}^{\infty}\left(1-\frac{1}{\sqrt{n}}\right)$ (iv) $\frac{3}{4}\cdot\frac{6}{7}\cdot\frac{9}{10}\ \ldots\ \frac{3n}{3n+1}\ \ldots$

Sol. (i) The given product is

$$\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right) = \prod_{n=2}^{\infty}(1-b_n) \text{ where } b_n = \frac{1}{n^2} \text{ and } n \ge 2 \text{ so that } 0 < b_n < 1.$$

$\therefore$ The product $\prod_{n=2}^{\infty}(1-b_n)$ and the series $\prod_{n=2}^{\infty} b_n$ converge or diverge together.

But the series $\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} \frac{1}{n^2}$ is convergent.

$\therefore$ the given product is convergent.

(ii) The given product is $\prod_{n=2}^{\infty}\left(1-\frac{1}{n}\right)=\prod_{n=2}^{\infty}(1-b_n)$ where $b_n=\frac{1}{n}$ and $n\geq 2$ so that $0<b_n<1$.

$\therefore$ The product $\prod_{n=2}^{\infty}(1-b_n)$ and the series $\sum_{n=2}^{\infty} b_n$ converge or diverge together.

But the series $\sum_{n=2}^{\infty} b_n=\sum_{n=2}^{\infty}\frac{1}{n}$ diverges to ∞.

$\therefore$ the given product diverges to zero.

(iii) *Please try yourself.* **[Ans.** Diverges to 0]

(iv) The given product is $\frac{3}{4}\cdot\frac{6}{7}\cdot\frac{9}{10}\cdots\cdots\frac{3n}{3n+1}\cdots$

$$=\left(1-\frac{1}{4}\right)\left(1-\frac{1}{7}\right)\left(1-\frac{1}{10}\right)\cdots\cdots\left(1-\frac{1}{3n+1}\right)\cdots\cdots$$

$$=\prod_{n=1}^{\infty}\left(1-\frac{1}{3n+1}\right)=\prod_{n=1}^{\infty}(1-b_n)$$

where $b_n=\frac{1}{3n+1}$ and $n\geq 1$ so that $0<b_n<1$.

$\therefore$ The product $\sum_{n=1}^{\infty}(1-b_n)$ and the series $\sum_{n=1}^{\infty} b_n$ converge or diverge together.

But the series $\sum_{n=1}^{\infty} b_n=\sum_{n=1}^{\infty}\frac{1}{3n+1}$ diverges to ∞.

$\therefore$ the given product diverges to zero.

Example 3. *Discuss the convergence of the infinite products :*

(i) $\prod_{n=1}^{\infty}\left(1+\sin^2\frac{\theta}{n}\right)$ (ii) $\prod_{n=1}^{\infty}\left(1+n\sin\frac{\theta}{n^2}\right)$

(iii) $\prod_{n=1}^{\infty}\left(1+\frac{x}{n^p}\right)$ *where x is a real number.*

Sol. (i) The given product is $\prod_{n=1}^{\infty}\left(1+\sin^2\frac{\theta}{n}\right)=\prod_{n=1}^{\infty}(1+a_n)$ where $a_n=\sin^2\frac{\theta}{n}\geq 0$

$\therefore$ The product $\prod_{n=1}^{\infty}(1+a_n)$ and the series $\sum_{n=1}^{\infty} a_n$ converge or diverge together.

Now
$$a_n=\sin^2\frac{\theta}{n}=\left(\frac{\theta}{n}-\frac{1}{3!}\cdot\frac{\theta^3}{n^3}+\frac{1}{5!}\cdot\frac{\theta^5}{n^5}-\cdots\cdots\right)^2$$

$$=\frac{\theta^2}{n^2}-\frac{2}{3!}\cdot\frac{\theta^4}{n^4}+\cdots\cdots=\frac{1}{n^2}\left(\theta^2-\frac{2}{3!}\cdot\frac{\theta^4}{n^2}+\cdots\cdots\right)$$

Take $b_n=\frac{1}{n^2}$, $\lim_{n\to\infty}\frac{a_n}{b_n}=\theta^2$

Since $\sum_{n=1}^{\infty} b_n$ is convergent, therefore, by comparison test, $\sum_{n=1}^{\infty} a_n$ is convergent.

Hence the given product is convergent.

(ii) The given product is $\prod_{n=1}^{\infty}\left(1+n\sin\frac{\theta}{n^2}\right)=\prod_{n=1}^{\infty}(1+a_n)$ where $a_n=n\sin\frac{\theta}{n^2}$

Now
$$a_n=n\sin\frac{\theta}{n^2}=n\left[\frac{\theta}{n^2}-\frac{1}{3!}\cdot\frac{\theta^3}{n^6}+\cdots\cdots\right]=\frac{1}{n}\left[\theta-\frac{1}{3!}\cdot\frac{\theta^3}{n^4}+\cdots\cdots\right]$$

Take $b_n=\frac{1}{n}$, $\lim_{n\to\infty}\frac{a_n}{b_n}=\theta$

If $\theta\neq 0$, then by comparison test, $\sum_{n=1}^{\infty} a_n$ is divergent since $\sum_{n=1}^{\infty} b_n$ is divergent.

If $\theta=0$, the product is obviously convergent.

(iii) The given product is $\prod_{n=1}^{\infty}\left(1+\frac{x}{n^p}\right)=\prod_{n=1}^{\infty}(1+a_n)$ where $a_n=\frac{x}{n^p}$

Now $\sum_{n=1}^{\infty} a_n=\sum_{n=1}^{\infty}\frac{x}{n^p}=x\sum_{n=1}^{\infty}\frac{1}{n^p}$ which is convergent if $p>1$ and divergent if $p\leq 1$.

Hence the given product is also convergent if $p>1$ and divergent if $p\leq 1$.

Example 4. *Show that* $(1+x)\left(1+\frac{x}{2}\right)\left(1+\frac{x}{3}\right)\cdots\cdots$ *diverges to* ∞ *or to 0 according as* $x>0$ *or* $x<0$.

Sol. The given product is $\prod_{n=1}^{\infty}\left(1+\frac{x}{n}\right)=\prod_{n=1}^{\infty}(1+a_n)$ where $a_n=\frac{x}{n}$.

Now $\sum_{n=1}^{\infty} a_n=\sum_{n=1}^{\infty}\frac{x}{n}=x\sum_{n=1}^{\infty}\frac{1}{n}$ which diverges to $+\infty$ if $x>0$ and $-\infty$ if $x<0$.

Hence the given product diverges to ∞ if $x>0$ and 0 if $x<0$.

Example 5. *Discuss the convergence of the products :*

(i) $\prod_{n=2}^{\infty}\left(1+\frac{(-1)^n}{n}\right)$ (ii) $\left(1-\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\left(1+\frac{1}{5}\right)\cdots\cdots$

(iii) $\left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\left(1-\frac{1}{5}\right)\cdots\cdots$ (iv) $\left(1+\frac{1}{\sqrt{2}}\right)\left(1-\frac{1}{\sqrt{3}}\right)\left(1+\frac{1}{\sqrt{4}}\right)\left(1-\frac{1}{\sqrt{5}}\right)$

(v) $\prod_{n=1}^{\infty}\left(1+\frac{(-1)^{n-1}}{\sqrt{n}}\right)$.

Sol. (i) The given product is $\prod_{n=2}^{\infty}\left(1+\frac{(-1)^n}{n}\right)=\prod_{n=2}^{\infty}(1+a_n)$ where $a_n=\frac{(-1)^n}{n}$.

Now $\sum_{n=2}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is convergent by Leibnitz's test and $\sum_{n=2}^{\infty} a_n^2 = \sum_{n=2}^{\infty} \frac{1}{n^2}$ is also convergent.

$\therefore$ The given product is convergent.

(*ii*) The given product is $\left(1-\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\left(1+\frac{1}{5}\right)$ ......

$$= \prod_{n=2}^{\infty} \left(1+\frac{(-1)^{n-1}}{n}\right) = \prod_{n=2}^{\infty} (1+a_n) \quad \text{where } a_n = \frac{(-1)^{n-1}}{n}$$

Now see part (*i*). **[Ans.** Convergent]

(*iii*) The given product is $\prod_{n=2}^{\infty} \left(1+\frac{(-1)^n}{n}\right)$

Now see part (*i*). **[Ans.** Convergent]

(*iv*) The given product is $\left(1+\frac{1}{\sqrt{2}}\right)\left(1-\frac{1}{\sqrt{3}}\right)\left(1+\frac{1}{\sqrt{4}}\right)\left(1-\frac{1}{\sqrt{5}}\right)$ ......

$$= \prod_{n=2}^{\infty} \left(1+\frac{(-1)^n}{\sqrt{n}}\right) = \prod_{n=2}^{\infty} (1+a_n) \quad \text{where } a_n = \frac{(-1)^n}{\sqrt{n}}$$

Now $\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ is convergent by Leibnitz's test but $\sum_{n=2}^{\infty} a_n^2 = \sum_{n=2}^{\infty} \frac{1}{n}$ is divergent.

$\therefore$ The given product diverges to 0.

(*v*) *Please try yourself.* **[Ans.** Diverges to 0]

Example 6. *Discuss the convergence of the infinite product*

$$\left(1-\frac{1}{2}\right)\left(1+\frac{1}{2}\right)\left(1-\frac{1}{2}\right)\left(1+\frac{1}{2}\right)......$$

Sol. The given product is $\prod_{n=1}^{\infty} \left(1+(-1)^n \cdot \frac{1}{2}\right) = \prod_{n=1}^{\infty} (1+a_n)$ where $a_n = (-1)^n \cdot \frac{1}{2}$

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n \cdot \tfrac{1}{2} = \tfrac{1}{2} \sum_{n=1}^{\infty} (-1)^n = \tfrac{1}{2}(-1+1-1+1-......)$$

which oscillates between $-\frac{1}{2}$ and 0.

But $\sum_{n=1}^{\infty} a_n^2 = \sum_{n=1}^{\infty} \frac{1}{4} = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + ...$ which is divergent.

Hence the given product diverges to zero.

Example 7. *Discuss the convergence of* $\prod_{n=1}^{\infty} (1+(-1)^n)$.

Sol. Here $a_n = (-1)^n$

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n = -1+1-1+1- ......$$ which oscillates between -1 and 0.

But $\sum_{n=1}^{\infty} a_n^2 = \sum_{n=1}^{\infty} 1 = 1+1+1+ .....$ diverges to ∞.

Hence the given product diverges to zero.

Example 8. *Discuss the convergence of* $\frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} ...... \frac{2n-2}{2n-3} \cdot \frac{2n-2}{2n-1} \cdot \frac{2n}{2n-1}$ .....

Sol. The given product is $\prod_{n=1}^{\infty} (1+a_n)$

$$= (1+1)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1+\frac{1}{5}\right)......\left(1+\frac{1}{2n-3}\right)\left(1-\frac{1}{2n-1}\right)\left(1+\frac{1}{2n-1}\right)......$$

$\therefore$ $\sum_{n=1}^{\infty} a_n = 1 - \frac{1}{3} + \frac{1}{3} - \frac{1}{5} + \frac{1}{5} - ......$ which is convergent.

Also $\sum_{n=1}^{\infty} a_n^2 = 1 + \frac{1}{3^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{5^2} + ...$ which is convergent.

Hence the given product is convergent.

Example 9. *Show that the infinite product* $\prod_{n=2}^{\infty} \left(1+\frac{(-1)^n}{n^\alpha}\right)$ *is convergent if* $\alpha > \frac{1}{2}$.

Sol. The given product is $\prod_{n=2}^{\infty} \left(1+\frac{(-1)^n}{n^\alpha}\right) = \prod_{n=2}^{\infty} (1+a_n)$ where $a_n = \frac{(-1)^n}{n^\alpha}$.

Now $\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} \frac{(-1)^n}{n^\alpha}$ converges if $\alpha > 0$ (by Leibnitz's test).

Also $\sum_{n=2}^{\infty} a_n^2 = \sum_{n=2}^{\infty} \frac{1}{n^{2\alpha}}$ converges if $2\alpha > 1$ *i.e.*, if $\alpha > \frac{1}{2}$.

Hence the given product converges if $\alpha > \frac{1}{2}$.

Example 10. *Discuss the convergence of the product* $\prod_{n=1}^{\infty}\left[1+\left(\frac{nx}{n+1}\right)^n\right]$.

Sol. Here $a_n = \left(\frac{nx}{n+1}\right)^n$ $\therefore$ $(a_n)^{1/n} = \frac{nx}{n+1} = \frac{x}{1+\frac{1}{n}}$

$$\lim_{n\to\infty} (a_n)^{1/n} = x$$

$\therefore$ By Cauchy's root test, the series $\sum_{n=1}^{n} a_n$ is convergent if $x < 1$ and divergent if $x > 1$.

Hence the given product is convergent if $x < 1$ and divergent if $x > 1$.

If $x = 1$, then $\quad a_n = \left(\dfrac{n}{n+1}\right)^n = \dfrac{1}{\left(1+\dfrac{1}{n}\right)^n} \to \dfrac{1}{e}$ as $n \to \infty$.

$\therefore \displaystyle\sum_{n=1}^{\infty} a_n$ is divergent. Hence $\displaystyle\prod_{n=1}^{\infty} (1+a_n)$ is divergent.

Thus the given product is convergent if $x < 1$ and divergent if $x \geq 1$.

Example 11. *Discuss the convergence of the product*

$$\left[1+\left(\frac{x}{2}\right)^1\right]\left[1+\left(\frac{2x}{3}\right)^2\right]\left[1+\left(\frac{3x}{4}\right)^3\right] \ldots\ldots$$

Sol. The given product is $\displaystyle\prod_{n=1}^{\infty}\left[1+\left(\frac{nx}{n+1}\right)^n\right]$.

Now see example 10.

Example 12. *Show that the infinite product*

$e^{1+1^0} . e^{-1-1/2} . e^{1+1/3} . e^{-1-1/4}$ ..... *oscillates between 2 and 2e.*

Sol. Let $\quad P = e^{1+1} . e^{-1-1/2} . e^{1+1/3} . e^{-1-1/4} \ldots\ldots$

then $\quad \log P = (1+1) - (1+\frac{1}{2}) + (1+\frac{1}{3}) - (1+\frac{1}{4}) + \ldots\ldots$

which oscillates between log 2 and 1 + log 2, *i.e.*, between log 2 and log (2e).

$\therefore$ P oscillates between 2 and 2e.

Example 13. *Test for absolute convergence the following infinite products :*

(*i*) $\displaystyle\prod_{n=1}^{\infty} \cos\frac{\theta}{n}$ $\qquad$ (*ii*) $\displaystyle\prod_{n=1}^{\infty}\left[\frac{\sin\frac{x}{n}}{\frac{x}{n}}\right]$.

Sol. (*i*) Here $\quad 1 + a_n = \cos\dfrac{\theta}{n} = 1 - \dfrac{1}{2!}\cdot\dfrac{\theta^2}{n^2} + \dfrac{1}{4!}\cdot\dfrac{\theta^4}{n^4} - \ldots\ldots$

$$\Rightarrow \quad a_n = -\frac{1}{2!}\cdot\frac{\theta^2}{n^2} + \frac{1}{4!}\cdot\frac{\theta^4}{n^4} - \ldots\ldots = \frac{1}{n^2}\left(-\frac{\theta^2}{2!} + \frac{\theta^4}{4!\,n^2} - \ldots\ldots\right)$$

$$\Rightarrow \quad |a_n| = \frac{1}{n^2}\left|-\frac{\theta^2}{2!} + \frac{\theta^4}{4!\,n^2} - \ldots\ldots\right|$$

Comparing $\sum |a_n|$ with $\sum \dfrac{1}{n^2}$, we have $\displaystyle\lim_{n\to\infty}\frac{|a_n|}{\frac{1}{n^2}} = \frac{\theta^2}{2}$, a finite quantity.

But $\sum \dfrac{1}{n^2}$ is convergent, therefore, $\sum |a_n|$ is convergent.

$\Rightarrow$ Σa_n is absolutely convergent.

$\Rightarrow \quad \displaystyle\prod_{n=1}^{\infty}(1+a_n) = \prod_{n=1}^{\infty}\cos\frac{\theta}{n}$ is absolutely convergent for all values of θ.

(*ii*) Here $\quad 1 + a_n = \dfrac{\sin\frac{x}{n}}{\frac{x}{n}} = \dfrac{1}{\frac{x}{n}}\left[\dfrac{x}{n} - \dfrac{1}{3!}\cdot\dfrac{x^3}{n^3} + \dfrac{1}{5!}\cdot\dfrac{x^5}{n^5} - \ldots\ldots\right] = 1 - \dfrac{1}{3!}\cdot\dfrac{x^2}{n^2} + \dfrac{1}{5!}\cdot\dfrac{x^4}{n^4} - \ldots\ldots$

$$\Rightarrow \quad a_n = -\frac{1}{3!}\cdot\frac{x^2}{n^2} + \frac{1}{5!}\cdot\frac{x^4}{n^4} - \ldots\ldots = \frac{1}{n^2}\left(-\frac{x^3}{3!} + \frac{x^4}{5!\,n^2} - \ldots\ldots\right)$$

$$\Rightarrow \quad |a_n| = \frac{1}{n^2}\left|-\frac{x^2}{3!} + \frac{x^4}{5!\,n^2} - \ldots\ldots\right|$$

Comparing $\sum |a_n|$ with $\sum \dfrac{1}{n^2}$, we have $\displaystyle\lim_{n\to\infty}\frac{|a_n|}{\frac{1}{n^2}} = \frac{x^2}{3!}$, a finite quantity.

But $\sum \dfrac{1}{n^2}$ is convergent, therefore $\sum |a_n|$ is convergent.

$\Rightarrow$ Σa_n is absolutely convergent.

$\Rightarrow \quad \displaystyle\prod_{n=1}^{\infty}(1+a_n) = \prod_{n=1}^{\infty}\left[\frac{\sin\frac{x}{n}}{x/n}\right]$ is absolutely convergent for all values of x.

Example 14. *Prove that* $\displaystyle\prod_{n=1}^{\infty}\left(1+\frac{x}{n}\right)e^{-x/n}$ *is absolutely convergent for any real x.*

Sol. Here $\quad 1 + a_n = \left(1+\dfrac{x}{n}\right)e^{-x/n}$

$$= \left(1+\frac{x}{n}\right)\left(1-\frac{x}{n}+\frac{x^2}{2!\,n^2}-\frac{x^3}{3!\,n^3}+\ldots\ldots\right) = 1 - \frac{x^2}{n^2} + \frac{x^2}{2n^2} + \frac{x^3}{2n^3} - \frac{x^3}{6n^3} + \ldots\ldots$$

$$\Rightarrow \quad a_n = -\frac{x^2}{2n^2} + \frac{x^3}{2n^3} - \ldots\ldots = \frac{1}{n^2}\left(-\frac{x^2}{2} + \frac{x^3}{3n}\ldots\ldots\right)$$

$$\Rightarrow \quad |a_n| = \frac{1}{n^2}\left|-\frac{x^2}{2} + \frac{x^3}{3n}\ldots\ldots\right|$$

Comparing $\Sigma |a_n|$ with $\sum \dfrac{1}{n^2}$, we have $\displaystyle\lim_{n\to\infty}\frac{|a_n|}{\frac{1}{n^2}} = \frac{x^2}{2}$, a finite quantity.

But $\sum \dfrac{1}{n^2}$ is convergent, therefore, $\Sigma |a_n|$ is convergent.

$\Rightarrow$ Σa_n is absolutely convergent.

$\Rightarrow \quad \displaystyle\prod_{n=1}^{\infty}(1+a_n) = \prod_{n=1}^{\infty}\left(1+\frac{x}{n}\right)e^{-\frac{x}{n}}$ is absolutely convergent for all values of x.

Example 15. *Prove that* $\displaystyle\prod_{n=1}^{\infty}\left(1+\frac{x}{n\pi}\right)e^{-\frac{x}{n\pi}}$ *is absolutely convergent for all values of x.*

Sol. *Please try yourself.*

Example 16. *Show that* $\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)e^{-1/n}$ *is absolutely convergent.*

Sol. *Please try yourself.*

Example 17. *Show that* $\prod_{n=1}^{\infty}\left(1-\frac{1}{n^{2/3}}\right)e^{\frac{1}{n^{2/3}}}$ *is absolutely convergent.*

Sol. Here $$1+a_n=\left(1-\frac{1}{n^{2/3}}\right)e^{\frac{1}{n^{2/3}}}=\left(1-\frac{1}{n^{2/3}}\right)\left(1+\frac{1}{n^{2/3}}+\frac{1}{2!\,n^{4/3}}+\frac{1}{3!\,n^{6/3}}+\ldots\right)$$

$$=1-\frac{1}{n^{4/3}}+\frac{1}{2n^{4/3}}+\frac{1}{6n^{6/3}}-\frac{1}{2n^{6/3}}+\ldots$$

$$\Rightarrow \quad a_n=-\frac{1}{2n^{4/3}}-\frac{1}{3n^{6/3}}\ldots\ldots=\frac{1}{n^{4/3}}\left(-\frac{1}{2}-\frac{1}{3n^{2/3}}\ldots\right)$$

$$\Rightarrow \quad |a_n|=\frac{1}{n^{4/3}}\left|-\frac{1}{2}-\frac{1}{3n^{2/3}}\ldots\right|$$

Comparing $\sum|a_n|$ with $\sum\frac{1}{n^{4/3}}$, we have $\lim_{n\to\infty}\frac{|a_n|}{\frac{1}{n^{4/3}}}=\frac{1}{2}$, a finite quantity.

But $\sum\frac{1}{n^{4/3}}$ is convergent, therefore, $\sum|a_n|$ is convergent.

$\Rightarrow$ Σa_n is absolutely convergent.

$$\Rightarrow \quad \prod_{n=1}^{\infty}(1+a_n)=\prod_{n=1}^{\infty}\left(1-\frac{1}{n^{2/3}}\right)e^{\frac{1}{n^{2/3}}} \text{ is absolutely convergent.}$$

Example 18. *Test the convergence of the infinite product* $\prod_{n=1}^{\infty}\left(\frac{x+x^{2n}}{1+x^{2n}}\right)$.

Sol. Here $$1+a_n=\frac{x+x^{2n}}{1+x^{2n}}=\frac{(1+x^{2n})+(x-1)}{1+x^{2n}}$$

$$=1+\frac{x-1}{1+x^{2n}} \quad\Rightarrow\quad a_n=\frac{x-1}{1+x^{2n}}.$$

Now, when $|x|>1$, we have $|a_n|=\left|\frac{x-1}{1+x^{2n}}\right|=\frac{|x-1|}{1+x^{2n}}<\frac{|x-1|}{x^{2n}}$

and the series $\sum\frac{1}{x^{2n}}$ is convergent.

$\Rightarrow$ The series $\Sigma|a_n|$ is convergent. $\Rightarrow$ Σa_n is absolutely convergent.

Hence $\prod_{n=1}^{\infty}(1+a_n)$ is absolutely convergent.

If $|x|<1$, $\quad 1+a_n=\frac{x(1+x^{2n-1})}{1+x^{2n}}\to x$

so that a_n does not tend to zero. Hence the product is divergent.

If $x=1$, every factor is unity and hence the product is convergent.
If $x=-1$, every factor is zero and $a_n\to-1\neq0$.
Hence the product is divergent.

Example 19. *Discuss the convergence of the infinite product* $\prod_{n=1}^{\infty}\left(1+\frac{x^n}{x^{2n}+1}\right)$.

Sol. Here $$1+a_n=1+\frac{x^n}{x^{2n}+1} \text{ so that } a_n=\frac{x^n}{x^{2n}+1}$$

$$\left|\frac{a_n}{a_{n+1}}\right|=\left|\frac{x^n}{x^{2n}+1}\cdot\frac{x^{2n+2}+1}{x^{n+1}}\right|=\left|\frac{x^{2n+2}+1}{x(x^{2n}+1)}\right|$$

$\therefore$ If $|x|<1$, $\lim_{n\to\infty}\left|\frac{a}{a_{n+1}}\right|=\frac{1}{|x|}>1$

By ratio test, $\Sigma|a_n|$ converges and hence $\Pi(1+a_n)$ converges absolutely.

If $|x|>1$, $$\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|=\lim_{n\to\infty}\left|\frac{x^{2n+2}+1}{x^{2n+1}+x}\right|=\lim_{n\to\infty}\left|\frac{x+\frac{1}{x^{2n+1}}}{1+\frac{1}{x^{2n}}}\right|=|x|>1$$

By ratio test, $\Sigma|a_n|$ converges and hence $\Pi(1+a_n)$ converges absolutely.
If $x=1$, $a_n=\frac{1}{2}$ so that the series Σa_n is divergent and hence $\Pi(1+a_n)$ is also divergent.
If $x=-1$, the product becomes $(1-\frac{1}{2})(1+\frac{1}{2})(1-\frac{1}{2})(1+\frac{1}{2})$ ...... which diverges to zero.
(See example 6)

Example 20. *Show that* $\prod_{n=2}^{\infty}\left[1-\left(1-\frac{1}{n}\right)^{-n}x^{-n}\right]$ *converges absolutely for* $|x|>1$.

Sol. Here $$a_n=\left(1-\frac{1}{n}\right)^{-n}.x^{-n}$$

$$\frac{a_n}{a_{n+1}}=\frac{\left(1-\frac{1}{n}\right)^{-n}x^{-n}}{\left(1-\frac{1}{n+1}\right)^{-(n+1)}.x^{-(n+1)}}=\frac{\left(1-\frac{1}{n+1}\right)^{n+1}}{\left(1-\frac{1}{n}\right)^{n}}x$$

$$\therefore \quad \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|=\frac{e}{e}|x|=|x|>1$$

By ratio test, $\Sigma|a_n|$ is convergent. Hence the infinite product converges absolutely.

Example 21. *Show that* $\prod_{n=0}^{\infty}(1+x^{2^n})$ *converges to* $\frac{1}{1-x}$ *if* $|x|<1$.

Sol. $$P_n=\prod_{0}^{n-1}(1+x^{2^n})=(1+x)(1+x^2)(1+x^{2^2})\ldots\ldots(1+x^{2^{n-1}})$$

$$=\frac{1}{1-x}[(1-x)(1+x)(1+x^2)(1+x^{2^2})\ldots\ldots(1+x^{2^{n-1}})]$$

$$= \frac{1}{1-x}\,[(1-x^2)(1+x^2)(1+x^{2^2})\;......\;(1+x^{2^{n-1}})]$$

$$= \frac{1}{1-x}\,[(1-x^{2^2})(1+x^{2^2})\;......\;(1+x^{2^{n-1}})]$$

....

....

$$= \frac{1}{1-x}\,(1-x^{2^n})$$

If $|x| < 1$, $x^{2^n} \to 0$ as $n \to \infty \Rightarrow$ $P_n \to \frac{1}{1-x}$ as $n \to \infty$

Hence the infinite product $\sum_{n=0}^{\infty} (1 + x^{2^n})$ converges to $\frac{1}{1-x}$.

Example 22. *Show that* $\prod_{n=0}^{\infty}\left[1+\left(\frac{1}{2}\right)^{2^n}\right]$ *converges to 2.*

Sol. *Please try yourself.* [**Hint.** Here $x = \frac{1}{2}$]

9

Riemann Integration

9.1. INTRODUCTION

In elementary calculus, the process of integration is treated as the inverse operation of differentiation and the integral of a function is called an anti-derivative. Historically, however, the subject of integration was developed is connection with areas of plane regions. It was based on the concept of the limit of a sum when the number of terms in the sum tends to infinity and each term tends to zero. This notion of integral as summation is associated with intuitive dependence on geometrical concepts. The first satisfactory rigorous arithmetic treatment of definite integral was given by a German mathematician George Friedrich Bernhard Riemann (1826—1866). Many refinements and generalisations of the subject have appeared since then. However, we shall confine ourselves to the discussion of Riemann integration only.

We shall be dealing with closed finite intervals $[a, b]$ so that $(b - a) \in \mathrm{R}$ and $x \in [a, b]$ implies $a \le x \le b$. Moreover, all functions f will be assumed to be real valued function defined and bounded on $[a, b]$.

Thus $f : [a, b] \to \mathrm{R}$ and $|f(x)| \le k$, where k is a positive real number.

9.2. PARTITION OF A CLOSED INTERVAL

Let $\mathrm{I} = [a, b]$ be a finite closed interval. If $a = x_0 < x_1 < x_2 < ...... < x_n = b$, then the finite ordered set $\mathrm{P} = \{x_0, x_1, x_2, ..., x_n\}$ is called a partition of I.

The $n + 1$ points $x_0, x_1, x_2, ..., x_n$ are called partition points of P.

The n closed sub-intervals $\mathrm{I}_1 = [x_0, x_1]$, $\mathrm{I}_2 = [x_1, x_2]$, ..., $\mathrm{I}_r = [x_{r-1}, x_r]$, ..., $\mathrm{I}_n = [x_{n-1}, x_n]$ determined by P are called the segments of the partition P.

Clearly $\bigcup_{r=1}^{n} \mathrm{I}_r = \bigcup_{r=1}^{n} [x_{r-1}, x_r] = [a,b] = \mathrm{I}$

The length of the rth sub-interval $\mathrm{I}_r = [x_{r-1}, x_r]$ is denoted by δ_r. Thus $\delta_r = x_r - x_{r-1}$.

Note 1. Partition is also known as dissection or net.

Note 2. By changing the partition points, the partition can be changed and hence, there can be an infinite number of partitions of the interval I.

We shall denote by $\mathrm{P}[a, b]$ the set (or family) of all partitions of $[a, b]$.

9.3. NORM OF A PARTITION

The maximum of the lengths of the sub-intervals of a partition P is called the **norm or mesh** of the partition P and is denoted by $\|\mathrm{P}\|$ or μ (P).

Thus $\|\mathrm{P}\| = \max. \{\delta_r : r = 1, 2, ..., n\}$

$= \max. (x_r - x_{r-1} : r = 1, 2, ..., n\}$.

9.4. REFINEMENT OF A PARTITION

If P, P′ be two partitions of $[a, b]$ and $P \subset P'$, then the partition P′ is called a refinement of partition P on $[a, b]$. We also say P′ is finer than P.

Thus, if P′ is finer than P, then every point of P is used in the construction of P′ and P′ has at least one additional point.

If P_1, P_2 are two partitions of $[a, b]$, then $P_1 \subset P_1 \cup P_2$ and $P_2 \subset P_1 \cup P_2$. Therefore, $P_1 \cup P_2$ is called a **common refinement** of P_1 and P_2.

Note 1. If $P_1, P_2 \in P[a, b]$ and $P_1 \subset P_2$, then $\|P_2\| \le \|P_1\|$.

Note 2. If $P = \{x_0, x_1, x_2, ..., x_n\}$ is a partition of $[a, b]$,

then $$\sum_{r=1}^{n} \delta_r = \delta_1 + \delta_2 + ... + \delta_n = (x_1 - x_0) + (x_2 - x_1) + ... + (x_n - x_{n-1}) = x_n - x_0 = b - a.$$

9.5. UPPER AND LOWER DARBOUX SUMS

Let $f : [a, b] \to R$ be a bounded function and $P = \{a = x_0, x_1, x_2, ..., x_n = b\}$ be a partition of $[a, b]$.

Since f is bounded on $[a, b]$, f is also bounded on each of the sub-intervals. Let M, m be the supermum and infimum of f in $[a, b]$ and M_r, m_r be the supermum and infimum of f in the rth sub-interval $\quad I_r = [x_{r-1}, x_r]$; $r = 1, 2, ..., n$.

The sum $M_1\delta_1 + M_2\delta_2 + ... + M_r\delta_r + ... + M_n\delta_n = \sum_{r=1}^{n} M_r\delta_r$ is called the **upper Darboux sum of f** corresponding to the partition P and is denoted by U(P, f) or U(f, P).—

The sum $m_1\delta_1 + m_2\delta_2 + ... + m_r\delta_r + ... + m_n\delta_n = \sum_{r=1}^{n} m_r\delta_r$ is called the **lower Darboux sum of f** corresponding to the partition P and is denoted by L(P, f) or L(f, P).

Thus $$U(P, f) = \sum_{r=1}^{n} M_r\delta_r; \quad L(P, f) = \sum_{r=1}^{n} m_r\delta_r$$

Clearly, these sums depend upon the function f and the partition P, and do exist for every bounded function.

9.6. OSCILLATORY SUM

Let $f : [a, b] \to R$ be a bounded function and $P = \{a = x_0, x_1, x_2, ..., x_n = b\}$ be a partition of $[a, b]$.

Let m_r and M_r be the infimum and supermum of f on $I_r = [x_{r-1}, x_r]$, $r = 1, 2, 3, ..., n$. Then

$$U(P, f) - L(P, f) = \sum_{r=1}^{n} M_r\delta_r - \sum_{r=1}^{n} m_r\delta_r = \sum_{r=1}^{n} (M_r - m_r)\,\delta_r = \sum_{r=1}^{n} O_r\delta_r$$

where $O_r = M_r - m_r$ denotes the oscillation of f in I_r.

$U(P, f) - L(P, f) = \sum_{r=1}^{n} O_r\delta_r$ is called the oscillatory sum of f corresponding to the partition P and is denoted by $\omega(P, f)$.

Since $O_r = M_r - m_r \ge 0, r = 1, 2, ..., n$, each oscillatory sum consists of a finite number of non-negative terms.

$\therefore \quad \omega(P, f) \ge 0.$

Theorem 1. *If $f : [a, b] \to R$ is a bounded function and $P \in P[a, b]$, then*
$$m(b - a) \le L(P, f) \le U(P, f) \le M(b - a)$$
where m, M are the infimum and supermum of f on $[a, b]$.

Proof. Let $P = \{a = x_0, x_1, x_2, ..., x_n = b\}$ be a partition of $[a, b]$.

f is bounded on $[a, b]$ $\Rightarrow$ f is bounded on each sub-interval $[x_{r-1}, x_r]$, $r = 1, 2, ..., n$.

Let m_r and M_r be the infimum and supermum of f on $[x_{r-1}, x_r]$.

Clearly $\quad m \le m_r \le M_r \le M \Rightarrow m\delta_r \le m_r\delta_r \le M_r\delta_r \le M\delta_r$

$$\Rightarrow \quad \sum_{r=1}^{n} m\delta_r \le \sum_{r=1}^{n} m_r\delta_r \le \sum_{r=1}^{n} M_r\delta_r \le \sum_{r=1}^{n} M\delta_r$$

$$\Rightarrow \quad m\sum_{r=1}^{n} \delta_r \le L(P, f) \le U(P, f) \le M\sum_{r=1}^{n} \delta_r$$

$$\Rightarrow \quad m(b - a) \le L(P, f) \le U(P, f) \le M(b - a) \qquad \left[\because \sum_{r=1}^{n} \delta_r = b - a\right]$$

Note. The above theorem implies that L(P, f) and U(P, f) are bounded if f is bounded.

Theorem 2. *If $f : [a, b] \to R$ is a bounded function and $P \in P(a, b)$, then*
(i) $L(P, f) \le U(P, f)$ $\qquad$ (ii) $L(P, -f) = -U(P, f)$ and $U(P, -f) = -L(P, f)$.

Proof. Let $P = \{a = x_0, x_1, x_2, ..., x_n = b\}$ be any partition of $[a, b]$.

Let m, M be the infimum and supermum of f on $[a, b]$ and m_r, M_r be the infimum and supermum of f on $I_r = [x_{r-1}, x_r]$, $r = 1, 2, ..., n$.

(i) We have $\quad m_r \le M_r, \quad r = 1, 2, ..., n$

$$\Rightarrow \quad m_r\delta_r \le M_r\delta_r \Rightarrow \sum_{r=1}^{n} m_r\delta_r \le \sum_{r=1}^{n} M_r\delta_r \Rightarrow L(P, f) \le U(P, f).$$

(ii) f is bounded on $[a, b]$ $\Rightarrow$ $-f$ is bounded on $[a, b]$. m_r, M_r are the infimum and supermum of f on I_r.

$\Rightarrow \; -M_r, -m_r$ are the infimum and supermum of $-f$ on I_r. **(Note this step)**

$$\therefore \text{ By definition, } \; L(P, -f) = \sum_{r=1}^{n} (-M_r)\delta_r = -\sum_{r=1}^{n} M_r\delta_r = -U(P, f)$$

$$U(P, -f) = \sum_{r=1}^{n} (-m_r)\,\delta_r = -\sum_{r=1}^{n} m_r\delta_r = -L(P, f).$$

Theorem 3. *If $f : [a, b] \to R, g : [a, b] \to R$ are bounded functions and $P \in P[a, b]$, then*
(i) $U(P, f + g) \le U(P, f) + U(P, g)$ $\qquad$ (ii) $L(P, f + g) \ge L(P, f) + L(P, g)$
(iii) $\omega(P, f + g) \le \omega(P, f) + \omega(P, g)$.

Proof. Let $P = \{a = x_0, x_1, x_2, ..., x_n = b\}$ be any partition of $[a, b]$.

f, g are bounded on $[a, b]$ $\Rightarrow$ $f + g$ is bounded on $[a, b]$.

Let m_r', M_r' be the infimum and supermum of f on I_r,
m_r'', M_r'' be the infimum and supermum of g on I_r and
m_r, M_r be the infimum and supermum of $f+g$ on I_r.

(i) M_r', M_r'' are supermum of f, g on I_r

$\Rightarrow \quad f(x) \le M_r', g(x) \le M_r'' \quad \forall x \in I_r$

$\Rightarrow \quad f(x) + g(x) \le M_r' + M_r'' \quad \forall x \in I_r$

$\Rightarrow \quad (f+g)(x) \le M_r' + M_r'' \quad \forall x \in I_r$

$\Rightarrow$ $M_r' + M_r''$ is an upper bound of $f+g$ on I_r

But M_r is the least upper bound of $f+g$ on I_r

$\therefore \quad M_r \le M_r' + M_r''$ on I_r, $r = 1, 2, \ldots\ldots, n \;\Rightarrow\; M_r\delta_r \le M_r'\delta_r + M_r''\delta_r$

$$\Rightarrow \quad \sum_{i=1}^{\infty} M_r\delta_r \le \sum_{r=1}^{n} M_r'\delta_r + \sum_{r=1}^{n} M_r''\delta_r \;\Rightarrow\; U(P, f+g) \le U(P, f) + U(P, g).$$

(ii) m_r', m_r'' are infimum of f, g on I_r.

$\Rightarrow \quad f(x) \ge m_r', g(x) \ge m_r'' \quad \forall x \in I_r$

$\Rightarrow \quad f(x) + g(x) \ge m_r' + m_r'' \quad \forall x \in I_r$

$\Rightarrow \quad (f+g)(x) \ge m_r' + m_r'' \quad \forall x \in I_r$

$\Rightarrow$ $m_r' + m_r''$ is a lower bound of $f+g$ on I_r.

But m_r is the greatest lower bound of $f+g$ on I_r.

$\therefore \quad m_r \ge m_r' + m_r'' \;\Rightarrow\; m_r\delta_r \ge m_r'\delta_r + m_r''\delta_r$

$$\Rightarrow \quad \sum_{r=1}^{n} m_r\delta_r \ge \sum_{r=1}^{n} m_r'\delta_r + \sum_{r=1}^{n} m_r''\delta_r \;\Rightarrow\; L(P, f+g) \ge L(P, f) + L(P, g).$$

(iii) $\omega(P, f+g) = U(P, f+g) - L(P, f+g) \le [U(P, f) + U(P, g)] - [L(P, f) + L(P, g)]$

$\qquad = [U(P, f) - L(P, f)] + [U(P, g) - L(P, g)] = \omega(P, f) + \omega(P, g)$

$\therefore \quad \omega(P, f+g) \le \omega(P, f) + \omega(P, g)$.

Theorem 4. *If $f : [a, b] \to R$ is a bounded function and $P, P' \in P[a, b]$ such that $P \subset P'$ (i.e., P' is a refinement of P), then*

(i) $L(P, f) \le L(P', f)$

(ii) $U(P, f) \ge U(P', f)$

(iii) $L(P, f) \le L(P', f) \le U(P', f) \le U(P, f)$

(iv) $U(P, f) - L(P, f) \ge U(P', f) - L(P', f)$

(v) $\omega(P, f) \ge \omega(P', f)$.

Proof. Let P′ contain just one point ξ (say) more than

$$P = \{a = x_0, x_1, x_2, \ldots\ldots, x_{r-1}, x_r, \ldots\ldots, x_n = b\}$$

and $x_{r-1} < \xi < x_r$, then $\quad P' = \{a = x_0, x_1, x_2, \ldots\ldots, x_{r-1}, \xi, x_r, \ldots\ldots, x_n = b\}$

Let m_r', m_r'' and m_r be the infimum of f in the intervals $[x_{r-1}, \xi]$, $[\xi, x_r]$ and $[x_{r-1}, x_r]$ respectively.

Let M_r', M_r'' and M_r be the supermum of f in the intervals $[x_{r-1}, \xi]$, $[\xi, x_r]$ and $[x_{r-1}, x_r]$ respectively.

Then $\quad m_r \le m_r', m_r \le m_r''$

$\qquad M_r \ge M_r', M_r \ge M_r''$.

(i) The contribution to $L(P, f)$ and $L(P', f)$ of each sub-interval except $[x_{r-1}, x_r]$ is the same. Also the contribution of the sub-interval $[x_{r-1}, x_r]$ to $L(P, f)$ is $m_r(x_r - x_{r-1})$ and its contribution to $L(P', f)$ is

$$m_r'(\xi - x_{r-1}) + m_r''(x_r - \xi).$$

$\therefore \quad L(P', f) - L(P, f) = [m_r'(\xi - x_{r-1}) + m_r''(x_r - \xi)] - m_r(x_r - x_{r-1})$

$\qquad = m_r'(\xi - x_{r-1}) + m_r''(x_r - \xi) - m_r[(x_r - \xi) + (\xi - x_{r-1})]$

$\qquad = (m_r' - m_r)(\xi - x_{r-1}) + (m_r'' - m_r)(x_r - \xi)$

$\qquad \ge 0 \qquad [\because \; m_r' \ge m_r, m_r'' \ge m_r, x_{r-1} < \xi < x_r]$

$\Rightarrow \quad L(P, f) \le L(P', f)$

If P′ contains p points more than P, then repeating the above argument p times, we have

$$L(P, f) \le L(P', f).$$

[**Note.** The above theorem asserts that the insertion of extra points in a partition does not decrease the lower sum.

$\Rightarrow$ Any refinement of P does not lower the lower sum.]

(ii) The contribution to $U(P, f)$ and $U(P', f)$ of each sub-interval except $[x_{r-1}, x_r]$ is the same. Also the contribution of the sub-interval $[x_{r-1}, x_r]$ to $U(P, f)$ is $M_r(x_r - x_{r-1})$ and its contribution to $U(P', f)$ is

$$M_r'(\xi - x_{r-1}) + M_r''(x_r - \xi).$$

$\therefore \quad U(P, f) - U(P', f) = M_r(x_r - x_{r-1}) - [M_r'(\xi - x_{r-1}) + M_r''(x_r - \xi)]$

$\qquad = M_r[(x_r - \xi) + (\xi - x_{r-1})] - [M_r'(\xi - x_{r-1}) + M_r''(x_r - \xi)]$

$\qquad = (M_r - M_r'')(x_r - \xi) + (M_r - M_r')(\xi - x_{r-1})$

$\qquad \ge 0 \qquad [\because \; M_r \ge M_r', M_r \ge M_r'', x_{r-1} < \xi < x_r]$

$\Rightarrow \quad U(P, f) \ge U(P', f)$.

If P′ contains p points more than P, then repeating the above argument p times, we have

$$U(P, f) \ge U(P', f).$$

[**Note.** The above theorem asserts that the insertion of extra points in a partition does not increase the upper sum.

$\Rightarrow$ Any refinement of P does not raise the upper sum.]

(iii) Now $\quad L(P, f) \le L(P', f)$

$\qquad U(P, f) \ge U(P', f)$

Also $\quad L(P', f) \ge U(P', f)$

Combining the above three statements, we have

$$L(P, f) \le L(P', f) \le U(P', f) \le U(P, f). \qquad \ldots(1)$$

(iv) Since $\quad U(P, f) \ge U(P', f)$

and $\quad L(P, f) \le L(P', f) \qquad \ldots(2)$

$\Rightarrow \quad -L(P, f) \ge -L(P', f)$

Adding (1) and (2), $U(P, f) - L(P, f) \ge U(P', f) - L(P', f)$.

(Second Method)

As in part (iii), $\quad L(P, f) \le L(P', f) \le U(P', f) \le U(P, f)$

$\Rightarrow \quad U(P, f) - L(P, f) \ge U(P', f) - L(P', f)$

[i.e., on the number line, the distance between $U(P', f)$ and $L(P', f)$ cannot exceed the distance between $U(P, f)$ and $L(P, f)$.]

(v) It is another form of part (iv).

Theorem 5. *If P' is a refinement of P containing p points more than P and $|f(x)| \le k \quad \forall\ x \in [a, b]$, then*

(i) $L(P, f) \le L(P', f) \le L(P, f) + 2pk\delta$ (ii) $U(P, f) \ge U(P', f) \ge U(P, f) - 2pk\delta$

(iii) $\omega(P, f) - \omega(P', f) \le 4pk\delta$

where $\|P\| = \delta$.

Proof. Let P′ contain just one point ξ(say) more than

$$P = \{a = x_0, x_1, x_2, \ldots\ldots, x_{r-1}, x_r, \ldots\ldots, x_n = b\}$$

and $x_{r-1} < \xi < x_r$, then $\quad P' = \{a = x_0, x_1, x_2, \ldots\ldots, x_{r-1}, \xi, x_r, \ldots\ldots, x_n = b\}$

Let m_r', m_r'' and m_r be the infimum of f in the intervals $[x_{r-1}, \xi]$, $[\xi, x_r]$ and $[x_{r-1}, x_r]$ respectively.

Let M_r', M_r'' and M_r be the supermum of f in the intervals $[x_{r-1}, \xi]$, $[\xi, x_r]$ and $[x_{r-1}, x_r]$ respectively.

Since $\quad |f(x)| \le k \quad \forall\ x \in [a, b]$ i.e., $-k \le f(x) \le k \quad \forall\ x \in [a, b]$

$$\therefore \quad -k \le m_r \le m_r' \le k,\ -k \le m_r \le m_r'' \le k$$
$$-k \le M_r' \le M_r \le k,\ -k \le M_r'' \le M_r \le k$$
$$\Rightarrow \quad 0 \le m_r' - m_r \le 2k,\ 0 \le m_r'' - m_r \le 2k$$
$$0 \le M_r - M_r' \le 2k,\ 0 \le M_r - M_r'' \le 2k.$$

(i)
$$L(P', f) - L(P, f) = [m_r'(\xi - x_{r-1}) + m_r''(x_r - \xi)] - m_r(x_r - x_{r-1})$$
$$= m_r'(\xi - x_{r-1}) + m_r''(x_r - \xi) - m_r[(x_r - \xi) + (\xi - x_{r-1})]$$
$$= (m_r' - m_r)(\xi - x_{r-1}) + (m_r'' - m_r)(x_r - \xi)$$
$$\le 2k(\xi - x_{r-1}) + 2k(x_r - \xi) = 2k(x_r - x_{r-1}) = 2k\delta_r$$
$$\le 2k\delta \qquad [\because\ \delta_r \le \|P\| = \delta]$$
$$\Rightarrow \quad L(P', f) \le L(P, f) + 2k\delta$$

If P′ contains p points more than P, then introducing the additional points one by one and proceeding as above p times, we have $L(P', f) \le L(P, f) + 2pk\delta$

Also $\quad L(P, f) \le L(P', f)$

$$\therefore \quad L(P, f) \le L(P', f) \le L(P, f) + 2pk\delta.$$

(ii)
$$U(P, f) - U(P', f) = M_r(x_r - x_{r-1}) - [M_r'(\xi - x_{r-1}) + M_r''(x_r - \xi)]$$
$$= M_r[(x_r - \xi) + (\xi - x_{r-1})] - [M_r'(\xi - x_{r-1}) + M_r''(x_r - \xi)]$$
$$= (M_r - M_r'')(x_r - \xi) + (M_r - M_r')(\xi - x_{r-1})$$
$$\le 2k(x_r - \xi) + 2k(\xi - x_{r-1}) = 2k(x_r - x_{r-1}) = 2k\delta_r$$
$$\le 2k\delta \qquad [\because\ \delta_r \le \|P\| = \delta]$$
$$\Rightarrow \quad U(P', f) \ge U(P, f) - 2k\delta.$$

If P′ contains p points more than P, then introducing the additional points one by one and proceeding as above p times, we have $\quad U(P', f) \ge U(P, f) - 2pk\delta$

Also $\quad U(P, f) \ge U(P', f)$

$$\therefore \quad U(P, f) \ge U(P', f) \ge U(P, f) - 2pk\delta.$$

(iii) Now $\quad U(P, f) - U(P', f) \le 2pk\delta \quad$ and $\quad L(P', f) - L(P, f) \le 2pk\delta$

Adding, $\quad [U(P, f) - L(P, f)] - [U(P', f) - L(P', f)] \le 4pk\delta \quad \Rightarrow \quad \omega(P, f) - \omega(P', f) \le 4pk\delta.$

Theorem 6. *If $P_1, P_2 \in P[a, b]$, then*

(i) $L(P_1, f) \le U(P_2, f)$ (ii) $L(P_2, f) \le U(P_1, f)$

i.e., no lower sum can exceed any upper sum.

Proof. Let $P = P_1 \cup P_2$ be a common refinement of P_1 and P_2.

(i) Since any refinement does not lower the lower sum and does not raise the upper sum.

$$\therefore \quad L(P_1, f) \le L(P, f) \quad \text{and} \quad U(P, f) \le U(P_2, f)$$

Also $\quad L(P, f) \le U(P, f)$

Combining, we have $\quad L(P_1, f) \le L(P, f) \le U(P, f) \le U(P_2, f)$

$$\Rightarrow \quad L(P_1, f) \le U(P_2, f).$$

(ii) *Please try yourself.*

9.7. UPPER AND LOWER RIEMANN INTEGRALS

Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then for every $P \in P[a, b]$, we have

$$m(b - a) \le L(P, f) \le U(P, f) \le M(b - a)$$

where m and M are the infimum and supermum of f on $[a, b]$.

Thus for every $P \in P[a, b]$, we have $L(P, f) \le M(b - a)$ and $U(P, f) \ge m(b - a)$.

$\Rightarrow$ The set $\{L(P, f)\}_{P \in P[a, b]}$ of lower sums is bounded above by $M(b - a)$ and, therefore, has the least upper bound.

The set $\{U(P, f)\}_{P \in P[a, b]}$ of upper sums is bounded below by $m(b - a)$ and, therefore, has the greatest lower bound.

Lower Riemann Integral of f on [a, b] is defined as sup $\{L(P, f)\}_{P \in P[a, b]}$ and is denoted by $\underline{\int_a^b} f(x)\,dx$.

Upper Riemann Integral of f on [a, b] is defined as inf $\{U(P, f)\}_{P \in P[a, b]}$ is denoted by $\overline{\int_a^b} f(x)\,dx$.

9.8. RIEMANN INTEGRAL

A bounded function f is said to be Riemann integrable (or simply R-integrable) on $[a, b]$ if its lower and upper Riemann integrals are equal i.e., if $\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx$.

The common value of these integrals is called the Riemann integral of f on $[a, b]$ and is denoted by

$$\int_a^b f(x)\,dx.$$

The interval $[a, b]$ is called the range of integration. The numbers a and b are called the lower and upper limits of integration respectively.

Note 1. Riemann integral is based on the notion of bounds and is subject to two conditions :

(i) f is bounded on the interval, and (ii) the interval is closed.

2. The family of all bounded functions which are R-integrable on the closed intervals $[a, b]$ is denoted by $R[a, b]$. If f is R-integrable on $[a, b]$, then we write $f \in R[a, b]$.

3. f is R-integrable on $[a, b] \Rightarrow$ (i) f is bounded on $[a, b]$

(ii) $$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx.$$

9.9. THEOREM 1

If $f : [a, b] \to R$ is a bounded function, then $\underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx$.

Proof. Let $P_1, P_2 \in P[a, b]$, then

$$L(P_1, f) \le U(P_2, f) \qquad (\because \text{ no lower sum can exceed any upper sum})$$

This is true for each $P_1 \in P[a, b]$. Keeping P_2 fixed, the set $\{L(P_1, f)\}_{P_1 \in P[a,b]}$ has an upper bound $U(P_2, f)$.

Also $\quad \sup\{L(P_1, f)\}_{P_1 \in P[a,b]} = \underline{\int_a^b} f(x)\,dx$

Since supermum ≤ any upper bound

$$\therefore \quad \underline{\int_a^b} f(x)\,dx \le U(P_2, f)$$

This is true for each $P_2 \in P[a, b]$. Thus the set $\{U(P_2, f)\}_{P_2 \in P[a,b]}$ has a lower bound $\underline{\int_a^b} f(x)\,dx$.

But $\quad \inf\{U(P_2, f)\}_{P_2 \in P[a,b]} = \overline{\int_a^b} f(x)\,dx$.

Since any lower bound ≤ infimum.

$$\therefore \quad \underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx.$$

Theorem 2. *If $f : [a, b] \to R$ is bounded function, then*

$$m(b-a) \le \underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx \le M(b-a)$$

where m and M are the infimum and supermum of f on [a, b].

Proof. For every $P \in P[a, b]$, we have $m(b-a) \le L(P, f) \le U(P, f) \le M(b-a)$ ...(1)

Now $\quad \sup\{L(P, f)\}_{P \in P[a,b]} = \underline{\int_a^b} f(x)\,dx$

$$\Rightarrow \quad L(P, f) \le \underline{\int_a^b} f(x)\,dx \qquad ...(2)$$

$$\inf\{U(P, f)\}_{P \in P[a,b]} = \overline{\int_a^b} f(x)\,dx \Rightarrow \overline{\int_a^b} f(x)\,dx \le U(P, f) \qquad ...(3)$$

Also $\quad \underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx \qquad ...(4)$

From (1), (2), (3) and (4), we have

$$m(b-a) \le L(P, f) \le \underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx \le U(P, f) \le M(b-a)$$

$$\Rightarrow \quad m(b-a) \le \underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx \le M(b-a).$$

Theorem 3. *If $f \in R[a, b]$, then*

(i) $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$ if $b \ge a$ (ii) $m(b-a) \ge \int_a^b f(x)\,dx \ge M(b-a)$ if $b \le a$

where m and M are the infimum and supermum of f on [a, b].

Proof. For $a = b$, the result is trivial.

If $b > a$, then for every $P \in P[a, b]$, we have

$$m(b-a) \le L(P, f) \le U(P, f) \le M(b-a) \qquad ...(1)$$

Now $\quad \sup\{L(P, f)\}_{P \in P[a,b]} = \underline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx \qquad | \because f \in R[a, b]$

$$\Rightarrow \quad L(P, f) \le \int_a^b f(x)\,dx \qquad ...(2)$$

Also $\quad \inf\{U(P, f)\}_{P \in P[a,b]} = \overline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx \qquad | \because f \in R[a, b]$

$$\Rightarrow \quad \int_a^b f(x)\,dx \le U(P, f) \qquad ...(3)$$

From (1), (2) and (3), we have

$$m(b-a) \le L(P, f) \le \int_a^b f(x)\,dx \le U(P, f) \le M(b-a)$$

$$\therefore \quad m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$

If $b < a$, then $a > b$.

∴ Interchanging a and b in the above result, we have

$$m(a-b) \le \int_b^a f(x)\,dx \le M(a-b)$$

$$\Rightarrow \quad -m(a-b) \ge -\int_a^b f(x)\,dx \ge -M(a-b)$$

$$\Rightarrow \quad m(b-a) \ge \int_a^b f(x)\,dx \ge M(b-a).$$

ILLUSTRATIVE EXAMPLES

Example 1. *Let $f(x) = x$ for $x \in [0, 1]$ and let $P = \left\{0, \frac{1}{3}, \frac{2}{3}, 1\right\}$ be a partition of [0, 1]. Compute U(P, f) and L(P, f).*

Sol. Partition P divides the interval [0, 1] into sub-intervals

$$I_1 = \left[0, \frac{1}{3}\right], I_2 = \left[\frac{1}{3}, \frac{2}{3}\right], I_3 = \left[\frac{2}{3}, 1\right]$$

$$\therefore \quad \delta_1 = \frac{1}{3} - 0 = \frac{1}{3};\ \delta_2 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3};\ \delta_3 = 1 - \frac{2}{3} = \frac{1}{3}$$

Since $f(x) = x$ is increasing on [0, 1].

$$\therefore \quad M_1 = \frac{1}{3}, m_1 = 0\ ;\ M_2 = \frac{2}{3}, m_2 = \frac{1}{3}, M_3 = 1, m_3 = \frac{2}{3}.$$

$$\therefore \quad U(P, f) = \sum_{r=1}^{3} M_r\,\delta_r = M_1\delta_1 + M_2\delta_2 + M_3\delta_3$$

$$= \frac{1}{3}\cdot\frac{1}{3} + \frac{2}{3}\cdot\frac{1}{3} + 1\cdot\frac{1}{3} = \frac{1}{3}\left(\frac{1}{3} + \frac{2}{3} + 1\right) = \frac{2}{3}$$

$$L(P, f) = \sum_{r=1}^{3} m_r\delta_r = m_1\delta_1 + m_2\delta_2 + m_3\delta_3 = 0\cdot\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{3} + \frac{2}{3}\cdot\frac{1}{3} = \frac{1}{3}.$$

Example 2. *Compute L(P, f) and U(P, f) for the function f defined by $f(x) = x^2$ on [0, 1] and* $P = \left\{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1\right\}$.

Sol. Partition P divides the interval [0, 1] into sub-intervals

$$I_1 = \left[0, \frac{1}{4}\right], I_2 = \left[\frac{1}{4}, \frac{2}{4}\right], I_3 = \left[\frac{2}{4}, \frac{3}{4}\right], I_4 = \left[\frac{3}{4}, 1\right];\ \delta_1 = \delta_2 = \delta_3 = \delta_4 = \frac{1}{4}.$$

Since $f(x) = x^2$ is increasing on [0, 1]

$$\therefore \quad m_1 = 0, M_1 = \frac{1}{16};\ m_2 = \frac{1}{16}, M_2 = \frac{4}{16}, m_3 = \frac{4}{16}, M_3 = \frac{9}{16}, m_4 = \frac{9}{16}, M_4 = 1.$$

$$\therefore \quad L(P, f) = \sum_{r=1}^{4} m_r\delta_r = m_1\delta_1 + m_2\delta_2 + m_3\delta_3 + m_4\delta_4$$

$$= \left(0 + \frac{1}{16} + \frac{4}{16} + \frac{9}{16}\right) \times \frac{1}{4} = \frac{7}{32}$$

$$U(P, f) = \sum_{r=1}^{4} M_r\delta_r = M_1\delta_1 + M_2\delta_2 + M_3\delta_3 + M_4\delta_4$$

$$= \left(\frac{1}{16} + \frac{4}{16} + \frac{9}{16} + 1\right) \times \frac{1}{4} = \frac{15}{32}.$$

Example 3. *If f is defined on [a, b] by $f(x) = k \quad \forall \ x \in [a, b]$ where k is constant, then $f \in$ **R[a, b]** and $\int_a^b f(x)\,dx = k(b-a)$.*

Or

A constant function is R-integrable.

Sol. Let $P = \{a = x_0, x_1, x_2, \ldots\ldots, x_n = b\}$ be any partition of $[a, b]$. Then for any sub-interval $[x_{r-1}, x_r]$; $r = 1, 2, \ldots, n$, we have

$$M_r = m_r = k \qquad [\because \ f(x) = k = \text{constant}]$$

$$\therefore \quad U(P, f) = \sum_{r=1}^{n} M_r(x_r - x_{r-1}) = \sum_{r=1}^{n} k(x_r - x_{r-1}) = k\sum_{r=1}^{n}(x_r - x_{r-1}) = k(b-a) = \text{constant}$$

and $$L(P, f) = \sum_{r=1}^{n} m_r(x_r - x_{r-1}) = \sum_{r=1}^{n} k(x_r - x_{r-1}) = k\sum_{r=1}^{n}(x_r - x_{r-1}) = k(b-a) = \text{constant}$$

$$\therefore \quad \underline{\int_a^b} f(x)\,dx = \sup\{L(P, f)\}_{P \in P[a,b]} = k(b-a)$$

$$\overline{\int_a^b} f(x)\,dx = \inf\{U(P, f)\}_{P \in P[a,b]} = k(b-a)$$

Since $$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx = k(b-a)$$

$$\therefore \quad f \in \mathbf{R[a, b]} \quad \text{and} \quad \int_a^b f(x)\,dx = k(b-a).$$

Example 4. *If f is defined on [0, 1] by $f(x) = x \ \forall \ x \in [0, 1]$ then $f \in$ **R[0, 1]** and $\int_0^1 f(x)\,dx = \frac{1}{2}$.*

Sol. Let $P = \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{r-1}{n}, \frac{r}{n}, \ldots, \frac{n}{n} = 1\right\}$ be *any partition* of [0, 1]. Then for any sub-interval

$$I_r = \left[\frac{r-1}{n}, \frac{r}{n}\right], r = 1, 2, \ldots, n.$$

We have $$M_r = \frac{r}{n}, m_r = \frac{r-1}{n} \quad \text{and} \quad \delta_r = \frac{r}{n} - \frac{r-1}{n} = \frac{1}{n}$$

$$\therefore \quad U(P, f) = \sum_{r=1}^{n} M_r\delta_r = \sum_{r=1}^{n} \frac{r}{n}\cdot\frac{1}{n}$$

$$= \frac{1}{n^2}\sum_{r=1}^{n} r = \frac{1}{n^2}(1 + 2 + 3 + \ldots\ldots + n) = \frac{1}{n^2}\cdot\frac{n(n+1)}{2} = \frac{n+1}{2n}$$

$$L(P, f) = \sum_{r=1}^{n} m_r\delta_r = \sum_{r=1}^{n} \frac{r-1}{n}\cdot\frac{1}{n} = \frac{1}{n^2}\sum_{r=1}^{n}(r-1)$$

$$= \frac{1}{n^2}[0 + 1 + 2 + \ldots\ldots + (n+1)] = \frac{1}{n^2}\cdot\frac{(n-1)\cdot n}{2} = \frac{n-1}{2n}$$

$$\therefore \quad \underline{\int_0^1} f(x)\,dx = \sup\{L(P, f)\}_{P \in P[0,1]} = \lim_{n\to\infty}\left[\frac{n-1}{2n}\right] = \lim_{n\to\infty}\frac{1}{2}\left(1 - \frac{1}{n}\right) = \frac{1}{2}$$

and $$\overline{\int_0^1} f(x)\,dx = \inf\{U(P, f)\}_{P \in P[0,1]} = \lim_{n\to\infty}\left[\frac{n+1}{2n}\right] = \lim_{n\to\infty}\frac{1}{2}\left(1 + \frac{1}{n}\right) = \frac{1}{2}$$

Since $\int_0^1 f(x)\,dx = \overline{\int_0^1} f(x)\,dx = \frac{1}{2}$

$\therefore$ $f \in \mathbf{R}[0, 1]$ and $\int_0^1 f(x)\,dx = \frac{1}{2}$.

Example 5. *If f is defined on $[o, a]$; $a > o$ by $f(x) = x^2$ $\forall$ $x \in [o, a]$, then $f \in R[o, a]$ and $\int_o^a f(x)\,dx = \frac{a^3}{3}$.*

Sol. Let $$P = \left\{0, \frac{a}{n}, \frac{2a}{n}, \ldots\ldots, \frac{(r-1)\,a}{n}, \frac{ra}{n}, \ldots\ldots, \frac{na}{n} = a\right\}$$

be any partition of $[o, a]$. Then for any sub-interval $I_r = \left[\frac{(r-1)\,a}{n}, \frac{ra}{n}\right]$, $r = 1, 2, \ldots\ldots, n$ we have

$$M_r = \frac{r^2a^2}{n^2},\ m_r = \frac{(r-1)^2a^2}{n^2} \quad (\because\ f(x) = x^2 \text{ is increasing on } [0, a])$$

Also, $$\delta_r = \frac{a}{n}$$

$\therefore$ $$U(P, f) = \sum_{r=1}^{n} M_r\delta_r = \sum_{r=1}^{n} \frac{r^2a^2}{n^2}\cdot\frac{a}{n}$$

$$= \frac{a^3}{n^3}\sum_{r=1}^{n} r^2 = \frac{a^3}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6n^2}\cdot a^3$$

$$L(P, f) = \sum_{r=1}^{n} m_r\delta_r = \sum_{r=1}^{n} \frac{(r-1)^2a^2}{n^2}\cdot\frac{a}{n}$$

$$= \frac{a^3}{n^3}\sum_{r=1}^{n}(r-1)^2 = \frac{a^3}{n^3}\cdot\frac{(n-1)(n)(2n-1)}{6} = \frac{(n-1)(2n-1)}{6n^2}\cdot a^3$$

$\therefore$ $$\underline{\int_0^a} f(x)\,dx = \sup\{L(P, f)\}_{P\in P[0,a]}$$

$$= \lim_{n\to\infty}\left[\frac{(n-1)(2n-1)}{6n^2}\,a^3\right] = \lim_{n\to\infty}\frac{a^3}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right) = \frac{a^3}{3}$$

and $$\overline{\int_0^a} f(x)\,dx = \inf\{U(P, f)\}_{P\in P[0,a]}$$

$$= \lim_{n\to\infty}\left[\frac{(n+1)(2n+1)}{6n^2}\,a^3\right] = \lim_{n\to\infty}\frac{a^3}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) = \frac{a^3}{3}$$

Since $\underline{\int_o^a} f(x)\,dx = \overline{\int_o^a} f(x)\,dx = \frac{a^3}{3}$

$\therefore$ $f \in \mathbf{R[o, a]}$ and $\int_o^a f(x)\,dx = \frac{a^3}{3}$.

Example 6. *If f is defined on $[o, a]$, $a > o$ by $f(x) = x^3$ $\forall$ $x \in [o, a]$, then*

$$f \in \mathbf{R[o, a]} \text{ and } \int_o^a f(x)\,dx = \frac{a^4}{4}.$$

Sol. Let $P = \left\{o, \frac{a}{n}, \frac{2a}{n}, \ldots\ldots, \frac{(r-1)a}{n}, \frac{ra}{n}, \ldots\ldots, \frac{na}{n} = a\right\}$ be any partition of $[o, a]$. Then for any sub-interval $I_r = \left[\frac{(r-1)a}{n}, \frac{ra}{n}\right]$, $r = 1, 2, \ldots\ldots, n$, we have

$$M_r = \frac{r^3a^3}{n^3},\ m_r = \frac{(r-1)^3a^3}{n^3} \quad [\because\ f(x) = x^3 \text{ is increasing on } [o, a]]$$

Also $$\delta_r = \frac{a}{n}$$

$\therefore$ $$U(P, f) = \sum_{r=1}^{n} M_r\delta_r = \sum_{r=1}^{n} \frac{r^3a^3}{n^3}\cdot\frac{a}{n}$$

$$= \frac{a^4}{n^4}\sum_{r=1}^{n} r^3 = \frac{a^4}{n^4}\cdot\frac{n^2(n+1)^2}{4} = \frac{(n+1)^2}{n^2}\cdot\frac{a^4}{4}$$

$$L(P, f) = \sum_{r=1}^{n} m_r\delta_r = \sum_{r=1}^{n} \frac{(r-1)^3a^3}{n^3}\cdot\frac{a}{n}$$

$$= \frac{a^2}{n^4}\sum_{r=1}^{n}(r-1)^3 = \frac{a^4}{n^4}\cdot\frac{(n-1)^2n^2}{4} = \frac{(n-1)^2}{n^2}\cdot\frac{a^4}{4}$$

$\therefore$ $$\underline{\int_o^a} f(x)\,dx = \sup\{L(P, f)\}_{P\in P[0,a]}$$

$$= \lim_{n\to\infty}\left[\frac{(n-1)^2}{n^2}\cdot\frac{a^4}{4}\right] = \lim_{n\to\infty}\frac{a^4}{4}\left(1-\frac{1}{n}\right)^2 = \frac{a^4}{4}$$

and $$\overline{\int_o^a} f(x)\,dx = \inf\{U(P, f)\}_{P\in P[o,a]}$$

$$= \lim_{n\to\infty}\left[\frac{(n+1)^2}{n^2}\cdot\frac{a^4}{4}\right] = \lim_{n\to\infty}\frac{a^4}{4}\left(1+\frac{1}{n}\right)^2 = \frac{a^4}{4}$$

Since $\underline{\int_o^a} f(x)\,dx = \overline{\int_o^a} f(x)\,dx = \frac{a^4}{4}$

$\therefore$ $f \in \mathbf{R[o, a]}$ and $\int_o^a f(x)\,dx = \frac{a^4}{4}$.

Example 7. *Let $f(x) = \sin x$ for $x \in \left[0, \frac{\pi}{2}\right]$ and let $P = \left\{0, \frac{\pi}{2n}, \frac{2\pi}{2n}, \ldots\ldots, \frac{n\pi}{2n}\right\}$ be the partition of $\left[0, \frac{\pi}{2}\right]$.*

Compute U(P, f) and L(P, f). Hence prove that $f \in R\left[0, \frac{\pi}{2}\right]$.

Sol. Here
$$P = \left\{0, \frac{\pi}{2n}, \frac{2\pi}{2n}, \ldots\ldots, \frac{(r-1)\pi}{2n}, \frac{r\pi}{2n}, \ldots\ldots, \frac{n\pi}{2n} = \frac{\pi}{2}\right\}$$

For any sub-interval $I_r = \left[\frac{(r-1)\pi}{2\pi}, \frac{r\pi}{2n}\right], r = 1, 2, \ldots, n$

$$M_r = \sin\frac{r\pi}{2n}, \quad m_r = \sin\frac{(r-1)\pi}{2n}$$

$$\left| \because \ f(x) = \sin x \text{ is increasing on } \left[0, \frac{\pi}{2}\right] \right.$$

Also
$$\delta_r = \frac{r\pi}{2n} - \frac{(r-1)\pi}{2n} = \frac{\pi}{2n}$$

$$\therefore \quad U(P, f) = \sum_{r=1}^{n} M_r \delta_r = \sum_{r=1}^{n} \sin\frac{r\pi}{2n} \cdot \frac{\pi}{2n} = \frac{\pi}{2n}\left[\sin\frac{\pi}{2n} + \sin\frac{2\pi}{2n} + \ldots + \sin\frac{n\pi}{2n}\right]$$

$$= \frac{\pi}{2n} \cdot \frac{\sin\left[\frac{\pi}{2n} + \frac{n-1}{2} \cdot \frac{\pi}{2n}\right] \sin\left(\frac{n}{2} \cdot \frac{\pi}{2n}\right)}{\sin\left(\frac{1}{2} \cdot \frac{\pi}{2n}\right)}$$

$$\left[\because \ \sin\alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + \ldots\ldots \text{ to } n \text{ terms} = \frac{\sin\left(\alpha + \frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\right]$$

$$= \frac{\pi}{2n} \cdot \frac{\sin\frac{(n+1)\pi}{4\pi} \sin\frac{\pi}{4}}{\sin\frac{\pi}{4n}} = \frac{\pi}{2\sqrt{2}n} \cdot \frac{\sin\frac{(n+1)\pi}{4n}}{\sin\frac{\pi}{4n}}$$

$$= \frac{\pi}{2\sqrt{2}n} \cdot \frac{\sin\left(\frac{\pi}{4} + \frac{\pi}{4n}\right)}{\sin\frac{\pi}{4n}} = \frac{\pi}{2\sqrt{2}n} \cdot \frac{\sin\frac{\pi}{4}\cos\frac{\pi}{4n} + \cos\frac{\pi}{4}\sin\frac{\pi}{4n}}{\sin\frac{\pi}{4n}}$$

$$= \frac{\pi}{2\sqrt{2}n} \cdot \frac{1}{\sqrt{2}} \cdot \left[\cot\frac{\pi}{4n} + 1\right] = \frac{\pi}{4n}\left[\cot\frac{\pi}{4n} + 1\right]$$

$$L(P, f) = \sum_{r=1}^{n} m_r \delta_r = \sum_{r=1}^{n} \sin\frac{(r-1)\pi}{2n} \cdot \frac{\pi}{2n}$$

$$= \frac{\pi}{2n}\left[0 + \sin\frac{\pi}{2n} + \sin\frac{2\pi}{2n} + \ldots\ldots + \sin\frac{(n-1)\pi}{2n}\right]$$

$$= \frac{\pi}{2n} \cdot \frac{\sin\left[\frac{\pi}{2n} + \frac{n-2}{2} \cdot \frac{\pi}{2n}\right] \sin\left(\frac{n-1}{2} \cdot \frac{\pi}{2n}\right)}{\sin\left(\frac{1}{2} \cdot \frac{\pi}{2n}\right)}$$

$$= \frac{\pi}{2n} \cdot \frac{\sin\frac{\pi}{4}\sin\left(\frac{\pi}{4} - \frac{\pi}{4n}\right)}{\sin\frac{\pi}{4n}} = \frac{\pi}{2\sqrt{2}n} \cdot \frac{\sin\frac{\pi}{4}\cos\frac{\pi}{4n} - \cos\frac{\pi}{4}\sin\frac{\pi}{4n}}{\sin\frac{\pi}{4n}}$$

$$= \frac{\pi}{2\sqrt{2}n} \cdot \frac{1}{\sqrt{2}} \cdot \left[\cot\frac{\pi}{4n} - 1\right] = \frac{\pi}{4n}\left[\cot\frac{\pi}{4n} - 1\right]$$

$$\therefore \quad \int_0^{\pi/2} f(x)\,dx = \sup\{L(P, f)\}_{P \in P[0, \pi/2]}$$

$$= \lim_{n \to \infty} \frac{\pi}{4n}\left[\cot\frac{\pi}{4n} - 1\right] = \lim_{n \to \infty}\left[\frac{\frac{\pi}{4n}}{\tan\frac{\pi}{4n}} - \frac{\pi}{4n}\right] = 1$$

and
$$\overline{\int_0^{\frac{\pi}{2}}} f(x)\,dx = \inf\{U(P, f)\}_{P \in P[0, \pi/2]}$$

$$= \lim_{n \to \infty} \frac{\pi}{4n}\left[\cot\frac{\pi}{4n} + 1\right] = \lim_{n \to \infty}\left[\frac{\frac{\pi}{4n}}{\tan\frac{\pi}{4n}} + \frac{\pi}{4n}\right] = 1$$

Since $\underline{\int_0^{\pi/2}} f(x)\,dx = \overline{\int_0^{\frac{\pi}{2}}} f(x)\,dx = 1$

$$\therefore \quad f \in \mathbf{R}\left[0, \frac{\pi}{2}\right] \quad \text{and} \quad \int_0^{\pi/2} f(x)\,dx = 1.$$

Example 8. *Show by an example that every bounded function need not be R-integrable.*

Sol. Consider a function f defined on $[0, 1]$ by $f(x) = \begin{cases} 0, & \text{when } x \text{ is rational} \\ 1, & \text{when } x \text{ is irrational} \end{cases}$

Clearly, $f(x)$ is bounded in $[0, 1]$ because $0 \le f(x) \le 1 \quad \forall\, x \in [0, 1]$

If $P = \{0, x_0, x_1, x_2, \ldots\ldots, x_n = 1\}$ is any partition of $[0, 1]$, then for any sub-interval $I_r = [x_{r-1}, x_r]$, $r = 1, 2, \ldots, n$, we have $M_r = 1, m_r = 0$

$$\therefore \quad U(P, f) = \sum_{r=1}^{n} M_r \delta_r = \sum_{r=1}^{n} 1 . (x_r - x_{r-1}) = x_n - x_0 = 1$$

and
$$L(P, f) = \sum_{r=1}^{n} m_r \delta_r = \sum_{r=1}^{n} 0\,(x_r - x_{r-1}) = 0$$

$\therefore \quad \underline{\int_0^1} f(x)\,dx = \sup\{L(P, f)\}_{P \in P[0,1]} = 0$

and $\quad \overline{\int_0^1} f(x)\,dx = \inf\{U(P, f)_{P \in P[0,1]}\} = 1$

Since $\quad \underline{\int_0^1} f(x)\,dx \neq \overline{\int_0^1} f(x)\,dx,\ f \notin \mathbf{R}[0, 1].$

Example 9. *If f(x) be defined [0, 1]. as follows :* $f(x) = \begin{cases} 1, & \text{when } x \text{ is rational} \\ -1, & \text{when } x \text{ is irrational} \end{cases}$. *Show that f is not R-integrable over [0, 1].*

Sol. *Please try yourself.*

[**Hint.** For any sub-interval $I_r = [x_{r-1}, x_r]$, $r = 1, 2, 3, \ldots\ldots, n$, we have $M_r = 1$, $m_r = -1$].

Example 10. *If f be a function defined on* $\left[0, \frac{\pi}{4}\right]$ *by* $f(x) = \begin{cases} \cos x, & \text{if } x \text{ is rational} \\ \sin x, & \text{if } x \text{ is irrational} \end{cases}$, *then* $f \notin \mathbf{R}\left[0, \frac{\pi}{4}\right]$.

Sol. Let $P = \left\{0, \frac{\pi}{4n}, \frac{2\pi}{4n}, \ldots\ldots, \frac{(r-1)\pi}{4n}, \frac{r\pi}{4n}, \ldots\ldots, \frac{n\pi}{4\pi} = \frac{\pi}{4}\right\}$ be any partition of $\left[0, \frac{\pi}{4}\right]$. Then for any sub-interval $\left[\frac{(r-1)\pi}{4n}, \frac{r\pi}{4n}\right]$, $r = 1, 2, \ldots\ldots, n$, we have

$$M_r = \cos\frac{(r-1)\pi}{4n} \text{ and } m_r = \sin\frac{(r-1)\pi}{4n} \qquad \left\{\because \cos x \geq \sin x \text{ on } \left[0, \frac{\pi}{4}\right]\right\}$$

Also $\quad \delta_r = \frac{\pi}{4\pi}$

$$\therefore \quad U(P, f) = \sum_{r=1}^{n} M_r \delta_r = \sum_{r=1}^{n} \frac{\pi}{4n} \cos\frac{(r-1)\pi}{4n}$$

$$= \frac{\pi}{4n}\left[\cos 0 + \cos\frac{\pi}{4n} + \ldots\ldots + \cos\frac{(n-1)\pi}{4n}\right]$$

$$= \frac{\pi}{4n} \cdot \frac{\cos\left[0 + \frac{n-1}{2} \cdot \frac{\pi}{4n}\right] \sin\left(\frac{n}{2} \cdot \frac{\pi}{4n}\right)}{\sin\left(\frac{1}{2} \cdot \frac{\pi}{4n}\right)} = \frac{\pi}{4n} \cdot \frac{\cos\frac{(n-1)\pi}{8n} \cdot \sin\frac{\pi}{8}}{\sin\frac{\pi}{8n}}$$

and $\quad L(P, f) = \sum_{r=1}^{n} m_r \delta_r = \sum_{r=1}^{n} \frac{\pi}{4n} \sin\frac{(r-1)\pi}{4n}$

$$= \frac{\pi}{4n}\left[\sin 0 + \sin\frac{\pi}{4n} + \ldots\ldots + \sin\frac{(n-1)\pi}{4n}\right]$$

$$= \frac{\pi}{4n} \cdot \frac{\sin\left[0 + \frac{n-1}{2} \cdot \frac{\pi}{4n}\right] \sin\left(\frac{n}{2} \cdot \frac{\pi}{4n}\right)}{\sin\left(\frac{1}{2} \cdot \frac{\pi}{4n}\right)} = \frac{\pi}{4n} \cdot \frac{\sin\frac{(n-1)\pi}{8n} \sin\frac{\pi}{8}}{\sin\frac{\pi}{8n}}$$

$$\therefore \quad \underline{\int_0^{\pi/4}} f(x)\,dx = \sup\{L(P, f)\}_{P \in P[0, \pi/4]}$$

$$= \lim_{n \to \infty}\left[\frac{\pi}{4n} \cdot \frac{\sin\frac{(n-1)\pi}{8n} \sin\frac{\pi}{8}}{\sin\frac{\pi}{8n}}\right] = \lim_{n \to \infty}\left[\frac{\frac{\pi}{8n}}{\sin\frac{\pi}{8n}} \cdot 2\sin\left(\frac{\pi}{8} - \frac{\pi}{8n}\right)\sin\frac{\pi}{8}\right]$$

$$= 1 \times 2\sin^2\frac{\pi}{8} = 1 - \cos\frac{\pi}{4} = 1 - \frac{1}{\sqrt{2}}$$

and $\quad \overline{\int_0^{\frac{\pi}{4}}} f(x)\,dx = \inf\{U(P, f)\}_{P \in P[0, \pi/4]}$

$$= \lim_{n \to \infty}\left[\frac{\pi}{4n} \cdot \frac{\cos\frac{(n-1)\pi}{8n} \sin\frac{\pi}{8}}{\sin\frac{\pi}{8n}}\right] = \lim_{n \to \infty}\left[\frac{\frac{\pi}{8n}}{\sin\frac{\pi}{8n}} \cdot 2\cos\left(\frac{\pi}{8} - \frac{\pi}{8n}\right)\sin\frac{\pi}{8}\right]$$

$$= 1 \times 2\cos\frac{\pi}{8}\sin\frac{\pi}{8} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

Since $\quad \underline{\int_0^{\frac{\pi}{4}}} f(x)\,dx \neq \overline{\int_0^{\frac{\pi}{4}}} f(x)\,dx, \quad \therefore \quad f \notin R\left[0, \frac{\pi}{4}\right].$

Example 11. *Show that f(x) = 2x + 1 is integrable on [1, 2] and* $\int_1^2 (2x+1)\,dx = 4$.

Sol. Consider a partition $P = \left\{1, 1 + \frac{1}{n}, 1 + \frac{2}{n}, \ldots\ldots, 1 + \frac{n}{n}\right\}$ dividing the interval [1, 2] into n equal parts.

Then for any sub-interval

$$I_r = \left[1 + \frac{r-1}{n}, 1 + \frac{r}{n}\right], \ r = 1, 2, \ldots\ldots, n, \text{ we have}$$

$$m_r = 2\left(1 + \frac{r-1}{n}\right) + 1 = 3 + \frac{2(r-1)}{n}$$

$$M_r = 2\left(1 + \frac{r}{n}\right) + 1 = 3 + \frac{2r}{n} \qquad \{\because f(x) = 2x + 1 \text{ is increasing on } [1, 2]\}$$

$$\delta_r = \frac{1}{n}$$

$$L(P, f) = \sum_{r=1}^{n} m_r \delta_r = \sum_{r=1}^{m} \left[3 + \frac{2(r-1)}{n}\right] \cdot \frac{1}{n}$$

$$= \frac{3}{n}.n + \frac{2}{n^2}\sum_{r=1}^{n}(r-1) = 3 + \frac{2}{n^2}.\frac{(n-1)n}{2} = 3 + \frac{n-1}{n} = 4 - \frac{1}{n}$$

$$U(P, f) = \sum_{r=1}^{n} M_r\delta_r = \sum_{r=1}^{n}\left(3+\frac{2r}{n}\right).\frac{1}{n}$$

$$= \frac{3}{n}.n + \frac{2}{n^2}\sum_{r=1}^{n} r = 3 + \frac{2}{n^2}.\frac{n(n+1)}{2} = 3 + \frac{n+1}{n} = 4 + \frac{1}{n}$$

$\therefore$ $$\underline{\int_1^2} f(x)\,dx = \sup\{L(P, f)\}_{P\in P[1,2]} = \lim_{n\to\infty}\left(4-\frac{1}{n}\right) = 4$$

and $$\overline{\int_1^2} f(x)\,dx = \inf\{U(P, f)\}_{P\in P[1,2]} = \lim_{n\to\infty}\left(4+\frac{1}{n}\right) = 4$$

Since $\underline{\int_1^2} f(x)\,dx = \overline{\int_1^2} f(x)\,dx = 4$

$\therefore$ $f \in R[1, 2]$ and $\int_1^2 f(x)\,dx = 4.$

Example 12. *Prove that* $\int_1^2 f(x)\,dx = 7$, *where* $f(x) = 2x + 4$.

Sol. *Please try yourself.*

Example 13. *Prove that* $f(x) = 3x + 1$ *is integrable on* $[1, 2]$ *and* $\int_1^2 (3x+1)\,dx = \frac{11}{2}$.

Sol. *Please try yourself.*

Example 14. *Show that* $f(x) = 2 - 3x$ *is integrable on* $[1, 3]$ *and* $\int_1^3 (2-3x)\,dx = -8$.

Sol. Consider a partition $P = \left\{1, 1+\frac{2}{n}, 1+\frac{4}{n}, \ldots\ldots, 1+\frac{2n}{n}\right\}$ dividing the interval $[1, 3]$ into n equal parts.

Then for any sub-interval $I_r = \left[1+\frac{2(r-1)}{n}, 1+\frac{2r}{n}\right]$, $r = 1, 2, \ldots\ldots, n$, we have

$$m_r = 2 - 3\left(1+\frac{2r}{n}\right) = -1 - \frac{6r}{n}$$

$$M_r = 2 - 3\left[1+\frac{2(r-1)}{n}\right] = -1 - \frac{6(r-1)}{n}$$

$[\because\ f(x) = 2 - 3x$ is decreasing on $[1, 3]]$

$$\delta_r = \frac{2}{n}$$

$\therefore$ $$L(P, f) = \sum_{r=1}^{n} m_r\delta_r = \sum_{r=1}^{n}\left(-1-\frac{6r}{n}\right).\frac{2}{n}$$

$$= -\frac{2}{n}.n - \frac{12}{n^2}\sum_{r=1}^{n} r = -2 - \frac{12}{n^2}.\frac{n(n+1)}{2} = -2 - \frac{6(n+1)}{n} = -8 - \frac{6}{n}$$

$$U(P, f) = \sum_{r=1}^{n} M_r\delta_r = \sum_{r=1}^{n}\left[-1-\frac{6(r-1)}{n}\right].\frac{2}{n}$$

$$= -\frac{2}{n}.n - \frac{12}{n^2}\sum_{r=1}^{n}(r-1) = -2 - \frac{12}{n^2}.\frac{(n-1)n}{2} = -2 - \frac{6(n-1)}{n} = -8 + \frac{6}{n}$$

$\therefore$ $$\underline{\int_1^3} f(x)\,dx = \sup\{L(P, f)\}_{P\in P[1,3]} = \lim_{n\to\infty}\left(-8-\frac{6}{n}\right) = -8$$

and $$\overline{\int_1^3} f(x)\,dx = \inf\{U(P, f)\}_{P\in P[1,3]} = \lim_{n\to\infty}\left(-8+\frac{6}{n}\right) = -8$$

Since $\underline{\int_1^3} f(x)\,dx = \overline{\int_1^3} f(x)\,dx = -8$

$\therefore$ $f \in R[1, 3]$ and $\int_1^3 f(x)\,dx = -8.$

Example 15. *Show that* $f(x) = x$ *is integrable on* $[a, b]$ *and* $\int_a^b f(x)\,dx = \frac{1}{2}(b^2 - a^2)$.

Sol. Consider a partition $P = \left\{a, a+\frac{h}{n}, a+\frac{2h}{n}, \ldots\ldots, a+\frac{nh}{n}\right\}$ where $h = b - a$, dividing the interval $[a, b]$ into n equal parts.

Then for any sub-interval $I_r = \left[a+\frac{(r-1)h}{n}, a+\frac{rh}{n}\right]$, $r = 1, 2, \ldots\ldots, n$, we have

$$m_r = a + \frac{(r-1)h}{n},\ M_r = a + \frac{rh}{n},\ \delta_r = \frac{h}{n}.$$

$\therefore$ $$L(P, f) = \sum_{r=1}^{n} m_r\delta_r = \sum_{r=1}^{n}\left[a+\frac{(r-1)h}{n}\right].\frac{h}{n}$$

$$= \frac{ah}{n}.n + \frac{h^2}{n^2}\sum_{r=1}^{n}(r-1) = ah + \frac{h^2}{n^2}.\frac{(n-1)n}{2} = ah + \frac{h^2}{2}\left(1-\frac{1}{n}\right)$$

$$U(P, f) = \sum_{r=1}^{n} M_r\delta_r = \sum_{r=1}^{n}\left(a+\frac{rh}{n}\right).\frac{h}{n}$$

$$= \frac{ah}{n}.n + \frac{h^2}{n^2}\sum_{r=1}^{n} r = ah + \frac{h^2}{n^2}.\frac{n(n+1)}{2} = ah + \frac{h^2}{2}\left(1+\frac{1}{n}\right)$$

$\therefore$ $$\underline{\int_a^b} f(x)\,dx = \sup\{L(P, f)\}_{P\in P[a,b]} = \lim_{n\to\infty}\left[ah + \frac{h^2}{2}\left(1-\frac{1}{n}\right)\right]$$

$$= ah + \frac{h^2}{2} = \frac{h}{2}(2a+h) = \frac{b-a}{2}(2a+b-a) = \frac{1}{2}(b^2-a^2)$$

and $\int_a^{\bar{b}} f(x)\,dx = \inf\{U(P,f)\}_{P\in P[a,b]} = \lim_{n\to\infty}\left[ah + \frac{h^2}{2}\left(1+\frac{1}{n}\right)\right] = ah + \frac{h^2}{2} = \frac{1}{2}(b^2 - a^2)$

Since $\int_{\underline{a}}^b f(x)\,dx = \int_a^{\bar{b}} f(x)\,dx = \frac{1}{2}(b^2 - a^2)$

$\therefore \quad f \in R[a, b]$ and $\int_a^b f(x)\,dx = \frac{1}{2}(b^2 - a^2)$.

Example 16. *Show that $f(x) = x^2$ is integrable on $[1, 4]$ and $\int_1^4 x^2\,dx = 21$.*

Sol. *Please try yourself.*

Example 17. *Let f be defined on [0, 1] by* $f(x) = \begin{cases} \frac{1}{2}, \text{ when } x \in Q \\ \frac{1}{3}, \text{ when } x \in R - Q. \end{cases}$

Then show that f is bounded but not R-integrable on [0, 1].

Sol. *Please try yourself.*

[Hint. $m_r = \frac{1}{3}$, $M_r = \frac{1}{2}$]

Example 18. *A function f is bounded on [a, b]. Show that (i) when k is a positive constant,*

$$\int_a^{\bar{b}} kf\,dx = k\int_a^{\bar{b}} f\,dx \text{ and } \int_{\underline{a}}^b kf\,dx = k\int_{\underline{a}}^b f\,dx$$

and (ii) when k is a negative constant, $\int_a^{\bar{b}} kf\,dx = k\int_{\underline{a}}^b f\,dx$ *and* $\int_{\underline{a}}^b kf\,dx = k\int_a^{\bar{b}} f\,dx.$

Also deduce that if f ~~is integrable~~ on [a, b], then so is kf, where k is a constant and $\int_a^b kf\,dx = k\int_a^b f\,dx.$

Sol. Consider a partition $P = \{a = x_0, x_1, x_2, ..., x_n = b\}$ of the interval $[a, b]$.

Let m_r and M_r be the infimum and supermum of f on $I_r = [x_{r-1}, x_r]$.

Let m_r' and M_r' be the infimum and supermum of kf and $I_r = [x_{r-1}, x_r]$.

(*i*) Since $k > 0$, $m_r' = km_r$ and $M_r' = kM_r$

$$U(P, kf) = \sum_{r=1}^n M_r'\delta_r = \sum_{r=1}^n kM_r\,\delta_r = k\sum_{r=1}^n M_r\,\delta_r = kU(P, f)$$

Similarly, $L(P, kf) = kL(P, f)$

$$\therefore \quad \int_a^{\bar{b}} kf\,dx = \inf\{U(P, kf)\}_{P\in P[a,b]} = \inf\{k\,U(P,f)\}_{P\in P[a,b]}$$

$$= k\inf\{U(P,f)\}_{P\in P[a,b]} = k\int_a^{\bar{b}} f\,dx$$

and $$\int_{\underline{a}}^b kf\,dx = \sup\{L(P,kf)\}_{P\in P[a,b]} = \sup\{kL(P,f)\}_{P\in P[a,b]}$$

$$= k\sup\{L(P,f)\}_{P\in P[a,b]} = k\int_{\underline{a}}^b f\,dx.$$

(*ii*) Since $k < 0$, $m_r' = kM_r$ and $M_r' = km_r$

$$U(P, kf) = \sum_{r=1}^n M_r'\delta_r = \sum_{r=1}^n km_r\,\delta_r = k\sum_{r=1}^n m_r\delta_r = kL(P, f)$$

Similarly, $L(P, kf) = kU(P, f)$

$$\therefore \quad \int_a^{\bar{b}} kf\,dx = \inf\{U(P,kf)\}_{P\in P[a,b]} = \inf\{kL(P,f)\}_{P\in P[a,b]}$$

$$= k\sup\{L(P,f)\}_{P\in P[a,b]} = k\int_{\underline{a}}^b f\,dx$$

and $$\int_{\underline{a}}^b kf\,dx = \sup\{L(P,kf)\}_{P\in P[a,b]} = \sup\{kU(P,f)\}_{P\in P[a,b]}$$

$$= k\inf\{U(P,f)\}_{P\in P[a,b]} = k\int_a^{\bar{b}} f\,dx.$$

If f is integrable ~~on [a, b], then~~ $\int_{\underline{a}}^b f\,dx = \int_a^{\bar{b}} f\,dx = \int_a^b f\,dx$

$\therefore$ From parts (*i*) and (*ii*), we have $\int_a^{\bar{b}} kf\,dx = \int_{\underline{a}}^b kf\,dx = k\int_a^b f\,dx$

$\Rightarrow$ kf is integrable on $[a, b]$ and $\int_a^b kf\,dx = k\int_a^b f\,dx.$

9.10. DARBOUX'S THEOREM

If $f : [a, b] \to R$ is a bounded function, then for each $\varepsilon > 0$, there exists a $\delta > 0$ such that

(*i*) $U(P, f) < \int_a^{\bar{b}} f(x)\,dx + \varepsilon$ (*ii*) $L(P, f) > \int_{\underline{a}}^b f(x)\,dx - \varepsilon$

for each $P \in P[a, b]$ with $\|P\| < \delta$.

Proof. Since f is bounded on $[a, b]$, there exists a real number $k > 0$ such that

$$|f(x)| \le k \;\forall\; x \in [a, b].$$

(*i*) By definition $\int_a^{\bar{b}} f(x)\,dx = \inf\{U(P,f)\}_{P\in P[a,b]}$

$\therefore$ For each $\varepsilon > 0$, there exists a partition $P_1 = \{a = x_0, x_1, x_2, ..., x_p = b\}$ such that

$$U(P_1, f) < \int_a^{\bar{b}} f(x)\,dx + \frac{\varepsilon}{2} \quad ...(1)$$

The partition P_1 has $(p - 1)$ points excluding the end points a and b. Choose $\delta > 0$ such that

$$2k(p-1)\delta = \frac{\varepsilon}{2} \quad ...(2)$$

Let P be any partition with $\|P\| < \delta$. Then P may contain some or none of the partition points x_r, $r = 1, 2, ..., p - 1$ belonging to P_1.

If $P_2 = P \cup P_1$, then P_2 is finer than P and contains at the most $p - 1$ additional points.

$\therefore \quad U(P, f) - 2k(p-1)\delta \le U(P_2, f) \le U(P_1, f)$

$$< \int_a^{\bar{b}} f(x)\,dx + \frac{\varepsilon}{2} \qquad \text{[Using (1)]}$$

$$\Rightarrow \quad U(P, f) < 2k(p-1)\,\delta + \int_a^{\bar{b}} f(x)\,dx + \frac{\varepsilon}{2}$$

$$\Rightarrow \quad U(P, f) < \frac{\varepsilon}{2} + \int_a^{\bar{b}} f(x)\,dx + \frac{\varepsilon}{2} \qquad \text{[Using (2)]}$$

$$\Rightarrow \quad U(P, f) < \int_a^{\bar{b}} f(x)\,dx + \varepsilon \text{ for any partition P with } \| P \| < \delta.$$

(*ii*) By definition $\int_{\underline{a}}^{b} f(x)\,dx = \sup \{L(P, f)\}_{P \in P[a, b]}$

$\therefore$ For each $\varepsilon > 0$, there exists a partition $P_1 = \{a = x_0, x_1, x_2, ..., x_p = b\}$ such that

$$L(P_1, f) > \int_{\underline{a}}^{b} f(x)\,dx - \frac{\varepsilon}{2} \qquad ...(3)$$

The partition P_1 has $(p-1)$ points excluding the end points a and b. Choose $\delta > 0$ such that

$$2k(p-1)\delta = \frac{\varepsilon}{2} \qquad ...(4)$$

Let P be any partition with $\| P \| < \delta$. Then P may contain some or none of the partition points x_r, $r = 1, 2, ..., p-1$ belonging to P_1.

If $P_2 = P \cup P_1$, then P_2 is finer than P and contains at most $p-1$ additional points.

$\therefore$ $L(P, f) + 2k(p-1)\,\delta \geq L(P_2, f) \geq L(P_1, f)$

$$> \int_{\underline{a}}^{b} f(x)\,dx - \frac{\varepsilon}{2} \qquad \text{[Using (3)]}$$

$$\Rightarrow \quad L(P, f) > \int_{\underline{a}}^{b} f(x)\,dx - \frac{\varepsilon}{2} - 2k(p-1)\delta$$

$$\Rightarrow \quad L(P, f) > \int_{\underline{a}}^{b} f(x)\,dx - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} \qquad \text{[Using (4)]}$$

$$\Rightarrow \quad L(P, f) > \int_{\underline{a}}^{b} f(x)\,dx - \varepsilon \text{ for any partition P with } \| P \| < \delta.$$

Note. $U(P, f) < \int_a^{\bar{b}} f(x)\,dx + \varepsilon$ is clear even by the definition of infimum but then it implies that there exists at least one partition P with this property. The importance of Darboux's theorem is the existence of an infinite number of partitions P with $\| P \| < \delta$ where δ is a positive number depending on the choice of ε.

Corollary 1. For $\varepsilon > 0$, there exists $\delta > 0$ such that $U(P, f) < \int_a^{\bar{b}} f(x)\,dx + \frac{\varepsilon}{2}$

and $\quad L(P, f) > \int_{\underline{a}}^{b} f(x)\,dx - \frac{\varepsilon}{2}$ or $-L(P, f) < -\int_{\underline{a}}^{b} f(x)\,dx + \frac{\varepsilon}{2}$

$\therefore \quad \omega(P, f) = U(P, f) - L(P, f)$

$$< \int_a^{\bar{b}} f(x)\,dx - \int_{\underline{a}}^{b} f(x)\,dx + \varepsilon \text{ for any partition P with } \| P \| < \delta.$$

Corollary 2. For each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$U(P, f) < \int_a^{\bar{b}} f(x)\,dx + \varepsilon \text{ with } \| P \| < \delta$$

$$\Rightarrow \quad -\varepsilon < 0 \leq U(P, f) - \int_a^{\bar{b}} f(x)\,dx < \varepsilon \quad \Rightarrow \quad \left| U(P, f) - \int_a^{\bar{b}} f(x)\,dx \right| < \varepsilon$$

$$\therefore \quad \lim_{\| P \| \to 0} U(P, f) = \int_a^{\bar{b}} f(x)\,dx$$

As the norm of a partition P becomes small, the number (n) of sub-intervals becomes large in such a way that $n \to \infty$ as $\| P \| \to 0$.

$$\therefore \quad \int_a^{\bar{b}} f(x)\,dx = \lim_{\| P \| \to 0} U(P, f) = \lim_{n \to \infty} U(P, f).$$

Similarly, $\quad \int_{\underline{a}}^{b} f(x)\,dx = \lim_{\| P \| \to 0} L(P, f) = \lim_{n \to \infty} L(P, f)$

The above results in the limit form are very useful in working out problems on integrability of functions.

9.11. NECESSARY AND SUFFICIENT CONDITION FOR INTEGRABILITY

Theorem 1. *A bounded function f is integrable on* $[a, b]$ *if and only if for each* $\varepsilon > 0$, *there exists a partition P of* $[a, b]$ *such that* $U(P, f) - L(P, f) < \varepsilon$.

Proof. (The condition is necessary)

Let f be integrable on $[a, b]$ so that $\int_{\underline{a}}^{b} f(x)\,dx = \int_a^{\bar{b}} f(x)\,dx = \int_a^b f(x)\,dx$

Let $\varepsilon > 0$ be given

Since $\quad \int_a^{\bar{b}} f(x)\,dx = \inf \{U(P, f)\}_{P \in P[a, b]}$ and $\int_{\underline{a}}^{b} f(x)\,dx = \sup \{L(P, f)\}_{P \in P[a, b]}$

therefore, there exist partitions P_1 and P_2 of $[a, b]$ such that

$$U(P_1, f) < \int_a^{\bar{b}} f(x)\,dx + \frac{\varepsilon}{2} = \int_a^b f(x)\,dx + \frac{\varepsilon}{2} \qquad ...(1)$$

and

$$L(P_2, f) > \int_{\underline{a}}^{b} f(x)\,dx - \frac{\varepsilon}{2} = \int_a^b f(x)\,dx - \frac{\varepsilon}{2} \qquad ...(2)$$

Let $P = P_1 \cup P_2$, then $U(P, f) \leq U(P_1, f) < \int_a^b f(x)\,dx + \frac{\varepsilon}{2} < L(P_2, f) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$

$\leq L(P, f) + \varepsilon$ [using (1) and (2)]

$\Rightarrow \quad U(P, f) - L(P, f) < \varepsilon$

Conversely. (The condition is sufficient)

Let $\varepsilon > 0$ be given. Let P be a partition of $[a, b]$ such that $U(P, f) - L(P, f) < \varepsilon \qquad ...(3)$

Since $\quad L(P, f) \leq \int_{\underline{a}}^{b} f(x)\,dx \leq \int_a^{\bar{b}} f(x)\,dx \leq U(P, f)$

$$\therefore \int_a^{\bar{b}} f(x)\,dx - \int_{\underline{a}}^{b} f(x)\,dx \leq U(P, f) - L(P, f) < \varepsilon \qquad \text{[by (3)]}$$

But $\varepsilon > 0$ is arbitrary

$$\therefore \quad \overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx = 0 \quad \Rightarrow \quad \overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx$$

$\Rightarrow$ f is integrable.

Theorem 2. *A bounded function f is integrable on $[a, b]$ if and only if for each $\varepsilon > 0$, there corresponds a $\delta > 0$ such that for every partition P of $[a, b]$ with $\|P\| < \delta$, $U(P, f) - L(P, f) < \varepsilon$.*

Proof. (The condition is necessary)

Let f be integrable on $[a, b]$ so that $\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx$

Let $\varepsilon > 0$ be given. By Darboux's theorem, there exists $\delta > 0$, such that for each partition P of $[a, b]$, with $\|P\| < \delta$,

$$U(P, f) < \overline{\int_a^b} f(x)\,dx + \frac{\varepsilon}{2} = \int_a^b f(x)\,dx + \frac{\varepsilon}{2} \quad ...(1)$$

and

$$L(P, f) > \underline{\int_a^b} f(x)\,dx - \frac{\varepsilon}{2} = \int_a^b f(x)\,dx - \frac{\varepsilon}{2}$$

or

$$-L(P, f) < -\int_a^b f(x)\,dx + \frac{\varepsilon}{2} \quad ...(2)$$

Adding (1) and (2), we have $U(P, f) - L(P, f) < \varepsilon$ for each partition P with $\|P\| < \delta$.

Conversely. (The condition is sufficient)

Let $\varepsilon > 0$ be given. Then for each partition P with $\|P\| < \delta$ (where δ is a positive number depending on ε).

$$U(P, f) - L(P, f) < \varepsilon \quad ...(3)$$

Also, for any partition P, $\quad L(P, f) \le \underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx \le U(P, f)$

$$\Rightarrow \quad \overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx \le U(P, f) - L(P, f) < \varepsilon \quad \text{[by (3)]}$$

But $\varepsilon > 0$ is arbitrary

$$\therefore \quad \overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx = 0 \quad \Rightarrow \quad \overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx$$

$\Rightarrow$ f is integrable.

Corollary 1. *(Based on Theorem 1)*

A bounded function f is integrable on $[a, b]$ if and only if there exists a number I such that for each $\varepsilon > 0$, there exists a partition P of $[a,b]$ such that $|U(P, f) - I| < \varepsilon$ and $|I - L(P, f)| < \varepsilon$.

Proof. Necessary part

Let f be integrable on $[a, b]$.

Let $\varepsilon > 0$ be given. Then $\exists$ a partition P such that

$$U(P, f) - L(P, f) < \varepsilon \quad \Rightarrow \quad -\varepsilon < 0 \le U(P, f) - L(P, f) < \varepsilon$$

$\Rightarrow \quad |U(P, f) - L(P, f)| < \varepsilon$

If I is a number lying between $L(P, f)$ and $U(P, f)$, then

$$|U(P, f) - I| < |U(P, f) - L(P, f)| < \varepsilon$$

and

$$|I - L(P, f)| < |U(P, f) - L(P, f)| < \varepsilon.$$

Sufficient part

Let $\varepsilon > 0$ be given. Then $\exists$ a partition P such that $|U(P, f) - I| < \frac{\varepsilon}{2}$ and $|I - L(P, f)| < \frac{\varepsilon}{2}$

$$\text{Now } |U(P, f) - L(P, f)| = |U(P, f) - I + I - L(P, f)|$$

$$\le |U(P, f) - I| + |I - L(P, f)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\Rightarrow$ f is integrable on $[a, b]$.

Corollary 2. *(Based on Theorem 2)*

A bounded function f is integrable on $[a, b]$ if and only if there is a number I such that for each $\varepsilon > 0$, there exists a $\delta > 0$ such that for all partitions P with $\|P\| < \delta$,

$$|U(P, f) - I| < \varepsilon \text{ and } |I - L(P, f)| < \varepsilon.$$

Proof. Necessary part

Let f be integrable on $[a, b]$,

Let $\varepsilon > 0$ be given. Then $\exists$ a $\delta > 0$ such that for every partition P of $[a, b]$ with $\|P\| < \delta$,

$$|U(P, f) - L(P, f)| < \varepsilon$$

If I is a number lying between $L(P, f)$ and $U(P, f)$, then

$$|U(P, f) - I| < |U(P, f) - L(P, f)| < \varepsilon$$

and

$$|I - L(P, f)| < |U(P, f) - L(P, f)| < \varepsilon$$

Sufficient part

Let $\varepsilon > 0$ be given. Then for every partition P with $\|P\| < \delta$ (where δ is a positive number depending on ε)

$$|U(P, f) - I| < \frac{\varepsilon}{2} \quad \text{and} \quad |I - L(P, f)| < \frac{\varepsilon}{2}$$

$$\text{Now } |U(P, f) - L(P, f)| = |U(P, f) - I + I - L(P, f)|$$

$$\le |U(P, f) - I| + |I - L(P, f)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\Rightarrow$ f is integrable on $[a, b]$.

9.12. SOME CLASSES OF BOUNDED INTEGRABLE FUNCTIONS

A bounded function $f : [a, b] \to R$ is integrable on $[a, b]$ if

(*i*) f is continuous on $[a, b]$ (*ii*) f is monotonic on $[a, b]$

(*iii*) f has a finite number of points of discontinuity on $[a, b]$

(*iv*) the set of points of discontinuity of f on $[a, b]$ has a finite number of limit points.

Now we prove these assertions in the following theorems.

Theorem 1. *If $f : [a, b] \to R$ is continuous on $[a, b]$, then f is integrable on $[a, b]$.*

Proof. f is continuous on **closed interval** $[a, b]$

$\Rightarrow$ f is uniformly continuous on $[a, b]$

$\Rightarrow$ For each $\varepsilon > 0, \exists$ a $\delta > 0$ such that $| f(x') - f(x'') | < \dfrac{\varepsilon}{b-a}$ ...(1)

for all $x', x'' \in [a, b]$ and $| x' - x'' | < \delta$

Let $P = \{a = x_0, x_1, x_2, ..., x_n = b\}$ be a partition of $[a, b]$ such that $\| P \| < \delta$.

Since f is continuous on $[a, b]$, therefore, f is bounded on $[a, b]$.

$\Rightarrow$ f is continuous on $I_r = [x_{r-1}, x_r]$ and attains its infimum m_r and supermum M_r at some points c_r and d_r of $[x_{r-1}, x_r]$ so that $m_r = f(c_r)$ and $M_r = f(d_r)$

Since $| c_r - d_r | \le | x_r - x_{r-1} | = \delta_r < \delta$ and $c_r, d_r \in [x_{r-1}, x_r] \subset [a, b]$

$\Rightarrow$ c_r, d_r satisfy the conditions imposed on x', x'' in (1).

$\therefore$ From (1), $| f(c_r) - f(d_r) | < \dfrac{\varepsilon}{b-a}$

But $| f(c_r) - f(d_r) | = | m_r - M_r | = M_r - m_r$

$\therefore$ $M_r - m_r < \dfrac{\varepsilon}{b-a}$

Now
$$U(P, f) - L(P, f) = \sum_{r=1}^{n} (M_r - m_r)\delta_r$$
$$< \sum_{r=1}^{n} \left(\frac{\varepsilon}{b-a}\right)\delta_r = \frac{\varepsilon}{b-a} \sum_{r=1}^{n} \delta_r = \frac{\varepsilon}{b-a}(b-a) = \varepsilon$$

$\Rightarrow$ $U(P, f) - L(P, f) < \varepsilon$ with $\| P \| < \delta$

$\Rightarrow$ f is integrable on $[a, b]$.

Note. There exist functions which are integrable but not continuous. So continuity is a sufficient but ~~not necessary~~ condition.

Theorem 2. *If $f : [a, b] \to R$ is monotonic on $[a, b]$, then f is integrable on $[a, b]$.*

Proof. Let f be monotonically increasing on $[a, b]$, then

$$f(a) \le f(x) \le f(b) \quad \forall \ x \in [a, b]$$

$\Rightarrow$ f is bounded on $[a, b]$ and $\inf f = f(a)$ and $\sup f = f(b)$

Let $\varepsilon > 0$ be given and $P = \{a = x_0, x_1, x_2, ..., x_n = b\}$ be a partition $[a, b]$ such that

$$\delta_r < \frac{\varepsilon}{f(b) - f(a) + 1} \quad \text{for } r = 1, 2, ..., n.$$

Let m_r and M_r be the infimum and supermum of f on $I_r = [x_{r-1}, x_r]$.

Since f is monotonically increasing, $m_r = f(x_{r-1})$ and $M_r = f(x_r)$

Now
$$U(P, f) - L(P, f) = \sum_{r=1}^{n} (M_r - m_r)\,\delta_r$$
$$= \sum_{r=1}^{n} [f(x_r) - f(x_{r-1})]\,\delta_r < \sum_{r=1}^{n} [f(x_r) - f(x_{r-1})] \cdot \frac{\varepsilon}{f(b) - f(a) + 1}$$
$$= \frac{\varepsilon}{f(b) - f(a) + 1} \sum_{r=1}^{n} [f(x_r) - f(x_{r-1})] = \frac{\varepsilon}{f(b) - f(a) + 1} [f(x_n) - f(x_0)]$$

$$= \frac{\varepsilon}{f(b) - f(a) + 1} \{f(b) - f(a)\} = \frac{f(b) - f(a)}{f(b) - f(a) + 1}\,\varepsilon < \varepsilon$$

$\therefore$ for each $\varepsilon > 0, \exists$ a partition P such that $U(P, f) - L(P, f) < \varepsilon$

$\Rightarrow$ f is integrable on $[a, b]$.

Similarly, when f is monotonically decreasing on $[a, b]$, we can prove that f is integrable on $[a, b]$.

Hence, f is monotonic on $[a, b]$ $\Rightarrow$ f is integrable on $[a, b]$.

Theorem 3. *If the set of points of discontinuity of a bounded function $f : [a, b] \to R$ is finite, then f is integrable on $[a, b]$.*

Proof. Let $c_1, c_2, ..., c_p$ be the finite number of points of discontinuity of f on $[a, b]$ such that

$$c_1 < c_2 < ...... < c_p.$$

Let $\varepsilon > 0$ be given.

Enclose the points $c_1, c_2, ..., c_p$ in p non-overlapping sub-intervals

$$I_1 = [a_1, b_1], I_2 = [a_2, b_2], ..., I_p = [a_p, b_p]$$

such that the sum of their lengths $= \sum_{i=1}^{p} (b_i - a_i)$ is $< \dfrac{\varepsilon}{2(M - m)}$, where m and M are the infimum and supermum of f on $[a, b]$.

Since the oscillation of f in each of these sub-intervals is $\le M - m$, therefore, the total contribution of these p sub-intervals to the oscillatory sum is

$$\sum_{i=1}^{p} (M_i - m_i)(b_i - a_i) \le (M - m) \cdot \frac{\varepsilon}{2(M - m)} = \frac{\varepsilon}{2}$$

The $(p + 1)$ sub-intervals in $[a, b]$ that are formed by deleting the above p sub-intervals are

$$I_1' = [a, a_1], I_2' = [b_1, a_2], I_3' = [b_2, a_3], ..., I_p' = [b_{p-1}, a_p], I_{p+1}' = [b_p, b].$$

f is continuous on each of these sub-intervals. Therefore, there exists a partition P_r of I_r', $r = 1, 2, ..., p + 1$ such that the part of the oscillatory sum arising from each of these $(p + 1)$ sub-intervals is

$$< \frac{\varepsilon}{2(p + 1)}$$

$\therefore$ The total contribution of these $(p + 1)$ sub-intervals to the oscillatory sum is

$$< \frac{\varepsilon}{2(p + 1)} \cdot (p + 1) = \frac{\varepsilon}{2}$$

Thus, for the partition $P = \{a, ..., a_1, b_1, ..., a_2, b_2, ..., a_p, b_p, ..., b\}$ of $[a, b]$, we have

$$U(P, f) - L(P, f) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Since for each $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that $U(P, f) - L(P, f) < \varepsilon$

Hence f is integrable on $[a, b]$.

Note. There are integrable functions having an infinite number of points of discontinuity in $[a, b]$.

Theorem 4. *If the set of points of discontinuity of a bounded function $f : [a, b] \to R$ has a finite number of limit points, then f is integrable on $[a, b]$.*

Proof. Let $c_1, c_2, ..., c_p$ be the finite number of limit points of the set of points of discontinuity of f on $[a, b]$ such that $c_1 < c_2 < ...... < c_p$.

Let $\varepsilon > 0$ be given.

Enclose the points $c_1, c_2, ..., c_p$ in p non-overlapping sub-intervals

$$I_1 = [a_1, b_1], I_2 = [a_2, b_2], ..., I_p = [a_p, b_p] \text{ such that the sum of their lengths} = \sum_{i=1}^{p} (b_i - a_i)$$

is $< \dfrac{\varepsilon}{2(M-m)}$, where m and M are the infimum and supermum of f on $[a, b]$.

Since the oscillation of f in each of these sub-intervals is $\le M - m$, therefore, the total contribution of these p sub-intervals to the oscillatory sum is $\sum_{i=1}^{p} (M_i - m_i)(b_i - a_i)$

$$\le (M-m) \cdot \frac{\varepsilon}{2(M-m)} = \frac{\varepsilon}{2}$$

The $(p + 1)$ sub-intervals in $[a, b]$ that are formed by deleting the above p sub-intervals are

$$I_1' = [a, a_1], I_2' = [b_1, b_2], I_3' = [b_2, a_3], ..., I_p' = [b_{p-1}, a_p],$$
$$I_{p+1}' = [b_p, b].$$

In each of these sub-intervals, f has only a finite number of points of discontinuity [for otherwise the interval which contains an infinite number of points of discontinuity of f will have a limit point (by Bolzano Weierestrass Theorem). Therefore, each of these sub-intervals can be further sub-divided such that the part of the oscillatory sum arising from each of these $(p + 1)$ sub-intervals is

$$< \frac{\varepsilon}{2(p+1)}$$

$\therefore$ The total contribution of these $(p + 1)$ sub-intervals to the oscillatory sum is

$$< \frac{\varepsilon}{2(p+1)} \cdot (p+1) = \frac{\varepsilon}{2}.$$

Thus, for the partition $P = \{a, ..., a_1, b_1, ..., a_2, b_2, ..., a_p, b_p, ..., b\}$ of $[a, b]$, we have

$$U(P, f) - L(P, f) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Since for each $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that

$$U(P, f) - L(P, f) < \varepsilon$$

Hence f is integrable on $[a, b]$.

9.13. RIEMANN SUM

Let f be a real valued function defined on $[a, b]$.

Let $P = \{a = x_0, x_1, x_2, ..., x_n = b\}$ be a partition of $[a, b]$.

Let $\xi_r \in [x_{r-1}, x_r]$, $r = 1, 2, ..., n$. Then the sum $\sum_{r=1}^{n} f(\xi_r)\delta_r$ is called a Riemann sum of f on $[a, b]$ relative to P.

Since ξ_r is any arbitrary point of $[x_{r-1}, x_r]$, therefore, corresponding to each partition P of $[a, b]$, there exist infinitely many Riemann sums.

9.14. INTEGRAL AS THE LIMIT OF A SUM (*Second definition of integrability*)

A function $f : [a, b] \to R$ is said to be integrable on $[a, b]$ if for each $\varepsilon > 0$, there exists a $\delta > 0$ and a number I such that for every partition $P = \{a = x_0, x_1, x_2, ..., x_n = b\}$ of $[a, b]$ with $\| P \| < \delta$ and $\xi_r \in [x_{r-1}, x_r]$ arbitrarily, $\left| \sum_{r=1}^{n} f(\xi_r)\delta_r - I \right| < \varepsilon$.

The number I is the Riemann integral of f on $[a, b]$ *i.e.*, $I = \int_a^b f(x)\,dx$

Thus a function f is integrable ~~on $[a, b]$~~ if $\lim_{\|P\| \to 0} \sum_{r=1}^{n} f(\xi_r)\delta_r$ exists and is independent of the choice of sub-interval $[x_{r-1}, x_r]$ and of the point $\xi_r \in [x_{r-1}, x_r]$.

This limit, if it exists, is $I = \int_a^b f(x)\,dx$.

9.15. EQUIVALENCE OF THE TWO DEFINITIONS OF RIEMANN INTEGRAL

Definition 1. A bounded function $f : [a, b] \to R$ is said to be integrable on $[a, b]$ if its lower and upper integrals are equal and the common value of these integrals is called the Riemann integral of f on $[a, b]$.

Thus $$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx.$$

Definition 2. A function $f : [a, b] \to R$ is said to be integrable on $[a, b]$ if for each $\varepsilon > 0$, there exists a $\delta > 0$ and a number I such that for every partition $P = \{a = x_0, x_1, x_2, ..., x_n = b\}$ of $[a, b]$ with $\| P \| < \delta$ and $\xi_r \in [x_{r-1}, x_r]$ arbitrarily,

$$\left| \sum_{r=1}^{n} f(\xi_r)\,\delta_r - I \right| < \varepsilon \text{ where } I = \int_a^b f(x)\,dx.$$

Definition 1 $\Rightarrow$ Definition 2

Let a bounded function f be integrable on $[a, b]$ according to definition 1, so that

$$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx$$

Let $\varepsilon > 0$ be given.

Then, by Darbourx's theorem, there exists a $\delta > 0$ such that for every partition P with $\| P \| < \delta$.

$$U(P, f) < \overline{\int_a^b} f(x)\,dx + \varepsilon = \int_a^b f(x)\,dx + \varepsilon \qquad ...(1)$$

and
$$L(P, f) > \int_{\underline{a}}^{b} f(x)\,dx - \varepsilon = \int_a^b f(x)\,dx - \varepsilon \qquad \ldots(2)$$

If m_r, M_r be the infimum and supermum of f on $[x_{r-1}, x_r]$, then for $\xi_r \in [x_{r-1}, x_r]$, we have

$$m_r \le f(\xi_r) \le M_r \quad \Rightarrow \quad m_r\delta_r \le f(\xi_r)\,\delta_r \le M_r\delta_r$$

$$\Rightarrow \quad \sum_{r=1}^{m} m_r\delta_r \le \sum_{r=1}^{n} f(\xi_r)\,\delta_r \le \sum_{r=1}^{n} M_r\delta_r \quad \Rightarrow \quad L(P, f) \le \sum_{r=1}^{n} f(\xi_r)\delta_r \le U(P, f) \qquad \ldots(3)$$

From (1), (2) and (3), we have $\int_a^b f(x)\,dx - \varepsilon < L(P, f) \le \sum_{r=1}^{n} f(\xi_r)\,\delta_r \le U(P, f) < \int_a^b f(x)\,dx + \varepsilon$

$$\Rightarrow \quad I - \varepsilon < \sum_{r=1}^{n} f(\xi_r)\,\delta_r < I + \varepsilon \quad \text{where} \quad I = \int_a^b f(x)\,dx$$

$$\Rightarrow \quad \left|\sum_{r=1}^{n} f(\xi_r)\delta_r - I\right| < \varepsilon.$$

$\Rightarrow$ f is integrable according to definition 2.

Conversely. Definition 2 $\Rightarrow$ Definition 1

Let f be integrable on $[a, b]$ according to definition 2.

We shall show that f is bounded on $[a,b]$ and its lower and upper integrals on $[a, b]$ are equal.

If possible, let f be not bounded on $[a, b]$.

By definition 2, for $\varepsilon = 1$, there exists $\delta > 0$ and a number I such that for each partition P of $[a, b]$ with $\|P\| < \delta$,

$$\left|\sum_{r=1}^{n} f(\xi_r)\delta_r - I\right| < 1, \ \forall\ \xi_r \in [x_{r-1}, x_r]$$

$$\therefore \quad \left|\left|\sum_{r=1}^{n} f(\xi_r)\delta_r\right| - |I|\right| \le \left|\sum_{r=1}^{n} f(\xi_r)\delta_r - I\right| < 1$$

$$\Rightarrow \quad |I| - 1 < \left|\sum_{r=1}^{n} f(\xi_r)\delta_r\right| < |I| + 1$$

$$\Rightarrow \quad \left|\sum_{r=1}^{n} f(\xi_r)\delta_r\right| < |I| + 1 \ \forall\ \xi_r \in [x_{r-1}, x_r] \qquad \ldots(4)$$

Since f is not bounded on $[a, b]$, it is not bounded on at least one sub-interval of P, say $[x_{m-1}, x_m]$.

Taking $\xi_r = x_r$ for $r \ne m$, each term of $\sum_{r=1}^{n} f(\xi_r)\,\delta_r$ except $f(\xi_m)\,\delta_m$ is fixed,

i.e.,
$$\sum_{r=1}^{m-1} f(\xi_r)\delta_r + \sum_{r=m+1}^{n} f(\xi_r)\delta_r \text{ is fixed.}$$

Since f is not bounded on $[x_{m-1}, x_m]$, we can choose $\xi_m \in [x_{m-1}, x_m]$ such that

$$\left|\sum_{r=1}^{m-1} f(\xi_r)\,\delta_r + \sum_{r=m+1}^{n} f(\xi_r)\,\delta_r + f(\xi_m)\,\delta_m\right| < |I| + 1$$

i.e.,
$$\left|\sum_{r=1}^{n} f(\xi_r)\delta_r\right| > |I| + 1 \quad \text{which contradicts (4).}$$

$\therefore$ f cannot be unbounded on any sub-interval of $[a, b]$ and hence f is bounded on $[a, b]$.

Now, let $\varepsilon > 0$ be given. Then, by definition 2, there exists $\delta > 0$ and a number I such that for every partition P with $\|P\| < \delta$,

$$\left|\sum_{r=1}^{n} f(\xi_r)\delta_r - I\right| < \frac{\varepsilon}{2} \ \forall\ \xi_r \in [x_{r-1}, x_r]$$

$$\Rightarrow \quad I - \frac{\varepsilon}{2} < \sum_{r=1}^{n} f(\xi_r)\delta_r < I + \frac{\varepsilon}{2} \ \forall\ \xi_r \in [x_{r-1}, x_r] \qquad \ldots(5)$$

Let m_r, M_r be the infimum and supermum of f on $[x_{r-1}, x_r]$, then there exist points α_r, $\beta_r \in [x_{r-1}, x_r]$ such that

$$f(\alpha_r) < m_r + \frac{\varepsilon}{2(b-a)} \quad \text{and} \quad f(\beta_r) > M_r - \frac{\varepsilon}{2(b-a)}$$

$$\Rightarrow \quad \sum_{r=1}^{n} f(\alpha_r)\delta_r < \sum_{r=1}^{n} m_r\delta_r + \frac{\varepsilon}{2(b-a)}\sum_{r=1}^{n}\delta_r$$

and
$$\sum_{r=1}^{n} f(\beta_r)\delta_r > \sum_{r=1}^{n} M_r\delta_r - \frac{\varepsilon}{2(b-a)}\sum_{r=1}^{n}\delta_r$$

$$\Rightarrow \quad \sum_{r=1}^{n} f(\alpha_r)\delta_r < L(P, f) + \frac{\varepsilon}{2(b-a)}\cdot(b-a)$$

and
$$\sum_{r=1}^{n} f(\beta_r)\delta_r < U(P, f) - \frac{\varepsilon}{2(b-a)}\cdot(b-a)$$

$$\left.\begin{aligned} \Rightarrow \quad & \sum_{r=1}^{n} f(\alpha_r)\delta_r < L(P, f) + \frac{\varepsilon}{2} \\ \text{and} \quad & \sum_{r=1}^{n} f(\beta_r)\delta_r > U(P, f) - \frac{\varepsilon}{2} \end{aligned}\right] \qquad \ldots(6)$$

From (5) and (6), taking $\xi_r = \alpha_r$ and β_r, we have

$$I - \frac{\varepsilon}{2} < \sum_{r=1}^{n} f(\xi_r)\delta_r < L(P, f) + \frac{\varepsilon}{2}$$

and
$$I + \frac{\varepsilon}{2} > \sum_{r=1}^{n} f(\xi_r)\delta_r > U(P, f) - \frac{\varepsilon}{2}$$

$\Rightarrow \qquad I - \varepsilon < L(P, f)$ and $I + \varepsilon > U(P, f)$ for every partition P with $\| P \| < \delta$.

But $\qquad L(P, f) \leq \int_{\underline{a}}^{b} f(x)\, dx \leq \int_{a}^{\bar{b}} f(x)\, dx \leq U(P, f)$

$\therefore \qquad I - \varepsilon < \int_{\underline{a}}^{b} f(x)\, dx \leq \int_{a}^{\bar{b}} f(x)\, dx < I + \varepsilon$

$\Rightarrow \qquad \left| \int_{a}^{\bar{b}} f(x)\, dx - \int_{\underline{a}}^{b} f(x)\, dx \right| < (I + \varepsilon) - (I - \varepsilon) = 2\varepsilon$

Since $\varepsilon > 0$ is arbitrary.

$\therefore \qquad \int_{a}^{\bar{b}} f(x)\, dx - \int_{\underline{a}}^{b} f(x)\, dx = 0 \Rightarrow \int_{a}^{\bar{b}} f(x)\, dx = \int_{\underline{a}}^{b} f(x)\, dx$

$\Rightarrow$ f is integrable according to definition 1.

Note. From the above theorem, we conclude that f is integrable on $[a, b]$ if $\lim\limits_{\| P \| \to 0} \sum\limits_{r=1}^{n} f(\xi_r)\delta_r$ exists, where $\xi_r \in [x_{r-1}, x_r]$ and $\int_a^b f(x)\, dx = \lim\limits_{\| P \| \to 0} \sum\limits_{r=1}^{n} f(\xi_r)\delta_r$.

Cor. 1. *If f is integrable on $[a, b]$, then $\int_a^b f(x) dx = \lim\limits_{n \to \infty} \sum\limits_{r=1}^{n} hf(a + rh)$, where* $h = \dfrac{b-a}{n}$.

Proof. Let $P = \{a, a + h, a + 2h, \ldots\ldots, a + nh = b\}$ be a partition of $[a, b]$.

It divides the interval $[a, b]$ into n equal sub-intervals, each of length $h = \dfrac{b-a}{n}$.

$$\therefore \qquad \| P \| = \frac{b-a}{n}$$

As $\| P \| \to 0, n \to \infty$

The rth sub-interval $= [a + (r - 1) h, a + rh]$

Let ξ_r be such that $a + (r - 1) h \leq \xi_r \leq a + rh, r = 1, 2, \ldots\ldots, n$.

Then $\qquad \int_a^b f(x)\, dx = \lim\limits_{\| P \| \to 0} \sum\limits_{r=1}^{n} f(\xi_r)\delta_r = \lim\limits_{n \to \infty} \sum\limits_{r=1}^{n} f(a + rh)(h)$, taking $\xi_r = a + rh$

$$= \lim_{n \to \infty} \sum_{r=1}^{n} hf(a + rh).$$

Cor. 2. *If f is integrable on $[0, 1]$, then $\int_0^1 f(x)\, dx = \lim\limits_{n \to \infty} \dfrac{1}{n} \sum\limits_{r=1}^{n} f\left(\dfrac{r}{n}\right)$.*

Proof. $P = \left\{0, \dfrac{1}{n}, \dfrac{2}{n}, \ldots\ldots, \dfrac{n}{n} = 1\right\}$ be a partition of $[0, 1]$.

It divides the interval $[0, 1]$ into n equal sub-intervals, each of length $\dfrac{1}{n}$. Thus

$$\delta_r = \frac{1}{n}, r = 1, 2, \ldots\ldots, n.$$

$$\therefore \qquad \| P \| = \frac{1}{n}$$

As $\| P \| \to 0, n \to \infty$

The rth sub-interval $= \left[\dfrac{r-1}{n}, \dfrac{r}{n}\right]$

Let ξ_r be such that $\dfrac{r-1}{n} \leq \xi_r \leq \dfrac{r}{n}, r = 1, 2, \ldots\ldots, n$.

Then $\qquad \int_0^1 f(x)\, dx = \lim\limits_{\| P \| \to 0} \sum\limits_{r=1}^{n} f(\xi_r)\delta_r = \lim\limits_{n \to \infty} \sum\limits_{r=1}^{n} f\left(\dfrac{r}{n}\right)\left(\dfrac{1}{n}\right) \qquad \left(\text{taking } \xi_r = \dfrac{r}{n}\right)$

$$= \lim_{n \to \infty} \frac{1}{n} \sum_{r=1}^{n} f\left(\frac{r}{n}\right).$$

Note. To evaluate the limit of a sum

(*i*) write the limit of sum in the form $\lim\limits_{n \to \infty} \dfrac{1}{n} \sum\limits_{r=1}^{n} f\left(\dfrac{r}{n}\right)$.

(*ii*) replace $\dfrac{r}{n}$ by x and $\dfrac{1}{n}$ by dx. $\qquad$ (*iii*) replace $\lim\limits_{n \to \infty} \sum\limits_{r=1}^{n}$ by $\int_0^1$.

[Note that the limits of integration are the values of $\dfrac{r}{n}$ for the first and last terms as $n \to \infty$.]

Thus $\qquad \lim\limits_{n \to \infty} \dfrac{1}{n} \sum\limits_{r=1}^{n} f\left(\dfrac{r}{n}\right) = \int_0^1 f(x)\, dx.$

Cor. 3. *If f is integrable on $[a, b]$, then*

$$\int_a^b f(x) dx = \lim_{n \to \infty} \sum_{r=1}^{n} (ah^r - ah^{r-1}) f(ah^r) \text{ where } h = \left(\frac{b}{a}\right)^{1/n}.$$

Proof. Let $P = \{a, ah, ah^2, \ldots\ldots, ah^n = b\}$ be a partition of $[a, b]$ so that $h^n = \dfrac{b}{a}$

or $\quad h = \left(\dfrac{b}{a}\right)^{1/n} \qquad$ rth sub-interval $= [ah^{r-1}, ah^r]$

$$\delta_r = ah^r - ah^{r-1}$$

As $\| P \| \to 0, h \to 1$ and $n \to \infty$

Let ξ_r, be such that $ah^{r-1} \leq \xi_r \leq ah^r, r = 1, 2, \ldots\ldots, n$.

Then $\displaystyle\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} \sum_{r=1}^{n} f(\xi_r)\delta_r = \lim_{n\to\infty} \sum_{r=1}^{n} f(ah^r)(ah^r - ah^{r-1})$ (taking $\xi_r = ah^r$)

$$= \lim_{n\to\infty} \sum_{r=1}^{n} (ah^r - ah^{r-1}) f(ah^r).$$

ILLUSTRATIVE EXAMPLES

Example 1. *From definition, prove that*

(*i*) $\displaystyle\int_1^2 f(x)\,dx = 6$ *where* $f(x) = 2x + 3$ (*ii*) $\displaystyle\int_1^2 x\,dx = \frac{3}{2}$

(*iii*) $\displaystyle\int_0^1 (2x^2 - 3x + 5)\,dx = \frac{25}{6}$ (*iv*) $\displaystyle\int_1^3 (x^2 + 2x + 3)\,dx.$

Sol. (*i*) Since $f(x) = 2x + 3$ is bounded and continuous on $[1, 2]$.

$\therefore$ f is integrable on $[1, 2]$.

Consider a partition $P = \{1 = x_0, x_1, x_2, \ldots\ldots, x_n = 2\}$ of $[1, 2]$ dividing it into n equal sub-intervals, each of length

$$\frac{b-a}{n} = \frac{2-1}{n} = \frac{1}{n} \quad \text{so that} \quad \|P\| = \frac{1}{n} \to 0 \text{ as } n \to \infty.$$

Also $\quad x_r = 1 + \dfrac{r}{n}$ and $\delta_r = \dfrac{1}{n}, r = 1, 2, \ldots\ldots, n.$

$\therefore$ $\displaystyle\int_1^2 f(x)\,dx = \lim_{\|P\|\to 0} \sum_{r=1}^{n} f(\xi_r)\,\delta_r = \lim_{n\to\infty} \sum_{r=1}^{n} f(x_r)\,\delta_r$ (taking $\xi_r = x_r$)

$$= \lim_{n\to\infty} \sum_{r=1}^{n} f\left(1 + \frac{r}{n}\right) \cdot \frac{1}{n} = \lim_{n\to\infty} \sum_{r=1}^{n} \left[2\left(1+\frac{r}{n}\right) + 3\right] \frac{1}{n}$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \left(\frac{5}{n} + \frac{2r}{n^2}\right) = \lim_{n\to\infty} \left[\frac{5}{n} \cdot n + \frac{2}{n^2} \sum_{r=1}^{n} r\right]$$

$$= \lim_{n\to\infty} \left[5 + \frac{2}{n^2} \cdot \frac{n(n+1)}{2}\right] = \lim_{n\to\infty} \left[5 + \frac{n+1}{n}\right] = \lim_{n\to\infty} \left(6 + \frac{1}{n}\right) = 6.$$

(*ii*) *Please try yourself.*

(*iii*) Since $f(x) = 2x^2 - 3x + 5$ is bounded and continuous on $[0, 1]$, therefore, f is integrable on $[0, 1]$.

Consider a partition $P = \{0 = x_0, x_1, x_2, \ldots\ldots, x_n = 1\}$ of $[0, 1]$ dividing it into n equal sub-intervals, each of length

$$\frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n} \quad \text{so that } \|P\| \to 0 \text{ as } n \to \infty.$$

Also $\quad x_r = 0 + \dfrac{r}{n} = \dfrac{r}{n}$ and $\delta_r = \dfrac{1}{n}, r = 1, 2, \ldots\ldots, n$

$\therefore$ $\displaystyle\int_0^1 f(x)\,dx = \lim_{\|P\|\to 0} \sum_{r=1}^{n} f(\xi_r)\delta_r = \lim_{n\to\infty} \sum_{r=1}^{n} f(x_r)\,.\,\delta_r$ (taking $\xi_r = x_r$)

$$= \lim_{n\to\infty} \sum_{r=1}^{n} f\left(\frac{r}{n}\right) \cdot \frac{1}{n}$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \left[2\left(\frac{r}{n}\right)^2 - 3\left(\frac{r}{n}\right) + 5\right] \cdot \frac{1}{n} = \lim_{n\to\infty} \sum_{r=1}^{n} \left(\frac{2r^2}{n^3} - \frac{3r}{n^2} + \frac{5}{n}\right)$$

$$= \lim_{n\to\infty} \left[\frac{2}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} - \frac{3}{n^2} \cdot \frac{n(n+1)}{2} + \frac{5}{n} \cdot n\right]$$

$$= \lim_{n\to\infty} \left[\frac{1}{3}\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right) - \frac{3}{2}\left(\frac{n+1}{n}\right) + 5\right]$$

$$= \lim_{n\to\infty} \left[\frac{1}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) - \frac{3}{2}\left(1+\frac{1}{n}\right) + 5\right] = \frac{1}{3}(1)(2) - \frac{3}{2}(1) + 5 = \frac{25}{6}.$$

(*iv*) *Please try yourself.* $\left[\textbf{Ans. } \dfrac{68}{3}\right]$

Example 2. *Evaluate* $\displaystyle\int_{-1}^{1} f(x)\,dx$, *where* $f(x) = |x|$.

Sol. Since $\quad f(x) = |x| = \begin{cases} -x, \text{ when } x \le 0 \\ x, \text{ when } x > 0 \end{cases}$

$\therefore$ f is bounded and continuous on $[-1, 1]$

$\Rightarrow$ f is integrable on $[-1, 1]$.

Consider a partition $P = \{-1 = x_0, x_1, \ldots\ldots, x_n = 0, x_{n+1}, x_{n+2}, \ldots\ldots, x_{2n} = 1\}$ of $[-1, 1]$ dividing it into $2n$ equal sub-intervals, each of length $\dfrac{b-a}{2n} = \dfrac{1-(-1)}{2n} = \dfrac{1}{n}$ so that $\|P\| \to 0$ as $n \to \infty$.

Also $\quad x_r = -1 + \dfrac{r}{n}$ and $\delta_r = \dfrac{1}{n}, r = 1, 2, \ldots\ldots, 2n$

$\therefore$ $\displaystyle\int_{-1}^{1} f(x)\,dx = \lim_{\|P\|\to 0} \sum_{r=1}^{2n} f(\xi_r)\delta_r = \lim_{n\to\infty} \sum_{r=1}^{2n} f(x_r)\,.\,\delta_r$ (taking $\xi_r = x_r$)

$$= \lim_{n\to\infty} \left[\sum_{r=1}^{n} f(x_r)\delta_r + \sum_{r=n+1}^{2n} f(x_r)\delta_r\right]$$

$$= \lim_{n\to\infty} \left[\sum_{r=1}^{n} f\left(-1+\frac{r}{n}\right) \cdot \frac{1}{n} + \sum_{r=n+1}^{2n} f\left(-1+\frac{r}{n}\right) \cdot \frac{1}{n}\right]$$

$$= \lim_{n\to\infty} \left[\sum_{r=1}^{n} -\left(-1+\frac{r}{n}\right) \cdot \frac{1}{n} + \sum_{r=n+1}^{2n} \left(-1+\frac{r}{n}\right) \cdot \frac{1}{n}\right]$$

$$= \lim_{n\to\infty}\left[\sum_{r=1}^{n}\left(\frac{1}{n}-\frac{r}{n^2}\right)+\sum_{r=n+1}^{2n}\left(-\frac{1}{n}+\frac{r}{n^2}\right)\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{n}.n-\frac{1}{n^2}\sum_{r=1}^{n} r+\left(-\frac{1}{n}\right).n+\frac{1}{n^2}\sum_{r=n+1}^{2n} r\right]$$

$$= \lim_{n\to\infty}\left[-\frac{1}{n^2}.\frac{n(n+1)}{2}+\frac{1}{n^2}\left\{(n+1)+(n+2)+\ldots+2n\right\}\right]$$

$$= \lim_{n\to\infty}\left[-\frac{1}{2}\left(\frac{n+1}{n}\right)+\frac{1}{n^2}.\frac{n}{2}(n+1+2n)\right]$$

$$\left[\because \text{ in an A.P. } S_n=\frac{n}{2}(a+l)\right]$$

$$= \lim_{n\to\infty}\left[-\frac{1}{2}\left(1+\frac{1}{n}\right)+\frac{1}{2n}(3n+1)\right]$$

$$= \lim_{n\to\infty}\left[-\frac{1}{2}\left(1+\frac{1}{n}\right)+\frac{1}{2}\left(3+\frac{1}{n}\right)\right]=-\frac{1}{2}+\frac{3}{2}=1.$$

Example 3. *Evaluate* $\int_{-1}^{2} f(x)\,dx$, *where* $f(x)=|x|$.

Sol. Since $f(x)=|x|=\begin{cases}-x, \text{ when } x\le 0\\ x, \text{ when } x>0\end{cases}$

$\therefore$ f is bounded and continuous on $[-1, 2]$

$\Rightarrow$ f is integrable on $[-1, 2]$

Consider a partition $P=\{-1=x_0, x_1, x_2, \ldots\ldots, x_n=0, x_{n+1}, x_{n+2}, \ldots, x_{3n}=2\}$ of $[-1, 2]$ dividing it into $3n$ equal sub-intervals, each of length $\frac{b-a}{3n}=\frac{2-(-1)}{3n}=\frac{1}{n}$ so that $\|P\|\to 0$ as $n\to\infty$.

Also $\quad x_r=-1+\frac{r}{n}$ and $\delta_r=\frac{1}{n}$, $r=1, 2, \ldots\ldots, 3n$.

$$\therefore \int_{-1}^{2} f(x)\,dx=\lim_{\|P\|\to 0}\sum_{r=1}^{3n} f(\xi_r)\delta_r=\lim_{n\to\infty}\sum_{r=1}^{3n} f(x_r)\delta_r \qquad (\text{taking } \xi_r=x_r)$$

$$= \lim_{n\to\infty}\left[\sum_{r=1}^{n} f(x_r)\delta_r+\sum_{r=n+1}^{3n} f(x_r)\delta_r\right]$$

$$= \lim_{n\to\infty}\left[\sum_{r=1}^{n} f\left(-1+\frac{r}{n}\right).\frac{1}{n}+\sum_{r=n+1}^{3n} f\left(-1+\frac{r}{n}\right).\frac{1}{n}\right]$$

$$= \lim_{n\to\infty}\left[\sum_{r=1}^{n} -\left(-1+\frac{r}{n}\right).\frac{1}{n}+\sum_{r=n+1}^{3n}\left(-1+\frac{r}{n}\right).\frac{1}{n}\right]$$

$$= \lim_{n\to\infty}\left[\sum_{r=1}^{n}\left(\frac{1}{n}-\frac{r}{n^2}\right)+\sum_{r=n+1}^{3n}\left(-\frac{1}{n}+\frac{r}{n^2}\right)\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{n}.n-\frac{1}{n^2}\sum_{r=1}^{n} r+\left(-\frac{1}{n}\right).2n+\frac{1}{n^2}\sum_{r=n+1}^{3n} r\right]$$

$$= \lim_{n\to\infty}\left[1-\frac{1}{n^2}.\frac{n(n+1)}{2}-2+\frac{1}{n^2}\left\{(n+1)+(n+2)+\ldots+3n\right\}\right]$$

$$= \lim_{n\to\infty}\left[-1-\frac{1}{2}\left(\frac{n+1}{n}\right)+\frac{1}{n^2}.\frac{2n}{2}(n+1+3n)\right]$$

$$= \lim_{n\to\infty}\left[-1-\frac{1}{2}\left(1+\frac{1}{n}\right)+\left(4+\frac{1}{n}\right)\right]=-1-\frac{1}{2}+4=\frac{5}{2}.$$

Example 4. *Evaluate* $\int_{-2}^{1} f(x)\,dx$, *where* $f(x)=|x|$.

Sol. *Please try yourself.* $\left[\textbf{Ans. } \frac{5}{2}\right]$

Example 5. *Show that* $\int_0^a \sin x\,dx=1-\cos a$, *where a is a fixed real number.*

Sol. Since $f(x)=\sin x$ is bounded and continuous on $[0, a]$, therefore, f is integrable on $[0, a]$.

Consider a partition $P=\{0=x_0, x_1, x_2, \ldots\ldots, x_n=a\}$ of $[0, a]$ dividing it into n equal sub-intervals, each of length $\frac{a-0}{n}=\frac{a}{n}$ so that $\|P\|\to 0$ as $n\to\infty$.

Also $\quad x_r=0+\frac{ra}{n}=\frac{ra}{n}$ and $\delta_r=\frac{a}{n}$, $r=1, 2, \ldots\ldots, n$.

$$\therefore \int_0^a f(x)\,dx=\lim_{\|P\|\to 0}\sum_{r=1}^{n} f(\xi_r)\delta_r=\lim_{n\to\infty}\sum_{r=1}^{n} f(x_r)\delta_r \qquad (\text{taking } \xi_r=x_r)$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n} f\left(\frac{ra}{n}\right).\frac{a}{n}=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{a}{n}\sin\frac{ra}{n}$$

$$= \lim_{n\to\infty}\frac{a}{n}\left[\sin\frac{a}{n}+\sin\frac{2a}{n}+\ldots\ldots+\sin\frac{na}{n}\right]$$

$$= \lim_{n\to\infty}\frac{a}{n}.\frac{\sin\left(\frac{a}{n}+\frac{n-1}{2}.\frac{a}{n}\right)\sin\left(\frac{n}{2}.\frac{a}{n}\right)}{\sin\frac{a}{2n}}$$

$$\left[\because \sin\alpha+\sin(\alpha+\beta)+\sin(\alpha+2\beta)+\ldots\ldots \text{ to } n \text{ terms}=\frac{\sin\left(\alpha+\frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\right]$$

$$= \lim_{n\to\infty} 2 \cdot \frac{\frac{a}{2n}}{\sin \frac{a}{2n}} \cdot \sin \frac{a}{2}\left(\frac{2}{n} + \frac{n-1}{n}\right) \cdot \sin \frac{a}{2}$$

$$= \lim_{n\to\infty} 2 \cdot \frac{\frac{a}{2n}}{\sin \frac{a}{2n}} \cdot \sin \frac{a}{2}\left(1+\frac{1}{n}\right) \cdot \sin \frac{a}{2}$$

$$= 2 \times 1 \times \sin \frac{a}{2} \times \sin \frac{a}{2} \qquad \left[\because \text{ as } n \to \infty, \theta = \frac{a}{2n} \to 0 \text{ and } \lim_{\theta\to 0} \frac{\theta}{\sin\theta} = 1\right]$$

$$= 2\sin^2 \frac{a}{2} = 1 - \cos a.$$

Example 6. *Show that $\int_0^a \cos x\, dx = \sin a$, for a fixed number a.*

Sol. *Please try yourself.*

Example 7. *Prove that $\int_0^{\pi/2} \cos x\, dx = 1$.*

Sol. Since $f(x) = \cos x$ is bounded and continuous on $\left[0, \frac{\pi}{2}\right]$, therefore, f is integrable on $\left[0, \frac{\pi}{2}\right]$.

Consider a partition $P = \left\{0 = x_0, x_1, x_2, \ldots\ldots, x_n = \frac{\pi}{2}\right\}$ of $\left[0, \frac{\pi}{2}\right]$ dividing it into n equal sub-intervals, each of length $\frac{\frac{\pi}{2} - 0}{n} = \frac{\pi}{2n}$ so that $\| P \| \to 0$ as $n \to \infty$.

Also $\quad x_r = 0 + \frac{r\pi}{2n} = \frac{r\pi}{2n}$ and $\delta_r = \frac{\pi}{2n}$, $\quad r = 1, 2, \ldots\ldots, n$.

$$\therefore \int_0^{\pi/2} f(x)\, dx = \lim_{\|P\|\to 0} \sum_{r=1}^{n} f(\xi_r)\delta_r = \lim_{n\to\infty} \sum_{r=1}^{n} f(x_r)\delta_r \qquad (\text{taking } \xi_r = x_r)$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} f\left(\frac{r\pi}{2n}\right) \cdot \frac{\pi}{2n} = \lim_{n\to\infty} \sum_{r=1}^{n} \frac{\pi}{2n} \cos \frac{r\pi}{2n}$$

$$= \lim_{n\to\infty} \frac{\pi}{2n}\left[\cos\frac{\pi}{2n} + \cos\frac{2\pi}{2n} + \ldots\ldots + \cos\frac{n\pi}{2n}\right]$$

$$= \lim_{n\to\infty} \frac{\pi}{2n} \cdot \frac{\cos\left(\frac{\pi}{2n} + \frac{n-1}{2} \cdot \frac{\pi}{2n}\right)\sin\left(\frac{n}{2} \cdot \frac{\pi}{2n}\right)}{\sin\left(\frac{1}{2} \cdot \frac{\pi}{2n}\right)}$$

$$\left[\because \cos\alpha + \cos(\alpha+\beta) + \cos(\alpha + 2\beta) + \ldots\ldots \text{ to } n \text{ terms} = \frac{\cos\left(\alpha + \frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\right]$$

$$= \lim_{n\to\infty} 2 \cdot \frac{\frac{\pi}{4n}}{\sin \frac{\pi}{4n}} \cdot \cos \frac{\pi}{4}\left(\frac{2}{n} + \frac{n-1}{n}\right) \sin \frac{\pi}{4}$$

$$= \lim_{n\to\infty} 2 \cdot \frac{\frac{\pi}{4n}}{\sin \frac{\pi}{4n}} \cdot \cos \frac{\pi}{4}\left(1+\frac{1}{n}\right) \sin \frac{\pi}{4}$$

$$= 2 \times 1 \times \cos \frac{\pi}{4} \times \sin \frac{\pi}{4} = \sin \frac{\pi}{2} = 1.$$

Example 8. *Prove that $\int_0^{\pi/2} \sin x\, dx = 1$.*

Sol. *Please try yourself.*

Example 9. *Show that the greatest integer function $f(x) = [x]$ is integrable on $[0, 4]$ and*

$$\int_0^4 [x]\, dx = 6.$$

Sol. $\quad f(x) = [x]$ on $[0, 4]$ $\Rightarrow$ $f(x) = \begin{cases} 0 \text{ when } 0 \le x < 1 \\ 1 \text{ when } 1 \le x < 2 \\ 2 \text{ when } 2 \le x < 3 \\ 3 \text{ when } 3 \le x < 4 \end{cases}$

$\Rightarrow$ f is bounded and has *only four* points of finite discontinuity at 1, 2, 3, 4.

Since the points of discontinuity of f on $[0, 4]$ are finite in number, therefore, f is integrable on $[0, 4]$ and

$$\int_0^4 [x]\, dx = \int_0^1 [x]\, dx + \int_1^2 [x]\, dx + \int_2^3 [x]\, dx + \int_3^4 [x]\, dx$$

$$= \int_0^1 0\, dx + \int_1^2 1\, dx + \int_2^3 2\, dx + \int_3^4 3\, dx$$

$$= 0 + (2-1) + 2(3-2) + 3(4-3) = 6.$$

Example 10. *Show that the function f defined by $f(x) = \begin{cases} 0, \text{if } x \text{ is an integer} \\ 1, \text{otherwise} \end{cases}$ is integrable on $[0, m]$, m being a positive integer.*

Sol. $\quad f(x) = \begin{cases} 0, \text{ if } x = 0, 1, 2, \ldots\ldots, m \\ 1, \text{ if } r - 1 < x < r,\ r = 1, 2, \ldots\ldots, m \end{cases}$

$\Rightarrow$ f is bounded and has only $m + 1$ points of finite discontinuity at $0, 1, 2, \ldots\ldots, m$.

Since the points of discontinuity of f on $[0, m]$ are finite in number, therefore, f is integrable on $[0, m]$.

Note. $\quad \int_0^m f(x)\, dx = \int_0^1 f(x)\, dx + \int_1^2 f(x)\, dx + \ldots\ldots + \int_{m-1}^m f(x)\, dx$

$$= \int_0^1 1\, dx + \int_1^2 1\, dx + \ldots\ldots + \int_{m-1}^m 1\, dx$$

$$= (1-0) + (2-1) + \ldots\ldots + (m - (m-1)) = 1 + 1 + \ldots\ldots + 1 = m.$$

Example 11. *Show that the function f defined by*

$$f(x) = \frac{1}{2^n}, \quad \text{when } \frac{1}{2^{n+1}} < x \le \frac{1}{2^n}, \ (n = 0, 1, 2, \ldots\ldots)$$
$$f(a) = 0$$

is integrable on [0, 1], although it has an infinite number of points of discontinuity.

Also evaluate $\int_0^1 f(x)\,dx$.

Sol.

$$\begin{aligned} f(x) &= 1, && \text{when } \frac{1}{2} < x \le 1 \\ &= \frac{1}{2}, && \text{when } \frac{1}{2^2} < x \le \frac{1}{2} \\ &= \frac{1}{2^2}, && \text{when } \frac{1}{2^3} < x \le \frac{1}{2^2} \\ &\ \vdots \\ &= \frac{1}{2^{n-1}}, && \text{when } \frac{1}{2^n} < x \le \frac{1}{2^{n-1}} \\ &\ \vdots \\ &= 0, && \text{when } x = 0 \end{aligned}$$

Thus we notice that f is bounded and continuous on [0, 1] except at the points $0, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots\ldots$

The set of points of discontinuity of f on [0, 1] is $\left\{0, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots\ldots\right\}$ which has only one limit point 0.

Since the set of points of discontinuity of f on [0, 1] has a finite number of limit points, therefore, f is integrable on [0, 1].

Now
$$\int_{1/2^n}^1 f(x)\,dx = \int_{1/2}^1 f(x)\,dx + \int_{1/2^2}^{1/2} f(x)\,dx + \int_{1/2^3}^{1/2^2} f(x)\,dx + \ldots\ldots + \int_{1/2^n}^{1/2^{n-1}} f(x)\,dx$$
$$= \int_{1/2}^1 1\,dx + \int_{1/2^2}^{1/2} \frac{1}{2}\,dx + \int_{1/2^3}^{1/2^2} \frac{1}{2^2}\,dx + \ldots\ldots + \int_{1/2^n}^{1/2^{n-1}} \frac{1}{2^{n-1}}\,dx$$
$$= \left(1 - \frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{2} - \frac{1}{2^2}\right) + \frac{1}{2^2}\left(\frac{1}{2^2} - \frac{1}{2^3}\right) + \ldots\ldots + \frac{1}{2^{n-1}}\left(\frac{1}{2^{n-1}} - \frac{1}{2^n}\right)$$
$$= \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2^2}\right) + \frac{1}{2^2}\left(\frac{1}{2^3}\right) + \ldots\ldots + \frac{1}{2^{n-1}}\left(\frac{1}{2^n}\right)$$
$$= \frac{1}{2}\left[1 + \frac{1}{2^2} + \left(\frac{1}{2^2}\right)^2 + \ldots\ldots + \left(\frac{1}{2^2}\right)^{n-1}\right] = \frac{1}{2} \cdot \frac{1 - \left(\frac{1}{2^2}\right)^n}{1 - \frac{1}{2^2}} = \frac{2}{3}\left(1 - \frac{1}{4^n}\right)$$

Proceeding to the limit when $n \to \infty$, we get $\int_0^1 f(x)\,dx = \frac{2}{3}$.

Example 12. *Show that a function f defined on [0, 1] by* $f(x) = \begin{cases} \frac{1}{n}, & \frac{1}{n+1} < x \le \frac{1}{n}, \ (n = 1, 2, \ldots\ldots) \\ 0, & x = 0 \end{cases}$ *is integrable on [0, 1]. Also show that* $\int_0^1 f(x)\,dx = \frac{\pi^2}{6} - 1$.

Sol.

$$\begin{aligned} f(x) &= 1, \text{ when } \frac{1}{2} < x \le 1 \\ &= \frac{1}{2}, \text{ when } \frac{1}{3} < x \le \frac{1}{2} \\ &= \frac{1}{3}, \text{ when } \frac{1}{4} < x \le \frac{1}{3} \\ &\ \vdots \\ &= \frac{1}{n}, \text{ when } \frac{1}{n+1} < x \le \frac{1}{n} \\ &\ \vdots \\ &= 0, \text{ when } x = 0 \end{aligned}$$

Thus we notice that f is bounded and continuous on [0, 1] except at the points $0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\ldots$

The set of points of discontinuity of f on [0, 1] is $\{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\ldots\}$ which has only one limit point 0.

Since the set of points of discontinuity of f on [0, 1] has a finite number of limit points, therefore, f is integrable on [0, 1].

Now
$$\int_{1/(n+1)}^1 f(x)\,dx = \int_{1/2}^1 f(x)\,dx + \int_{1/3}^{1/2} f(x)\,dx + \int_{1/4}^{1/3} f(x)\,dx + \ldots\ldots + \int_{1/(n+1)}^{1/n} f(x)\,dx$$
$$= \int_{1/2}^1 1\,dx + \int_{1/3}^{1/2} \frac{1}{2}\,dx + \int_{1/4}^{1/3} \frac{1}{3}\,dx + \ldots\ldots + \int_{1/n+1}^{1/n} \frac{1}{n}\,dx$$
$$= \left(1 - \frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{2} - \frac{1}{3}\right) + \frac{1}{3}\left(\frac{1}{3} - \frac{1}{4}\right) + \ldots\ldots + \frac{1}{n}\left(\frac{1}{n} - \frac{1}{n+1}\right)$$
$$= \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots + \frac{1}{n^2}\right) - \left(\frac{1}{2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots\ldots + \frac{1}{n(n+1)}\right)$$
$$= \frac{\pi^2}{6} - \left[\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots\ldots + \left(\frac{1}{n} - \frac{1}{n+1}\right)\right]$$
$$= \frac{\pi^2}{6} - \left(1 - \frac{1}{n+1}\right)$$

Proceeding to the limit as $n \to \infty$, we get $\int_0^1 f(x)\,dx = \frac{\pi^2}{6} - 1.$

Example 13. *Show that the function f defined on $[0, 1]$ as $f(x) = 2rx$ if $\frac{1}{r+1} < x < \frac{1}{r}$, $r \in N$ is integrable over $[0, 1]$ and $\int_0^1 f(x)\,dx = \frac{\pi^2}{6}$.*

Sol.

$$f(x) = 2x, \text{ when } \frac{1}{2} < x < 1$$
$$= 4x, \text{ when } \frac{1}{3} < x < \frac{1}{2}$$
$$= 6x, \text{ when } \frac{1}{4} < x < \frac{1}{3}$$
$$\vdots$$
$$= 2(n-1)\,x, \text{ when } \frac{1}{n} < x < \frac{1}{n-1}$$
$$\vdots$$

Thus we notice that f is bounded and continuous on $[0, 1]$ except at the points $0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\ldots$

The set of points of discontinuity of f on $[0, 1]$ is $\{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\ldots\}$ which has only one limit point 0.

Since the set of points of discontinuity of f on $[0, 1]$ has a finite number of limit points, therefore, f is integrable on $[0, 1]$.

Now $\int_{1/n}^1 f(x)\,dx = \int_{1/2}^1 f(x)\,dx + \int_{1/3}^{1/2} f(x)\,dx + \int_{1/4}^{1/3} f(x)\,dx + \ldots\ldots + \int_{1/n}^{1/n-1} f(x)\,dx$

$$= \sum_{r=1}^{n-1} \int_{1/(r+1)}^{1/r} 2rx\,dx = \sum_{r=1}^{n-1} \left[rx^2\right]_{1/(r+1)}^{1/r} = \sum_{r=1}^{n-1} r\left[\frac{1}{r^2} - \frac{1}{(r+1)^2}\right]$$

$$= \sum_{r=1}^{n} \frac{2r+1}{r(r+1)^2} = \sum_{r=1}^{n-1}\left[\frac{1}{r} - \frac{1}{r+1} + \frac{1}{(r+1)^2}\right] \qquad \text{(Partial Fractions)}$$

$$= \sum_{r=1}^{n-1}\left(\frac{1}{r} - \frac{1}{r+1}\right) + \sum_{r=1}^{n-1} \frac{1}{(r+1)^2}$$

$$= \left[\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \ldots\ldots + \left(\frac{1}{n-1} - \frac{1}{n}\right)\right] + \left[\frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots + \frac{1}{n^2}\right]$$

$$= \left(1 - \frac{1}{n}\right) + \left(\frac{\pi^2}{6} - 1\right) \qquad \left[\because \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots + \frac{1}{n^2} = \frac{\pi^2}{6}\right]$$

$$= \frac{\pi^2}{6} - \frac{1}{n}$$

Proceeding to the limit as $n \to \infty$, we get $\int_0^1 f(x)\,dx = \frac{\pi^2}{6}$.

Example 14. *Show that* $\lim_{n\to\infty}\left[\frac{1}{n} + \frac{n^2}{(n+1)^3} + \frac{n^2}{(n+2)^3} + \ldots\ldots + \frac{1}{8n}\right] = \frac{3}{8}$.

Sol. $\lim_{n\to\infty}\left[\frac{1}{n} + \frac{n^2}{(n+1)^3} + \frac{n^2}{(n+2)^3} + \ldots\ldots + \frac{1}{8n}\right]$

$$= \lim_{n\to\infty}\left[\frac{n^2}{(n+0)^3} + \frac{n^2}{(n+1)^3} + \frac{n^2}{(n+2)^3} + \ldots\ldots + \frac{n^2}{(n+n)^3}\right]$$

$$= \lim_{n\to\infty}\sum_{r=0}^{n} \frac{n^2}{(n+r)^3} = \lim_{n\to\infty}\sum_{r=0}^{n} \frac{\frac{1}{n}}{\left(1+\frac{r}{n}\right)^3}$$

$$= \int_0^1 \frac{dx}{(1+x)^3} \qquad \left[\text{replacing } \frac{r}{n} \text{ by } x \text{ and } \frac{1}{n} \text{ by } dx\right]$$

$$= \left[\frac{-1}{2(1+x)^2}\right]_0^1 = -\frac{1}{2}\left(\frac{1}{4} - 1\right) = \frac{3}{8}.$$

Example 15. *Show that* $\lim_{n\to\infty}\left(\frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \frac{n}{n^2+3^2} + \ldots\ldots + \frac{1}{2n}\right) = \frac{\pi}{4}$.

Sol. $\lim_{n\to\infty}\left(\frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \frac{n}{n^2+3^2} + \ldots\ldots + \frac{1}{2n}\right)$

$$= \lim_{n\to\infty}\left[\frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \frac{n}{n^2+3^2} + \ldots\ldots + \frac{n}{n^2+n^2}\right]$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n} \frac{n}{n^2+r^2} = \lim_{n\to\infty}\sum_{r=1}^{n} \frac{\frac{1}{n}}{1+\left(\frac{r}{n}\right)^2}$$

$$= \int_0^1 \frac{dx}{1+x^2} \qquad \left[\text{replacing } \frac{r}{n} \text{ by } x \text{ and } \frac{1}{n} \text{ by } dx\right]$$

$$= \left[\tan^{-1} x\right]_0^1 = \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4}.$$

Example 16. *Show that*

(i) $\lim\limits_{n\to\infty} \frac{1}{n}\left[\left(\frac{1}{n}\right)^2+\left(\frac{2}{n}\right)^2+\ldots\ldots+\left(\frac{n}{n}\right)^2\right]=\frac{1}{3}$

(ii) $\lim\limits_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\ldots\ldots+\frac{1}{2n}\right]=\log_e 2$

(iii) $\lim\limits_{n\to\infty}\left[\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\ldots\ldots+\frac{1}{3n}\right]=\log_e 3.$

Sol. *Please try yourself.*

Example 17. *Show that* $\lim\limits_{n\to\infty}\frac{1}{n}\left[\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+\ldots\ldots+\sin\frac{n\pi}{n}\right]=\frac{2}{\pi}.$

Sol. $\lim\limits_{n\to\infty}\frac{1}{n}\left[\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+\ldots\ldots+\sin\frac{n\pi}{n}\right]=\lim\limits_{n\to\infty}\sum\limits_{r=1}^{n}\frac{1}{n}\sin\frac{r\pi}{n}$

$$=\int_0^1 \sin\pi x\,dx \qquad \left[\text{replacing } \frac{r}{n} \text{ by } x \text{ and } \frac{1}{n} \text{ by } dx\right]$$

$$=-\left[\frac{\cos\pi x}{\pi}\right]_0^1=-\frac{1}{\pi}(-1-1)=\frac{2}{\pi}.$$

Example 18. *Show that* $\lim\limits_{n\to\infty}\frac{1}{n}(e^{3/n}+e^{6/n}+e^{9/n}+\ldots\ldots+e^{3n/n})=\frac{1}{3}(e^3-1).$

Sol. *Please try yourself.*

Example 19. *Prove that* $\lim\limits_{n\to\infty}\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\ldots\ldots\left(1+\frac{4n}{n}\right)\right]^{1/n}=5\left[\frac{5}{e}\right]^4.$

Sol. Let $\quad L=\lim\limits_{n\to\infty}\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\ldots\ldots\left(1+\frac{4n}{n}\right)\right]^{1/n}$

$\therefore \quad \log L=\lim\limits_{n\to\infty}\frac{1}{n}\left[\log\left(1+\frac{1}{n}\right)+\log\left(1+\frac{2}{n}\right)+\ldots\ldots+\log\left(1+\frac{4n}{n}\right)\right]$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{4n}\log\left(1+\frac{r}{n}\right)=\int_0^4\log(1+x)\,dx$$

$$\left(\text{replacing } \frac{r}{n} \text{ by } x \text{ and } \frac{1}{n} \text{ by } dx\right)$$

[Note that the value of r/n is 0 and 4 for the first and last terms as $n\to\infty$]

$$=\left[\log(1+x).\,x\right]_0^4-\int_0^4\frac{1}{1+x}.\,x\,dx$$

$$=4\log 5-\int_0^4\left(1-\frac{1}{1+x}\right)dx=4\log 5-\left[x-\log(1+x)\right]_0^4$$

$$=4\log 5-[4-\log 5]=5\log 5-4=\log 5^5-\log e^4=\log\frac{5^5}{e^4}$$

$$\Rightarrow \qquad L=\frac{5^5}{e^4}=5\left[\frac{5}{e}\right]^4.$$

9.16. PROPERTIES OF RIEMANN INTEGRAL

Theorem 1. *If $f\in R[a,b]$ and $k\in R$, then $kf\in R[a,b]$ and $\int_a^b (kf)(x)\,dx=k\int_a^b f(x)\,dx$.*

Proof. If $k=0$, then theorem is obvious. So, let $k\neq 0$.

Since $\quad f\in R[a,b],\ \underline{\int_a^b} f(x)\,dx=\overline{\int_a^b} f(x)\,dx=\int_a^b f(x)\,dx$

Let $\quad P=\{a=x_0,x_1,x_2,\ldots\ldots,x_n=b\}$ be a partition of $[a,b]$.
Let m_r, M_r be the infimum and supermum of f on $I_r=[x_{r-1},x_r]$
f is bounded on $[a,b] \Rightarrow kf$ is bounded on $[a,b]$.
Let m_r', M_r' be the infimum and supermum of kf on $I_r=[x_{r-1},x_r]$.

Case 1. Let $k>0$.

Then $\qquad m_r'=km_r$ and $M_r'=kM_r$

$\therefore \qquad L(P,kf)=\sum\limits_{r=1}^{n}m_r'\delta_r=\sum\limits_{r=1}^{n}(km_r)\delta_r=k\sum\limits_{r=1}^{n}m_r\delta_r=kL(P,f)$

and $\qquad U(P,kf)=\sum\limits_{r=1}^{n}M_r'\delta_r=\sum\limits_{r=1}^{n}(kM_r)\delta_r=k\sum\limits_{r=1}^{n}M_r\delta_r=kU(P,f)$

$\therefore \quad \underline{\int_a^b}(kf)(x)\,dx=\sup\{L(P,kf)\}_{P\in P[a,b]}=\sup\{kL(P,f)\}_{P\in P[a,b]}$

$$=k\sup\{L(P,f)\}_{P\in P[a,b]}=k\underline{\int_a^b}f(x)\,dx=k\int_a^b f(x)\,dx$$

Also $\quad \overline{\int_a^b}(kf)(x)\,dx=\inf\{U(P,kf)\}_{P\in P[a,b]}=\inf\{kU(P,f)\}_{P\in P[a,b]}$

$$=k\inf\{U(P,f)\}_{P\in P[a,b]}=k\overline{\int_a^b}f(x)\,dx=k\int_a^b f(x)\,dx$$

$\therefore \quad \underline{\int_a^b}(kf)(x)\,dx=\overline{\int_a^b}(kf)(x)\,dx=k\int_a^b f(x)\,dx$

Hence $\qquad kf\in R[a,b]$ and $\int_a^b (kf)(x)\,dx=k\int_a^b f(x)\,dx.$

Case. 2. Let $k<0$.

Then $\qquad m_r'=kM_r$ and $M_r'=km_r$

$\therefore \quad L(P, kf) = \sum_{r=1}^{n} m_r'\delta_r = \sum_{r=1}^{n} (kM_r)\delta_r = k\sum_{r=1}^{n} M_r\delta_r = kU(P, f)$

and $\quad U(P, kf) = \sum_{r=1}^{n} M_r'\delta_r = \sum_{r=1}^{n} (km_r)\delta_r = k\sum_{r=1}^{n} m_r\delta_r = kL(P, f)$

$$\therefore \quad \underline{\int_a^b} (kf)(x)\,dx = \sup\{L(P, kf)\}_{P\in P[a,b]} = \sup\{kU(P, f)\}_{P\in P[a,b]}$$

$$= k\inf\{U(P, f)\}_{P\in P[a,b]} = k\overline{\int_a^b} f(x)\,dx = k\int_a^b f(x)\,dx$$

Also $\overline{\int_a^b} (kf)(x)\,dx = \inf\{U(P, kf)\}_{P\in P[a,b]} = \inf\{kL(P, f)\}_{P\in P[a,b]}$

$$= k\sup\{L(P, f)\}_{P\in P[a,b]} = k\underline{\int_a^b} f(x)\,dx = k\int_a^b f(x)\,dx$$

$$\therefore \quad \underline{\int_a^b} (kf)(x)\,dx = \overline{\int_a^b} (kf)(x)\,dx = k\int_a^b f(x)\,dx$$

Hence $\quad kf \in R[a, b]$ and $\int_a^b (kf)(x)\,dx = k\int_a^b f(x)\,dx.$

(Second Proof)

Since f is integrable on $[a, b]$, therefore, given $\varepsilon > 0$, there exists a partition

$$P = \{a = x_0, x_1, x_2, \ldots\ldots, x_n = b\} \text{ of } [a, b] \text{ such that } U(P, f) - L(P, f) < \frac{\varepsilon}{|k|} \qquad \ldots(1)$$

Let m_r, M_r be the infimum and supermum of f on $I_r = [x_{r-1}, x_r]$. f is bounded on $[a, b]$ $\Rightarrow$ kf is bounded on $[a, b]$.

Let m_r', M_r' be the infimum and supermum of kf on $I_r = [x_{r-1}, x_r]$.

$$m_r' = \begin{cases} km_r & \text{if } k>0 \\ kM_r & \text{if } k<0 \end{cases} \quad \text{and} \quad M_r' = \begin{cases} kM_r & \text{if } k>0 \\ km_r & \text{if } k<0 \end{cases}$$

$$\Rightarrow \quad L(P, kf) = \begin{cases} kL(P, f) & \text{if } k>0 \\ kU(P, f) & \text{if } k<0 \end{cases} \quad \text{and} \quad U(P, kf) = \begin{cases} kU(P, f) & \text{if } k>0 \\ kL(P, f) & \text{if } k<0 \end{cases}$$

$$\Rightarrow \quad U(P, kf) - L(P, kf) = \begin{cases} k(U(P, f) - L(P, f)) & \text{if } k>0 \\ -k(U(P, f) - L(P, f)) & \text{if } k<0 \end{cases}$$

$$= |k|\,(U(P, f) - L(P, f)) < \varepsilon \qquad \text{[from (1)]}$$

$\Rightarrow$ kf is integrable on $[a, b]$.

Also $\quad \int_a^b (kf)(x)\,dx = \overline{\int_a^b} (kf)(x)\,dx \qquad [\because\ kf \text{ is integrable}]$

$$= \inf\{U(P, kf)\}_{P\in P[a,b]} = \begin{cases} \inf\{kU(P, f)\}_{P\in P[a,b]} & \text{if } k>0 \\ \inf\{kL(P, f)\}_{P\in P[a,b]} & \text{if } k<0 \end{cases}$$

$$= \begin{cases} k\inf\{U(P, f)\}_{P\in P[a,b]} & \text{if } k>0 \\ k\sup\{L(P, f)\}_{P\in P[a,b]} & \text{if } k<0 \end{cases} = \begin{cases} k\overline{\int_a^b} f(x)\,dx & \text{if } k>0 \\ k\underline{\int_a^b} f(x)\,dx & \text{if } k<0 \end{cases}$$

$$= k\int_a^b f(x)\,dx, \text{ since } f \text{ is integrable.}$$

Theorem 2. *If $f \in R[a, b]$, then $|f| \in R[a, b]$ and* $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f|(x)\,dx$.

Proof. Since $f \in R[a, b]$, f is bounded on $[a, b]$

$\therefore$ there exists a positive number k such that

$$|f(x)| \le k \quad \forall \quad x \in [a, b] \Rightarrow |f|(x) \le k \quad \forall \quad x \in [a, b]$$

$\Rightarrow$ $|f|$ is bounded on $[a, b]$.

Since f is integrable on $[a, b]$, therefore, given $\varepsilon > 0$, there exists a partition

$$P = \{a = x_0, x_1, x_2, \ldots\ldots, x_n = b\} \text{ of } [a, b] \text{ such that } U(P, f) - L(P, f) < \varepsilon \qquad \ldots(1)$$

Now, let m_r, M_r be the infimum and supermum of f on $I_r = [x_{r-1}, x_r]$ and m_r', M_r' be the infimum and supermum of $|f|$ on I_r.

For all $\alpha, \beta \in I_r$, we have $\big|\,|f|(\alpha) - |f|(\beta)\,\big| = \big|\,|f(\alpha)| - |f(\beta)|\,\big|$

$$\le |f(\alpha) - f(\beta)| \le M_r - m_r$$

$\therefore \quad M_r' - m_r' \le M_r - m_r,\ r = 1, 2, \ldots\ldots, n$

$$U(P, |f|) - L(P, |f|) = \sum_{r=1}^{n} (M_r' - m_r')\,\delta_r \le \sum_{r=1}^{n} (M_r - m_r)\delta_r$$

$$= U(P, f) - L(P, f) < \varepsilon \qquad \text{[from (1)]}$$

$\therefore \quad |f| \in R[a, b]$

Since $\quad |f| = \max\{f, -f\}$

$\therefore \quad \left.\begin{array}{l} f(x) \le |f(x)| = |f|(x) \\ -f(x) \le |f(x)| = |f|(x) \end{array}\right\} \forall\, x \in [a, b]$ and

$$\Rightarrow \quad \int_a^b f(x)\,dx \le \int_a^b |f|(x)\,dx \qquad \ldots(2)$$

and $\quad \int_a^b -f(x)\,dx \le \int_a^b |f|(x)\,dx$ or $-\int_a^b f(x)\,dx \le \int_a^b |f|(x)\,dx \qquad \ldots(3)$

Combining (2) and (3), we have $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f|(x)\,dx.$

Remark. The converse of this theorem is not true. Thus, if $|f|$ is integrable on $[a, b]$, then f need not be integrable on $[a, b]$.

Consider a function $f: [a, b] \to R$ defined as $f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ -1, & \text{if } x \text{ is irrational} \end{cases}$

Let $P = \{a = x_0, x_1, x_2, \ldots\ldots, x_n = b\}$ be any partition of $[a, b]$.

Let m_r and M_r be the infimum and supermum of f on $I_r = [x_{r-1}, x_r]$, then

$$m_r = -1 \quad \text{and} \quad M_r = 1,\ r = 1, 2, \ldots\ldots, n.$$

$$L(P,f)=\sum_{r=1}^{n} m_r\delta_r=\sum_{r=1}^{n} -\delta_r=-(b-a)=a-b$$

$$U(P,f)=\sum_{r=1}^{n} M_r\delta_r=\sum_{r=1}^{n} \delta_r=b-a$$

$\therefore$ $$\underline{\int_a^b} f(x)\,dx=\sup\{L(P,f)\}_{P\in P[a,b]}=\sup\{(a-b)\}=a-b$$

$$\overline{\int_a^b} f(x)\,dx=\inf\{U(P,f)\}_{P\in P[a,b]}=\inf\{(b-a)\}=b-a$$

Since $\underline{\int_a^b} f(x)\,dx \neq \overline{\int_a^b} f(x)\,dx,\ f\notin R[a,b]$

But $|f|(x)=|f(x)|=1\ \forall\ x\in[a,b]$

Since $|f|$ is a constant function, $|f|\in R[a,b]$.

Theorem 3. *If $f, g\in R[a,b]$ then $f+g\in R[a,b]$ and*

$$\int_a^b (f+g)(x)dx=\int_a^b f(x)dx+\int_a^b g(x)\,dx.$$

Proof. f and g, being integrable, are bounded on $[a,b]$.

$\Rightarrow$ $f+g$ is bounded on $[a,b]$.

Let $P=\{a=x_0, x_1, \ldots\ldots, x_n=b\}$ be a partition of $[a,b]$.

Let m_r', M_r' be the infimum and supermum of f on I_r,

m_r'', M_r'' be the infimum and supermum of g on I_r, and

m_r, M_r be the infimum and supermum of $f+g$ on I_r.

Now M_r', M_r'' are superma of f, g on I_r

$\Rightarrow$ $f(x)\le M_r', g(x)\le M_r''\quad \forall\, x\in I_r$

$\Rightarrow$ $f(x)+g(x)\le M_r'+M_r''\quad \forall\ x\in I_r$

$\Rightarrow$ $(f+g)(x)\le M_r'+M_r''\quad \forall\ x\in I_r$

$\Rightarrow$ $M_r'+M_r''$ is an upper bound of $f+g$ on I_r.

But M_r is the least upper bound of $f+g$ on I_r.

$\therefore$ $M_r\le M_r'+M_r''$ on I_r, $r=1, 2, \ldots\ldots, n$

$\therefore$ $$U(P,f+g)=\sum_{r=1}^{n} M_r\delta_r\le\sum_{r=1}^{n}(M_r'+M_r'')\delta_r$$

$$=\sum_{r=1}^{n} M_r'\delta_r+\sum_{r=1}^{n} M_r''\delta_r=U(P,f)+U(P,g)$$

$\Rightarrow$ $U(P,f+g)\le U(P,f)+U(P,g)$

Similarly, we can prove that

$L(P,f+g)\ge L(P,f)+L(P,g)$

$\therefore$ $\omega(P,f+g)=U(P,f+g)-L(P,f+g)$

$\le[U(P,f)+U(P,g)]-[L(P,f)+L(P,g)]$

$=[U(P,f)-L(P,f)]+[U(P,g)-L(P,g)]=\omega(P,f)+\omega(P,g)$ ...(1)

Let $\varepsilon>0$ be given.

Since f and g are integrable on $[a,b]$, there exist partitions P_1 and P_2 of $[a,b]$ such that

$$\omega(P_1,f)<\frac{\varepsilon}{2}\quad\text{and}\quad\omega(P_2,g)<\frac{\varepsilon}{2}$$

Let $P'=P_1\cup P_2$, then $\omega(P',f)\le\omega(P_1,f)<\dfrac{\varepsilon}{2}$ ...(2)

and $\omega(P',g)\le\omega(P_2,g)<\dfrac{\varepsilon}{2}$ ...(3)

Using (1), we have $\omega(P',f+g)\le\omega(P',f)+\omega(P',g)<\dfrac{\varepsilon}{2}+\dfrac{\varepsilon}{2}=\varepsilon$ [by (2) and (3)]

$\Rightarrow$ $f+g$ is integrable on $[a,b]$.

Now, let $\varepsilon>0$ be given. Then there exist partitions P_1 and P_2 of $[a,b]$ such that

$$\int_a^b f(x)\,dx-\frac{\varepsilon}{2}=\underline{\int_a^b} f(x)\,dx-\frac{\varepsilon}{2}<L(P_1,f)$$

and $$\int_a^b g(x)\,dx-\frac{\varepsilon}{2}=\underline{\int_a^b} g(x)\,dx-\frac{\varepsilon}{2}<L(P_2,g)$$

If $P=P_1\cup P_2$, then $L(P_1,f)\le L(P,f)$ and $L(P_2,g)\le L(P,g)$

$\therefore$ $$\int_a^b f(x)\,dx+\int_a^b g(x)\,dx-\varepsilon<L(P_1,f)+L(P_2,g)$$

$$\le L(P,f)+L(P,g)\le L(P,f+g)$$

$$\le\underline{\int_a^b}(f+g)(x)\,dx=\int_a^b(f+g)(x)\,dx$$

($\because$ $f+g$ is integrable)

Since $\varepsilon>0$ is arbitrary, $\int_a^b f(x)\,dx+\int_a^b g(x)\,dx\le\int_a^b(f+g)(x)\,dx$ ...(4)

Replacing f by $-f$ and g by $-g$, we have from (4),

$$\int_a^b -f(x)\,dx+\int_a^b -g(x)\,dx\le\int_a^b -(f+g)(x)\,dx$$

$\Rightarrow$ $$\int_a^b f(x)\,dx+\int_a^b g(x)\,dx\ge\int_a^b (f+g)(x)\,dx \quad ...(5)$$

From (4) and (5), we get $\int_a^b (f+g)(x)\,dx=\int_a^b f(x)\,dx+\int_a^b g(x)\,dx.$

Cor. 1. *If $f, g\in R[a,b]$, then $f-g\in R[a,b]$*

and $$\int_a^b (f-g)(x)\,dx=\int_a^b f(x)dx-\int_a^b g(x)dx.$$

Cor. 2. *If $f, g\in R[a,b]$ and $\alpha,\beta\in R$, then $\alpha f+\beta g\in R[a,b]$.*

Proof. $f\in R[a,b], \alpha\in R \Rightarrow \alpha f\in R[a,b]$

$g\in R[a,b], \beta\in R \Rightarrow \beta g\in R[a,b]$

$\alpha f\in R[a,b], \beta g\in R[a,b] \Rightarrow \alpha f+\beta g\in R[a,b]$

Also $$\int_a^b(\alpha f+\beta g)(x)\,dx=\int_a^b(\alpha f)(x)dx+\int_a^b(\beta g)(x)\,dx=\alpha\int_a^b f(x)\,dx+\beta\int_a^b g(x)\,dx.$$

Theorem 4. *If $f \in R\,[a, b]$ then $f^2 \in R\,[a, b]$.*

Proof. $\quad f \in R[a, b] \Rightarrow |f| \in R[a, b]$

$\therefore$ f is bounded on $[a, b]$ $\Rightarrow$ $|f|$ is bounded on $[a, b]$ $\Rightarrow$ $|f|^2 = f^2$ is bounded on $[a, b]$

Since $f^2 = |f|^2$, without loss of generality, we can assume that $f \geq 0$.

Let $\sup f$ in $[a, b] = M$.

Let $\varepsilon > 0$ be given.

$f \in R[a, b] \Rightarrow$ there exists a partition P of $[a, b]$ such that

$$U(P, f) - L(P, f) < \frac{\varepsilon}{2M+1} \quad \text{or} \quad \sum_{r=1}^{n} (M_r - m_r)\delta_r < \frac{\varepsilon}{2M+1} \qquad \ldots(1)$$

where m_r, M_r are the infimum and supermum of f on I_r.

In $\quad I_r, \inf(f^2) = (\inf f)^2 = m_r^2$ and $\sup(f^2) = (\sup f)^2 = M_r^2$

$$\therefore \quad U(P, f^2) - L(P, f^2) = \sum_{r=1}^{n} (M_r^2 - m_r^2)\delta_r$$

$$= \sum_{r=1}^{n} (M_r + m_r)(M_r - m_r)\delta_r \leq \sum_{r=1}^{n} (M + M)(M_r - m_r)\delta_r$$

$$= 2M \sum_{r=1}^{n} (M_r - m_r)\delta_r < 2M \cdot \frac{\varepsilon}{2M+1} \qquad \text{[by (1)]}$$

$$< \varepsilon$$

$\Rightarrow$ for each $\varepsilon > 0$, we can find a partition P of $[a, b]$ such that $U(P, f^2) - L(P, f^2) < \varepsilon$

$\therefore$ f^2 is integrable on $[a, b]$.

Theorem 5. *If $f, g \in R[a, b]$ then $fg \in R[a, b]$.*

Proof. Since $f, g \in R[a, b]$, f and g are bounded on $[a, b]$

$\Rightarrow \quad \exists\, k > 0$ such that $|f(x)| \leq k$ and $|g(x)| \leq k \;\; \forall\, x \in [a, b]$

$\Rightarrow \quad |(fg)(x)| = |f(x)g(x)| = |f(x)|\,|g(x)| \leq k^2 \;\; \forall\, x \in [a, b]$

$\Rightarrow$ fg is bounded on $[a, b]$.

Now $f \in R[a, b] \Rightarrow$ for a given $\varepsilon > 0$, there exists a partition P_1 of $[a, b]$ such that

$$U(P_1, f) - L(P_1, f) < \frac{\varepsilon}{2k}$$

Also $g \in R[a, b] \Rightarrow$ for a given $\varepsilon > 0$, there exists a partition P_2 of $[a, b]$ such that

$$U(P_2, g) - L(P_2, g) < \frac{\varepsilon}{2k}.$$

Let $P = P_1 \cup P_2$, then $\quad U(P, f) - L(P, f) - U(P_1, f) - L(P_1, f) < \dfrac{\varepsilon}{2k}$

and $\quad U(P, g) - L(P, g) \leq U(P_2, g) \leq L(P_2, g) < \dfrac{\varepsilon}{2k} \qquad \ldots(1)$

Let m_r, M_r, m_r', M_r' and m_r'', M_r'' be the infimum and supermum of fg, f and g respectively on $I_r = [x_{r-1}, x_r]$.

For all $\alpha, \beta \in I_r$, we have

$|(fg)(\beta) - (fg)(\alpha)| = |f(\beta)g(\beta) - f(\alpha)g(\alpha)| = |f(\beta)g(\beta) - f(\alpha)g(\beta) + f(\alpha)g(\beta) - f(\alpha)g(\alpha)|$

$$= |g(\beta)(f(\beta) - f(\alpha)) + f(\alpha)(g(\beta) - g(\alpha))|$$

$$\leq |g(\beta)|\,|f(\beta) - f(\alpha)| + |f(\alpha)|\,|g(\beta) - g(\alpha)| \leq k(M_r' - m_r') + k(M_r'' - m_r'')$$

$\Rightarrow \quad M_r - m_r \leq k(M_r' - m_r') + k(M_r'' - m_r'')$

$\Rightarrow \quad \displaystyle\sum_{r=1}^{n} (M_r - m_r)\delta_r \leq k \sum_{r=1}^{n} (M_r' - m_r')\delta_r + k \sum_{r=1}^{n} (M_r'' - m_r'')\delta_r$

$\Rightarrow \quad U(P, fg) - L(P, fg) \leq k[U(P, f) - L(P, f)] + k[U(P, g) - L(P, g)]$

$$< k \cdot \frac{\varepsilon}{2k} + k \cdot \frac{\varepsilon}{2k} = \varepsilon \qquad \text{[by (1)]}$$

Thus for each $\varepsilon > 0$, we can find a partition P of $[a, b]$ such that $U(P, fg) - L(P, fg) < \varepsilon$

$\therefore \quad fg \in R[a, b]$.

(Second Proof)

We may write $\quad fg = \dfrac{1}{4}[(f+g)^2 - (f-g)^2]$

Now $\quad f, g \in R[a, b] \Rightarrow f+g, f-g \in R[a, b]$ (by Theorem 3 and Cor. 1)

$\Rightarrow \quad (f+g)^2, (f-g)^2 \in R[a, b]$ (by Theorem 4)

$\Rightarrow \quad (f+g)^2 - (f-g)^2 \in R[a, b]$ (by Theorem 3, Cor. 1)

$\Rightarrow \quad \dfrac{1}{4}[(f+g)^2 - (f-g)^2] \in R[a, b]$ (by Theorem 1)

$\Rightarrow \quad fg \in R[a, b]$.

Remark. Even though f, g are not integrable on $[a, b]$, fg may be integrable on $[a, b]$.

Consider $f : [a, b] \to R$ and $g : [a, b] \to R$ defined by

$$f(x) = \begin{cases} 0, & x \in Q \\ 1 & x \in R - Q \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 1, & x \in Q \\ 0 & x \in R - Q \end{cases}$$

Then f, g are not integrable on $[a, b]$, but

$(fg)(x) = f(x)g(x) = 0 \;\; \forall\, x \in [a, b]$ is a constant function.

$\therefore \quad fg \in R[a, b]$.

Theorem 6. *If $f \in R[a, b]$, and there exists $t > 0$ such that*

$$|f(x)| \geq t \quad \forall\, x \in [a, b], \text{ then } \frac{1}{f} \in R[a, b].$$

Proof. Since $\quad |f(x)| \geq t \quad \forall\, x \in [a, b]$.

$$\therefore \quad \frac{1}{|f(x)|} \leq \frac{1}{t} \quad \forall\, x \in [a, b], t > 0$$

$$\Rightarrow \quad \left|\frac{1}{f(x)}\right| \leq \frac{1}{t} \quad \forall\, x \in [a, b] \Rightarrow \left|\frac{1}{f}(x)\right| \leq \frac{1}{t} \quad \forall\, x \in [a, b]$$

$\Rightarrow \quad \dfrac{1}{f}$ is bounded on $[a, b]$.

Since $f \in R[a, b]$, for a given $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that

$$U(P, f) - L(P, f) < t^2 \varepsilon \qquad \ldots(1)$$

Let m_r', M_r' be the infimum and supermum of f on I_r and m_r, M_r be the infimum and supermum of $\dfrac{1}{f}$ on I_r.

For all $\alpha, \beta \in I_r$, we have

$$\left|\left(\frac{1}{f}\right)(\beta)-\left(\frac{1}{f}\right)(\alpha)\right| = \left|\frac{1}{f(\beta)}-\frac{1}{f(\alpha)}\right| = \frac{|f(\alpha)-f(\beta)|}{|f(\alpha)||f(\beta)|} \le \frac{M_r'-m_r'}{t^2}$$

$\therefore$ $$M_r - m_r \le \frac{1}{t^2}(M_r'-m_r')$$

$\Rightarrow$ $$\sum_{r=1}^{n}(M_r-m_r)\delta_r \le \frac{1}{t^2}\sum_{r=1}^{n}(M_r'-m_r')\delta_r$$

$\Rightarrow$ $$U\left(P,\frac{1}{f}\right)-L\left(P,\frac{1}{f}\right) \le \frac{1}{t^2}[U(P,f)-L(P,f)] < \frac{1}{t^2}\cdot t^2\varepsilon = \varepsilon \qquad \text{[by (1)]}$$

$\therefore$ for each $\varepsilon > 0$, we can find a partition P of $[a, b]$ such that

$$U\left(P,\frac{1}{f}\right)-L\left(P,\frac{1}{f}\right) < \varepsilon \Rightarrow \frac{1}{f} \in R[a, b].$$

Theorem 7. *If $f, g \in R[a, b]$ and there exists $t > 0$ such that*

$$|g(x)| \ge t \quad \forall \quad x \in [a, b], \text{ then } \frac{f}{g} \in R[a, b].$$

Proof. $f, g \in R[a, b] \Rightarrow f, g$ are bounded on $[a, b]$.

$\Rightarrow$ there exists a positive real number k such that

$$|f(x)| \le k, \; |g(x)| \le k \;\; \forall \;\; x \in [a, b]$$

$\therefore$ $$\forall\; x \in [a, b], \left|\left(\frac{f}{g}\right)(x)\right| = \left|\frac{f(x)}{g(x)}\right| \le \frac{k}{t}$$

$\Rightarrow$ $\frac{f}{g}$ is bounded on $[a, b]$.

Since f, g are integrable on $[a, b]$, for a given $\varepsilon > 0$, there exist partitions P_1 and P_2 of $[a, b]$ such that

$$\left.\begin{aligned} U(P_1, f) - L(P_1, f) &< \frac{t^2\varepsilon}{2k} \\ \text{and} \quad U(P_2, g) - L(P_2, g) &< \frac{t^2\varepsilon}{2k} \end{aligned}\right\} \qquad \ldots(1)$$

Let $P = P_1 \cup P_2$ be a refinement of P_1 and P_2, then using (1), we have

$$\left.\begin{aligned} U(P, f) - L(P, f) \le U(P_1, f) - L(P_1, f) &< \frac{t^2\varepsilon}{2k} \\ \text{and} \quad U(P, g) - L(P, g) \le U(P_2, g) - L(P_2, g) &< \frac{t^2\varepsilon}{2k} \end{aligned}\right\} \qquad \ldots(2)$$

Let m_r', M_r' be the infimum and supermum of f on I_r, m_r'', M_r'' be the infimum and supermum of g on I_r and m_r, M_r be the infimum and supermum of f/g on I_r.

For all $\alpha, \beta \in I_r$, we have

$$\left|\left(\frac{f}{g}\right)(\beta)-\left(\frac{f}{g}\right)(\alpha)\right| = \left|\frac{f(\beta)}{g(\beta)}-\frac{f(\alpha)}{g(\alpha)}\right| = \left|\frac{g(\alpha)f(\beta)-f(\alpha)g(\beta)}{g(\alpha)g(\beta)}\right|$$

$$= \frac{|g(\alpha)[f(\beta)-f(\alpha)]-f(\alpha)[g(\beta)-g(\alpha)]|}{|g(\alpha)||g(\beta)|}$$

$$\le \frac{|g(\alpha)|}{|g(\alpha)||g(\beta)|}\,|f(\beta)-f(\alpha)| + \frac{|f(\alpha)|}{|g(\alpha)||g(\beta)|}\,|g(\beta)-g(\alpha)|$$

$$\le \frac{k}{t^2}(M_r'-m_r') + \frac{k}{t^2}(M_r''-m_r'')$$

$\therefore$ $$M_r - m_r \le \frac{k}{t^2}(M_r'-m_r') + \frac{k}{t^2}(M_r''-m_r'')$$

$\Rightarrow$ $$\sum_{r=1}^{n}(M_r-m_r)\delta_r \le \frac{k}{t^2}\sum_{r=1}^{n}(M_r'-m_r')\delta_r + \frac{k}{t^2}\sum_{r=1}^{n}(M_r''-m_r'')\delta_r$$

$\Rightarrow$ $$U\left(P,\frac{f}{g}\right)-L\left(P,\frac{f}{g}\right) \le \frac{k}{t^2}[U(P,f)-L(P,f)] + \frac{k}{t^2}[U(P,g)-L(P,g)]$$

$$< \frac{k}{t^2}\cdot\frac{t^2\varepsilon}{2k} + \frac{k}{t^2}\cdot\frac{t^2\varepsilon}{2k} = \varepsilon \qquad \text{[by (2)]}$$

$\therefore$ $$\frac{f}{g} \in R[a, b].$$

Theorem 8. *If $f \in R[a, b]$ and $a < c < b$, then $f \in R[a, c]$, $f \in R[c, b]$ and*

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Proof. $f \in R[a, b] \Rightarrow f$ is bounded on $[a, b]$.

$\Rightarrow$ f is bounded on $[a, c]$ and $[c, b]$. $\qquad (\because \; a < c < b)$

Since $f \in R[a, b]$, for a given $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that $U(P, f) - L(P, f) < \varepsilon$

Let $P' = P \cup \{c\}$, then $L(P, f) \le L(P', f) \le U(P', f) \le U(P, f)$

$\Rightarrow$ $$U(P', f) - L(P', f) \le U(P, f) - L(P, f) < \varepsilon \qquad \ldots(1)$$

Let P_1, P_2 denote the set of points of P' on $[a, c]$, $[c, b]$ respectively, then P_1, P_2 are partitions on $[a, c]$ and $[c, b]$ respectively and $P' = P_1 \cup P_2$.

$\therefore$ $$U(P', f) = U(P_1, f) + U(P_2, f) \quad \text{and} \quad L(P', f) = L(P_1, f) + L(P_2, f)$$

$\Rightarrow$ $$[U(P_1, f) - L(P_1, f)] + [U(P_2, f) - L(P_2, f)] = U(P', f) - L(P', f) < \varepsilon \qquad \text{[by (1)]}$$

Since each of $[U(P_1, f) - L(P_1, f)]$ and $[U(P_2, f) - L(P_2, f)]$ is non-negative, each of these is less than ε

i.e., $$U(P_1, f) - L(P_1, f) < \varepsilon \quad \text{and} \quad U(P_2, f) - L(P_2, f) < \varepsilon$$

for partitions P_1, P_2 of $[a, c]$ and $[c, b]$ respectively.

Hence $f \in R[a, c]$ and $f \in R[c, b]$

Now $$U(P', f) = U(P_1, f) + U(P_2, f)$$

$\Rightarrow$ $$\inf U(P', f) = \inf U(P_1, f) + \inf U(P_2, f)$$

$\Rightarrow$ $$\overline{\int_a^b} f(x)\,dx = \overline{\int_a^c} f(x)\,dx + \overline{\int_c^b} f(x)\,dx$$

$\Rightarrow$ $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Since $f \in R[a, b]$, $f \in R[a, c]$ and $f \in R[c, b]$.

Cor. If f is integrable on $[a, b]$ then f is integrable on any sub-interval of $[a, b]$.

Theorem 9. *If $f \in R[a, c]$, $f \in R[c, b]$ and $a < c < b$ then $f \in R[a, b]$.*

Proof. Since $f \in R[a, c]$ and $f \in R[c, b]$

$\therefore$ given $\varepsilon > 0$, there exist partitions P_1 and P_2 of $[a, c]$ and $[c, b]$ respectively such that

$$\left.\begin{aligned} &U(P_1, f) - L(P_1, f) < \frac{\varepsilon}{2} \\ \text{and} \quad &U(P_2, f) - L(P_2, f) < \frac{\varepsilon}{2} \end{aligned}\right\} \qquad ...(1)$$

If $P = P_1 \cup P_2$, then P is a partition of $[a, b]$.

Also $U(P, f) - L(P, f)$

$$\begin{aligned} &= [U(P_1, f) + U(P_2, f)] - [L(P_1, f) + L(P_2, f)] \\ &= [U(P_1, f) - L(P_1, f)] + [U(P_2, f) - L(P_2, f)] \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \qquad [\text{by (1)}] \end{aligned}$$

$\Rightarrow \quad f \in R[a, b]$.

Theorem 10. *If $f \in R[a, b]$ and $f(x) \geq 0 \;\; \forall \; x \in [a, b]$ then $\int_a^b f(x)\,dx \geq 0$.*

Proof. $f \in R[a, b] \Rightarrow f$ is bounded on $[a, b]$.

Let m, M be the infimum and supermum of f on $[a, b]$.

Since $\quad f(x) \geq 0 \quad \forall x \in [a, b], m \geq 0$

For all partitions P of $[a, b]$, we have $L(P, f) \leq m(b - a) \geq 0$

$$\Rightarrow \quad \underline{\int_a^b} f(x)\,dx = \sup\{L(P, f)\}_{P \in P[a, b]} \geq 0$$

$$\text{But} \quad \underline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx \text{ since } f \in R[a, b]$$

$$\therefore \quad \int_a^b f(x)\,dx \geq 0.$$

Theorem 11. *If $f, g \in R[a, b]$ and $f(x) \geq g(x) \;\; \forall \; x \in [a, b]$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.*

Proof. $\quad f, g \in R[a, b] \Rightarrow f - g \in R[a, b]$

Also $\quad f(x) \geq g(x) \quad \forall \; x \in [a, b]$

$\Rightarrow \quad f(x) - g(x) \geq 0 \quad \forall \; x \in [a, b]$

$\Rightarrow \quad (f - g)(x) \geq 0 \quad \forall \; x \in [a, b]$

$\therefore$ By Theorem 10, $\int_a^b (f - g)(x)\,dx \geq 0$

$$\Rightarrow \quad \int_a^b (f(x) - g(x))\,dx \geq 0 \Rightarrow \int_a^b f(x)\,dx - \int_a^b g(x)\,dx \geq 0$$

$$\therefore \quad \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx.$$

Theorem 12. *If $f \in R[a, b]$ and m, M are the infimum and supermum of f in $[a, b]$, then*

$$\int_a^b f(x)\,dx = \mu(b - a) \text{ where } \mu \in [m, M].$$

Proof. For every partition P of $[a, b]$, we have

$$m(b - a) \leq L(P, f) \leq U(P, f) \leq M(b - a) \qquad ...(1)$$

$$\text{Now} \quad \sup\{L(P, f)\}_{P \in P[a, b]} = \underline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx \qquad [\because f \in R[a, b]]$$

$$\Rightarrow \quad L(P, f) \leq \int_a^b f(x)\,dx \qquad ...(2)$$

$$\text{Also} \quad \inf\{U(P, f)\}_{P \in P[a, b]} = \overline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx \qquad [\because f \in R[a, b]]$$

$$\Rightarrow \quad \int_a^b f(x)\,dx \leq U(P, f) \qquad ...(3)$$

From (1), (2) and (3), we have

$$m(b - a) \leq \int_a^b f(x)\,dx \leq M(b - a) \Rightarrow m \leq \frac{1}{b - a}\int_a^b f(x)\,dx \leq M \text{ for } a \neq b$$

$\Rightarrow \quad \frac{1}{b - a}\int_a^b f(x)\,dx$ is a number μ (say) lying between the bounds.

$$\Rightarrow \quad \frac{1}{b - a}\int_a^b f(x)\,dx = \mu \qquad \text{where } m \leq \mu \leq M$$

$$\Rightarrow \quad \int_a^b f(x)\,dx = \mu(b - a) \qquad \text{where } \mu \in [m, M]$$

For $a = b$, the result is trivially true.

Theorem 13. *If f is continuous on $[a, b]$, then there exists $c \in [a, b]$ such that*

$$\int_a^b f(x)\,dx = (b - a) f(c).$$

Proof. f is continuous on $[a, b]$.

$\Rightarrow \quad f$ is bounded on $[a, b]$ and $f \in R[a, b]$. If m, M are the infimum and supermum of f on $[a, b]$, then we know that

$$m(b - a) \leq \int_a^b f(x)\,dx \leq M(b - a)$$

$$\therefore \quad \exists \; \mu \in [m, M] \text{ such that } \int_a^b f(x)\,dx = \mu(b - a) \qquad ...(1)$$

Since f is continuous on $[a, b]$, it attains every value between its bounds m, M.

$\therefore \qquad \mu \in [m, M] \Rightarrow \exists$ a number $c \in [a, b]$ such that $f(c) = \mu$.

$\therefore$ From (1), we have $\int_a^b f(x)\,dx = (b - a) f(c)$.

Theorem 14. (First Mean Value Theorem)

If $f, g \in R[a, b]$ and g keeps the same sign on $[a, b]$ then there exists a number μ between the infimum and supermum of f on $[a, b]$ such that $\int_a^b f(x)\,g(x)\,dx = \mu \int_a^b g(x)\,dx$.

Proof. Let g be non-negative on $[a, b]$

Then $g(x) \geq 0 \quad \forall\ x \in [a, b]$

$f \in R[a, b] \Rightarrow f$ is bounded on $[a, b]$

$\therefore$ If m, M are the infimum and supermum of f on $[a, b]$, then

$$m \leq f(x) \leq M \quad \forall\ x \in [a, b]$$

Since $g(x) \geq 0 \quad \forall\ x \in [a, b]$

$\therefore$ $mg(x) \leq f(x)\, g(x) \leq Mg(x)$

$$\Rightarrow \int_a^b mg(x)\,dx \leq \int_a^b f(x)\,g(x)\,dx \leq \int_a^b Mg(x)\,dx$$

$$\Rightarrow m\int_a^b g(x)\,dx \leq \int_a^b f(x)\,g(x)\,dx \leq M\int_a^b f(x)\,dx$$

$\Rightarrow$ $\exists\ \mu \in [m, M]$ such that $\int_a^b f(x)\,g(x)\,dx = \mu\int_a^b g(x)$

If g be non-positive on $[a, b]$, then $g(x) \leq 0 \quad \forall\ x \in [a, b]$

$\therefore$ $m \leq f(x) \leq M \quad \forall\ x \in [a, b]$

$\Rightarrow$ $mg(x) \geq f(x)\, g(x) \geq Mg(x)$

$$\Rightarrow m\int_a^b g(x)\,dx \geq \int_a^b f(x)\,g(x)\,dx \geq M\int_a^b g(x)\,dx$$

$\Rightarrow$ $\exists\ \mu \in [m, M]$ such that $\int_a^b f(x) g(x)\,dx = \mu\int_a^b g(x)\,dx.$

Note. If we take $g(x) = 1 \quad \forall\ x \in [a, b]$, then $g \in R[a, b]$ and $g(x) > 0 \quad \forall\ x \in [a, b]$.

By the mean value theorem, we have

$$\int_a^b f(x)\,dx = \mu\int_a^b 1\,dx, \text{ where } \mu \in [m, M] \quad \text{or} \quad \int_a^b f(x)\,dx = \mu(b-a).$$

Cor. *If f is continuous on $[a, b]$, $g \in R[a, b]$ and g keeps the same sign on $[a, b]$, then there exists $c \in [a, b]$ such that $\int_a^b f(x) g(x)\,dx = f(c)\int_a^b g(x)\,dx$.*

Proof. f is continuous on $[a, b] \Rightarrow f \in R[a, b]$

By Theorem 14, there exists $\mu \in [m, M]$ such that

$$\int_a^b f(x)\,g(x)\,dx = \mu\int_a^b g(x)\,dx \quad \text{...(1)}$$

Since f is continuous on $[a, b]$, it attains every value between its bounds m, M.

$\therefore$ $\mu \in [m, M] \Rightarrow \exists$ a number $c \in [a, b]$ such that $f(c) = \mu$.

$\therefore$ From (1), we have $\int_a^b f(x)\,g(x)\,dx = f(c)\int_a^b g(x)\,dx.$

Primitive (Def.). *If f and F are two functions defined on $[a, b]$ and $F'(x) = f(x)$ $\forall\ x \in [a, b]$, then F is called a primitive of f on $[a, b]$.*

Theorem 15. (Fundamental Theorem of Integral Calculus)

If $f \in R\,[a, b]$ and F is a primitive of f on $[a, b]$, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

Proof. F is a primitive of f on $[a, b]$

$\Rightarrow$ $F'(x) = f(x) \quad \forall \quad x \in [a, b]$ ...(1)

Consider a partition $P = \{a = x_0, x_1, \ldots\ldots, x_n = b\}$ of $[a, b]$.

Since F is differentiable on $[a, b]$, it is differentiable (and hence continuous) on each sub-interval

$$I_r = [x_{r-1}, x_r],\ r = 1, 2, \ldots\ldots, n.$$

Applying Lagrange's Mean Value Theorem to F on each sub-interval $I_r = [x_{r-1}, x_r]$, $r = 1, 2, \ldots\ldots, n$, we have

$$F(x_r) - F(x_{r-1}) = (x_r - x_{r-1})\,F'(\xi_r) = f(\xi_p)\,\delta_r \quad \text{[by (1)]}$$

where $x_{r-1} < \xi_r < x_r,\ r = 1, 2, \ldots\ldots, n$

$$\Rightarrow \sum_{r=1}^{n} f(\xi_r)\,\delta_r = \sum_{r=1}^{n} [F(x_r) - F(x_{r-1})] = F(x_n) - F(x_0) = F(b) - F(a)$$

$$\therefore \lim_{\|P\|\to 0} \sum_{r=1}^{n} f(\xi_r)\,\delta_r = \lim_{\|P\|\to 0} [F(b) - F(a)] = F(b) - F(a)$$

But $\lim_{\|P\|\to 0} \sum_{r=1}^{n} f(\xi_r)\,\delta_r = \int_a^b f(x)\,dx$

$$\therefore \int_a^b f(x)\,dx = F(b) - F(a).$$

Remarks 1. The fundamental theorem does not state that if f is integrable, then f has a primitive on $[a, b]$. It only states that if f has a primitive on $[a, b]$, then this primitive can be used to evaluate $\int_a^b f(x)\,dx$.

2. A function may have a primitive without being integrable. Consider the functions F and f defined on $[-1, 1]$ as follows :

$$F(x) = \begin{cases} x^2 \sin\dfrac{1}{x^2}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases} \quad \text{and} \quad f(x) = \begin{cases} 2x\sin\dfrac{1}{x^2} - \dfrac{2}{x}\cos\dfrac{1}{x^2}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Clearly, $F'(x) = f(x) \Rightarrow$ F is a primitive of f on $[-1, 1]$.

But f is not integrable on $[-1, 1]$ because f is not bounded on $[-1, 1]$.

Example 1. *Prove that* $\dfrac{1}{\pi} \leq \displaystyle\int_0^1 \frac{\sin \pi x}{1+x^2}\,dx \leq \frac{2}{\pi}$.

Sol. Let $f(x) = \dfrac{1}{1+x^2}$ and $g(x) = \sin \pi x$, then f, g are continuous on $[0, 1]$ and hence integrable on $[0, 1]$.

Also $g(x) = \sin \pi x \geq 0$ on $[0, 1]$.

Since f is decreasing on $[0, 1]$, $\inf f = f(1) = \dfrac{1}{2}$ and $\sup f = f(0) = 1$

$\therefore$ By the first Mean Value Theorem, there exists $\mu \in \left[\dfrac{1}{2}, 1\right]$ such that

$$\int_0^1 f(x)\,g(x)\,dx = \mu\int_0^1 g(x)\,dx \quad i.e., \quad \int_0^1 \frac{\sin \pi x}{1+x^2}\,dx = \mu\int_0^1 \sin \pi x\,dx$$

But $$\int_0^1 \sin \pi x\,dx = -\left.\frac{\cos \pi x}{\pi}\right]_0^1 = \frac{2}{\pi}$$

$\therefore$ $$\int_0^1 \frac{\sin \pi x}{1+x^2}\,dx = \mu\cdot\frac{2}{\pi} \qquad \ldots(1)$$

Since f is continuous on $[0, 1]$, it attains every value between its bound $\frac{1}{2}$ and 1.

$\therefore$ $$\mu \in \left[\frac{1}{2}, 1\right] \Rightarrow \exists \text{ a number } c \in [0, 1] \text{ such that } f(c) = \mu.$$

From (1), $$f(c) = \mu = \frac{\pi}{2}\int_0^1 \frac{\sin \pi x}{1+x^2}\,dx$$

But $0 \le c \le 1$ and f is decreasing on $[0, 1]$

$\Rightarrow$ $$f(0) \ge f(c) \ge f(1) \Rightarrow \frac{1}{2} \le \frac{\pi}{2}\int_0^1 \frac{\sin \pi x}{1+x^2}\,dx \le 1 \quad \therefore \quad \frac{1}{\pi} \le \int_0^1 \frac{\sin \pi x}{1+x^2}\,dx \le \frac{2}{x}.$$

Example 2. *Prove that* $\dfrac{\pi^2}{9} \le \displaystyle\int_{\pi/6}^{\pi/2} \frac{x}{\sin x}\,dx \le \frac{2\pi^2}{9}$.

Sol. Let $f(x) = \dfrac{1}{\sin x}$ and $g(x) = x$, then f, g are continuous on $\left[\frac{\pi}{6}, \frac{\pi}{2}\right]$ and hence integrable on $\left[\frac{\pi}{6}, \frac{\pi}{2}\right]$.

Also $$g(x) = x > 0 \text{ on } \left[\frac{\pi}{6}, \frac{\pi}{2}\right].$$

Since f is decreasing on $\left[\frac{\pi}{6}, \frac{\pi}{2}\right]$, $\inf f = f\left(\frac{\pi}{2}\right) = 1$ and $\sup f = f\left(\frac{\pi}{6}\right) = 2$.

$\therefore$ by the first mean value theorem, there exists $\mu \in [1, 2]$ such that

$$\int_{\pi/6}^{\pi/2} f(x)\,g(x)\,dx = \mu\int_{\pi/6}^{\pi/2} g(x)\,dx \quad i.e., \quad \int_{\pi/6}^{\pi/2} \frac{x}{\sin x}\,dx = \mu\int_{\pi/6}^{\pi/2} x\,dx$$

But $$\int_{\pi/6}^{\pi/2} x\,dx = \left.\frac{x^2}{2}\right]_{\pi/6}^{\pi/2} = \frac{1}{2}\left(\frac{\pi^2}{4} - \frac{\pi^2}{36}\right) = \frac{\pi^2}{9}$$

$\therefore$ $$\int_{\pi/6}^{\pi/2} \frac{x}{\sin x}\,dx = \mu\cdot\frac{\pi^2}{9} \qquad \ldots(1)$$

Since f is continuous on $\left[\frac{\pi}{6}, \frac{\pi}{2}\right]$, it attains every value between its bounds 1 and 2.

$\therefore$ $$\mu \in [1, 2] \Rightarrow \exists \text{ a number } c \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right] \text{ such that } f(c) = \mu$$

From (1), $$f(c) = \mu = \frac{9}{\pi^2}\int_{\pi/6}^{\pi/2} \frac{x}{\sin x}\,dx$$

But $\dfrac{\pi}{6} \le c \le \dfrac{\pi}{2}$ and f is decreasing on $\left[\frac{\pi}{6}, \frac{\pi}{2}\right]$ $\Rightarrow$ $f\left(\frac{\pi}{6}\right) \ge f(c) \ge f\left(\frac{\pi}{2}\right)$ $\Rightarrow$ $1 \le \dfrac{9}{\pi^2}\displaystyle\int_{\pi/6}^{\pi/2} \frac{x}{\sin x}\,dx \le 2$

$\therefore$ $$\frac{\pi^2}{9} \le \int_{\pi/6}^{\pi/2} \frac{x}{\sin x}\,dx \le \frac{2\pi^2}{9}.$$

Example 3. *Prove that* $\dfrac{1}{3\sqrt{2}} \le \displaystyle\int_0^1 \frac{x^2}{\sqrt{1+x^2}}\,dx \le \frac{1}{3}$.

Sol. *Please try yourself.*

$\left[\textbf{Hint.} \text{ Take } f(x) = \dfrac{1}{\sqrt{1+x^2}}, g(x) = x^2.\right]$

10

Sequences and Series of Functions

10.1. INTRODUCTION

In chapters 4 and 5, we discussed the convergence of sequences and series *of real numbers*. In this chapter, we will discuss the convergence of sequences and series *of real-valued functions* defined on an interval.

10.2. SEQUENCES OF REAL-VALUED FUNCTIONS

Let f_n be a real-valued function defined on an interval I (or on a subset D of R) and for each $n \in$ N. Then

$$< f_1, f_2, f_3, \ldots\ldots, f_n, \ldots\ldots >$$

is called a sequence of real-valued functions on I. It is denoted by $\{f_n : I \to R, n \in N\}$ or briefly by $\{f_n\}$ or $< f_n >$.

For example :

(*i*) If f_n is a real-valued function defined by $f_n(x) = x^n$, $0 \le x \le 1$

then $$< f_1(x), f_2(x), f_3(x), \ldots\ldots > = < x^1, x^2, x^3, \ldots\ldots >$$

is a sequence of real-valued functions on [0, 1].

(*ii*) If f_n is a real-valued function defined by $f_n(x) = \dfrac{\sin nx}{n}$, $0 \le x \le 1$

then $$< f_1(x), f_2(x), f_3(x), \ldots\ldots > = < \sin x, \frac{\sin 2x}{2}, \frac{\sin 3x}{3}, \ldots\ldots >$$

is a sequence of real-valued functions on [0, 1].

If $< f_n >$ is a sequence of functions defined on I, then for $c \in$ I,

$< f_n(c) > = < f_1(c), f_2(c), \ldots\ldots, f_n(c), \ldots\ldots >$ is a sequence of real numbers.

For example, if $< f_n >$ is a sequence of functions defined by $f_n(x) = x^n$, $0 \le x \le 1$, then

$$< f_n(\tfrac{1}{2}) > = < f_1(\tfrac{1}{2}), f_2(\tfrac{1}{2}), f_3(\tfrac{1}{2}), \ldots\ldots, f_n(\tfrac{1}{2}), \ldots\ldots > = < \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots\ldots, \frac{1}{2^n}, \ldots\ldots >$$

is a sequence of real numbers corresponding to $\frac{1}{2} \in [0, 1]$.

Thus to each $x \in$ I, we have a sequence of real numbers.

10.3. POINTWISE CONVERGENCE OF A SEQUENCE OF FUNCTIONS

Let $< f_n >$ be a sequence of functions on I and $c \in$ I. Then the sequence of real numbers $< f_n(c) >$ may be convergent. In fact for each $c \in$ I, the corresponding sequence of real numbers may be convergent.

If $< f_n >$ is a sequence of real-valued functions on I and for each $x \in$ I, the corresponding sequence of real numbers is convergent, then we say the sequence $< f_n >$ converges pointwise. The limiting values of the sequences of real numbers corresponding to $x \in$ I define a function f called the limit function or simply the limit of the sequence $< f_n >$ of functions on I.

Definition. Let $< f_n >$ be a sequence of functions on I. If to each $x \in$ I and to each $\varepsilon > 0$, there corresponds to positive integer m such that $| f_n(x) - f(x) | < \varepsilon \ \forall n \ge m$ then we say that $< f_n >$ converges pointwise to the function f on I.

Note 1. $< f_n >$ converges pointwise to the function f on I.

$\Leftrightarrow \lim\limits_{n \to \infty} f_n(x) = f(x) \ \forall x \in$ I. $f(x)$ is called the limit function or simply the limit or the pointwise limit of $< f_n(x) >$ on I.

Note 2. The positive integer m depends on $x \in$ I and given $\varepsilon > 0$, *i.e.*, $m = m(x, \varepsilon)$.

Let us consider a few examples :

(*i*) Let $f_n(x) = x^n$, $x \in [0, 1]$

Since $$\lim_{n \to \infty} x^n = 0 \text{ for } 0 \le x < 1$$

we have $$\lim_{n \to \infty} f_n(x) = 0 \text{ for } 0 \le x < 1$$

When $x = 1$, the corresponding sequence $< f_n(1) > = < 1, 1, 1, \ldots\ldots >$ is a constant sequence converging to 1.

$$\therefore \quad \lim_{n \to \infty} f_n(x) = \begin{cases} 0 & \text{when } 0 \le x < 1 \\ 1 & \text{when } x = 1 \end{cases}$$

Hence $< f_n >$ converges pointwise on [0, 1].

$$f(x) = \begin{cases} 0 & \text{when } 0 \le x < 1 \\ 1 & \text{when } x = 1 \end{cases}$$ is the limit function of $< f_n(x) >$ on [0, 1].

(*ii*) Let $$f_n(x) = \frac{x}{1 + nx}, \ x \ge 0$$

Then for $x > 0$, $\lim\limits_{n \to \infty} f_n(x) = 0$

Also $f_n(0) = 0 \ \forall n \in$ N so that $< f_n(0) >$ converges to 0.

$$\therefore \quad \lim_{n \to \infty} f_n(x) = 0 \ \forall x \ge 0$$

Hence $< f_n >$ converges to zero pointwise on $[0, \infty)$ and $f(x) = 0$ is the limit function of $< f_n(x) >$ on $[0, \infty)$.

(*iii*) Let $$f_n(x) = \frac{nx}{1 + n^2x^2}, \ x \in R$$

For $x \ne 0$, $$f_n(x) = \frac{\frac{1}{nx}}{\frac{1}{n^2x^2} + 1} \to 0 \text{ as } n \to \infty$$

Also $$f_n(0) = 0 \ \forall n \in N$$

$$\therefore \quad \lim_{n \to \infty} f_n(x) = 0 \ \forall x \in R.$$

Hence $< f_n >$ converges to zero pointwise on R and $f(x) = 0$ is the limit function of $< f_n(x) >$ on R.

Note 3. For a sequence $<f_n>$ of functions, an important question is :

If each function of a sequence $< f_n >$ has a certain property such as continuity, differentiability or integrability, then to what extent is this property transferred to the limit function f? In fact, pointwise convergence is not strong enough to transfer any of the properties mentioned above from the terms f_n of $<f_n>$ to the limit function.

Let us consider a few examples :

(i) A sequence of continuous functions with a discontinuous limit function.

Consider the sequence $<f_n>$ where $f_n(x) = \dfrac{x^{2n}}{1+x^{2n}}$, $x \in \mathbf{R}$

Then $$f(x) = \lim_{n\to\infty} f_n(x) = \begin{cases} 0 & \text{if } |x|<1 \\ \frac{1}{2} & \text{if } |x|=1 \\ 1 & \text{if } |x|>1 \end{cases}$$

Here, each f_n is continuous on R but f is discontinuous at $x = \pm 1$.

(ii) A sequence of differentiable functions in which the limit of the derivatives is not equal to the derivative of the limit function.

Consider the sequence $<f_n>$ where $f_n(x) = \dfrac{\sin nx}{\sqrt{n}}$, $x \in \mathbf{R}$

Then $$f(x) = \lim_{n\to\infty} \frac{\sin nx}{\sqrt{n}} = 0 \quad \forall\, x \in \mathbf{R}$$

$\therefore$ $$f'(x) = 0 \quad \forall\, x \in \mathbf{R} \Rightarrow f'(0) = 0$$

But $$f_n'(x) = \sqrt{n}\cos nx$$

$\Rightarrow$ $$f_n'(0) = \sqrt{n} \to \infty \text{ as } n \to \infty$$

Thus, at $x = 0$, $\lim\limits_{n\to\infty} f_n'(x) \neq f'(0)$.

(iii) A sequence of functions in which the limit of integrals is not equal to the integral of the limit function.

Consider the sequence $<f_n>$ where $f_n(x) = nx(1-x^2)^n$, $x \in [0, 1]$

Then $$f_n(x) = 0 \text{ when } x = 0 \text{ or } 1$$

Also, if $0 < x < 1$, then $f(x) = \lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} nx(1-x^2)^n$ $\quad$ | Form $\infty \times 0$

$$= \lim_{n\to\infty} \frac{nx}{(1-x^2)^{-n}} \qquad \Big|\text{Form } \frac{\infty}{\infty}$$

$$= \lim_{n\to\infty} \frac{x}{-(1-x^2)^{-n}\log(1-x^2)} = \lim_{n\to\infty} \frac{-(x)(1-x^2)^n}{\log(1-x^2)} = 0$$

$\therefore$ $$f(x) = 0 \;\forall\, x \in [0, 1]$$

Now $$\int_0^1 f_n(x)\,dx = \int_0^1 nx(1-x^2)^n\,dx$$

$$= -\frac{n}{2}\int_0^1 (1-x^2)^n \cdot (-2x)\,dx = -\frac{n}{2}\left[\frac{(1-x^2)^{n+1}}{n+1}\right]_0^1 = \frac{n}{2(n+1)}$$

$\Rightarrow$ $$\lim_{n\to\infty}\int_0^1 f_n(x)\,dx = \lim_{n\to\infty}\frac{n}{2(n+1)} = \frac{1}{2}$$

Also $$\int_0^1 f(x)\,dx = \int_0^1 0\,dx = 0 \text{ so that } \lim_{n\to\infty}\int_0^1 f_n(x)\,dx \neq \int_0^1 f(x)\,dx.$$

The above few examples show that we need to investigate under what supplementary conditions these or other properties of the terms f_n of $<f_n>$ are transferred to the limit function f. A concept of great importance in this respect is that known as *uniform convergence.*

10.4. UNIFORM CONVERGENCE OF SEQUENCES OF FUNCTIONS

We know that a sequence $<f_n>$ of function on I converges pointwise to a function f if to each $x \in$ I and to each $\varepsilon > 0$, there corresponds a positive integer m such that $|f_n(x) - f(x)| < \varepsilon\ \forall\ n \geq m$

The positive integer m depends on $x \in$ I and given $\varepsilon > 0$, *i.e.*, $m = m(x, \varepsilon)$. It is not always possible to find an m which works for each $x \in$ I.

For example, consider the sequence $<f_n>$ defined by $f_n(x) = x^n$, $x \in [0, 1]$.

It converges pointwise to the function f on $[0, 1]$ where $f(x) = \begin{cases} 0, & \text{if } 0 \leq x < 1 \\ 1, & \text{if } x = 1 \end{cases}$

Let $\varepsilon = \frac{1}{2}$ be given.

Then for each $x \in [0, 1]$, there exists a positive integer m such that

$$|f_n(x) - f(x)| < \tfrac{1}{2} \;\forall\; n \geq m \qquad \ldots(1)$$

If $x = 0$, $f(x) = 0$ and $f_n(x) = 0 \;\forall\; n \in \mathbf{N}$

$\therefore$ $$|f_n(x) - f(x)| = |0 - 0| = 0 < \tfrac{1}{2} \;\forall\; n \geq 1$$

Thus (1) is true for $m = 1$

Similarly, (1) is true for $m = 1$ when $x = 1$.

If $$x = \frac{3}{4},\ f(x) = 0 \quad\text{and}\quad f_n(x) = \left(\frac{3}{4}\right)^n$$

$\therefore$ $$|f_n(x) - f(x)| = \left|\left(\frac{3}{4}\right)^n - 0\right| = \left(\frac{3}{4}\right)^n < \frac{1}{2} \;\forall\; n \geq 3$$

Thus (1) is true for $m = 3$.

Similarly, (1) is true for $m = 7$ when $x = \dfrac{9}{10}$.

Hence there is no single value of m for which (1) holds for all $x \in [0, 1]$. That is m depends both on x and ε.

Now consider the sequence $<f_n>$ defined by $f_n(x) = \dfrac{x}{1+nx}$, $x \geq 0$

It converges pointwise to zero, *i.e.*, $f(x) = 0$ for all $x \geq 0$.

Now $$0 \leq f_n(x) = \frac{x}{1+nx} \leq \frac{x}{nx} = \frac{1}{n}$$

$\therefore$ for any $\varepsilon > 0$, $|f_n(x) - f(x)| = |f_n(x)| \leq \dfrac{1}{n} < \varepsilon$

for all $x \in [0, \infty)$ provided $\dfrac{1}{n} < \varepsilon$ *i.e.*, $n > \dfrac{1}{\varepsilon}$

If m is a positive integer $> \dfrac{1}{\varepsilon}$, then $|f_n(x) - f(x)| < \varepsilon \;\forall\; n \geq m$ and $\forall\; x \in [0, \infty)$.

Thus, in this example, we can find an m which depends only on ε and not on $x \in [0, \infty)$. We say that the sequence $<f_n>$ is uniformly convergent to f on $[0, \infty)$.

Definition. *Let $<f_n>$ be a sequence of functions on I. Then $<f_n>$ is said to be uniformly convergent to a function f on I if to each $\varepsilon > 0$, there exists a positive integer m (depending only on ε) such that*

$$| f_n(x) - f(x) | < \varepsilon \ \forall x \geq m \text{ and } \forall x \in I.$$

The function f is called **uniform limit** of the sequence $<f_n>$ on I.

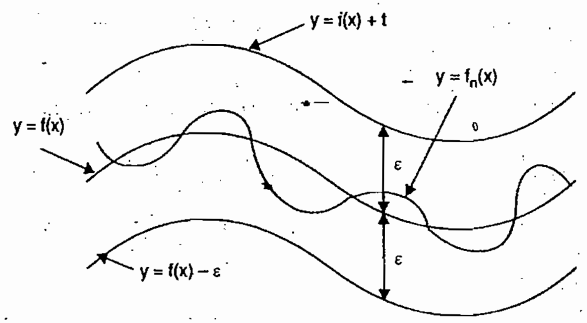
Geometrical Interpretation of Uniform Convergence

A sequence $< f_n(x) >$ of functions defined on I is said to be uniformly convergent to a function f on I if for each $\varepsilon > 0$, there exists a positive integer m (depending only on ε) such that

$$| f_n(x) - f(x) | < \varepsilon \quad \forall\ n \geq m \quad \text{and} \quad \forall x \in I$$

i.e.,

$$f(x) - \varepsilon < f_n(x) < f(x) + \varepsilon \quad \forall\ n \geq m \quad \text{and} \quad \forall x \in I.$$



This shows that the graph of $f_n(x)$ for all $n \geq m$ and for all $x \in$ I lies between the graphs of $f(x) - \varepsilon$ and $f(x) + \varepsilon$, *i.e.*, within a band of height 2ε situated symmetrically about the graph of f.

Note 1. In the definition of uniform convergence, $m \in$ N is the same for every $x \in$ I and depends only on given $\varepsilon > 0$.

Note. 2. If a sequence $<f_n>$ of functions defined on I converges uniformly to a function f on I, then the sequence $<f_n>$ converges pointwise to f also.

Thus uniform convergence $\Rightarrow$ pointwise convergence.

However, the converse is not true. For example, if $f_n(x) = x^n$, $x \in [0, 1]$, then the sequence $<f_n>$ converges pointwise to the function f on $[0, 1]$, where $f(x) = \begin{cases} 0, & \text{if } 0 \leq x < 1 \\ 1, & \text{if } x = 1 \end{cases}$ but $<f_n>$ does not converge uniformly on $[0, 1]$.

Note. 3. A sequence $<f_n>$ of functions defined on I does not converge uniformly to f on I iff there exists some $\varepsilon > 0$ such that there is no positive integer m for which the statement

'$| f_n(x) - f(x) | < \varepsilon\ \forall\ n \geq m$ and $\forall x \in$ I' holds.

Note 4. Uniform convergence is a property associated with an interval (or an infinite subset S of R) and not with a single point. On no account we speak of uniform convergence at a single point.

10.5. UNIFORMLY BOUNDED SEQUENCE OF FUNCTIONS

A sequence $<f_n>$ of functions defined on I is said to be uniformly bounded on I if there exists a positive real number K such that

$$| f_n(x) | < K \quad \forall\ n \in N \text{ and } \forall x \in I.$$

The number K is called a **uniform bound** for $<f_n>$ on I.

For example, if $f_n(x) = \sin nx$, $x \in$ R, then $| f_n(x) | = | \sin nx | \leq 1\ \forall\ n \in$ N and $\forall x \in$ R

$\therefore$ The sequence $<f_n>$ is uniformly bounded on R.

10.6. POINT OF NON-UNIFORM CONVERGENCE

Let $<f_n>$ be a sequence of functions defined on I. A point $x \in$ I said to be a point of non-uniform convergence if $<f_n>$ does not converge uniformly in any neighbourhood (however small) of x.

For example, if $f_n(x) = x^n$, $x \in [0, 1]$, then 1 is a point of non-uniform convergence of $<f_n>$.

10.7. THEOREM (Cauchy's criterion for uniform convergence)

A sequence $<f_n>$ of functions defined on I is uniformly convergent on I if and only if for each $\varepsilon > 0$ and for all $x \in$ I, there exists a positive integer m such that

$$| f_{n_1}(x) - f_{n_2}(x) | < \varepsilon\ \forall\ n_1, n_2 \geq m.$$

Proof. Necessary part. Let a sequence $<f_n>$ of functions defined on I be uniformly convergent on I.

Let $<f_n>$ converge uniformly to f on I.

Then for each $\varepsilon > 0$, there exists a positive integer m such that

$$| f_n(x) - f(x) | < \frac{\varepsilon}{2}\ \forall\ n \geq m \quad \text{and} \quad \forall x \in I.$$

If $n_1, n_2 \in$ N are such that $n_1, n_2 \geq m$, then

$$| f_{n_1}(x) - f(x) | < \frac{\varepsilon}{2}\ \forall x \in I \qquad \ldots(1)$$

and

$$| f_{n_2}(x) - f(x) | < \frac{\varepsilon}{2}\ \forall x \in I \qquad \ldots(2)$$

$$\therefore \quad | f_{n_1}(x) - f_{n_2}(x) | = | f_{n_1}(x) - f(x) + f(x) - f_{n_2}(x) | = | (f_{n_1}(x) - f(x)) - (f_{n_2}(x) - f(x)) |$$

$$\leq | f_{n_1}(x) - f(x) | + | f_{n_2}(x) - f(x) |$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\ \forall\ n_1, n_2 \geq m \text{ and } \forall x \in I \qquad \text{[by (1) and (2)]}$$

Sufficiency part. Let $<f_n>$ be a sequence of functions defined on I such that for each $\varepsilon > 0$, there exists a positive integer m such that

$$| f_{n_1}(x) - f_{n_2}(x) | < \varepsilon \quad \forall\ n_1, n_2 \geq m \quad \text{and} \quad \forall x \in I \qquad \ldots(3)$$

From (3), we find that for each $x \in$ I, the sequence $<f_n(x)>$ of real numbers is a Cauchy sequence and hence $<f_n(x)>$ is convergent. Thus the sequence $<f_n>$ is pointwise convergent.

Let the sequence $<f_n>$ converge pointwise to the function f on I.

Then $\quad \lim\limits_{n \to \infty} f_n(x) = f(x) \quad \forall x \in I \qquad \ldots(4)$

Putting $n_1 = n$ and keeping n fixed, from (3), we have $| f_n(x) - f_{n_2}(x) | < \varepsilon\ \forall\ n, n_2 \geq m$ and $\forall x \in$ I.

Also, from (4), as $n_2 \to \infty$, we have $f_{n_2}(x) \to f(x)$

$$\therefore \quad | f_n(x) - f(x) | < \varepsilon \quad \forall\ n \geq m \quad \text{and} \quad \forall x \in I$$

$\Rightarrow$ $<f_n>$ converges uniformly to f on I.

Note. The above theorem can also be stated as follows :

'A sequence $<f_n>$ of functions defined on I is uniformly convergent on I if and only if for each $\varepsilon > 0$ and for all $x \in$ I, there exists a positive integer m such that for any integer $p \geq 1$,

$$| f_{n+p}(x) - f_n(x) | < \varepsilon \ \forall \ n \geq m.$$

ILLUSTRATIVE EXAMPLES—A

Example 1. *Show that the sequence $< f_n >$ where $f_n(x) = x^n$ is uniformly convergent on $[0, k]$, $k < 1$ but only pointwise convergent on $[0, 1]$.*

Sol. Here, $\quad f(x) = \lim\limits_{n \to \infty} f_n(x) = \begin{cases} 0, & \text{if } 0 \leq x < 1 \\ 1, & \text{if } x = 1 \end{cases}$

Thus, the sequence $< f_n >$ converges pointwise to f on $[0, 1]$.

To see whether the sequence $< f_n >$ is uniformly convergent, let $\varepsilon > 0$ be given.

For $0 < x < 1$, $\quad | f_n(x) - f(x) | = | x^n - 0 | = x^n < \varepsilon$

if $\quad \dfrac{1}{x^n} > \dfrac{1}{\varepsilon} \quad$ or if $\quad n \log \dfrac{1}{x} > \log \dfrac{1}{\varepsilon}$

or if $\quad n > \dfrac{\log \frac{1}{\varepsilon}}{\log \frac{1}{x}} \qquad \left| \begin{array}{l} \text{Note that } 0 < x < 1 \\ \Rightarrow \dfrac{1}{x} > 1 \text{ so that } \log \dfrac{1}{x} > 0 \end{array} \right.$

The number $\dfrac{\log \frac{1}{\varepsilon}}{\log \frac{1}{x}}$ increases with x having maximum value $\dfrac{\log \frac{1}{\varepsilon}}{\log \frac{1}{k}}$ on $(0, k]$, $k < 1$.

Choose a positive integer m just $\geq \dfrac{\log \frac{1}{\varepsilon}}{\log \frac{1}{k}}$, then

$$| f_n(x) - f(x) | < \varepsilon \quad \forall \ n \geq m \quad \text{and} \quad 0 < x < 1.$$

At $x = 0$, $\qquad | f_n(x) - f(x) | = | 0 - 0 | = 0 < \varepsilon \quad \forall \ n \geq 1$

Thus, there exists a positive integer m such that

$$| f_n(x) - f(x) | < \varepsilon \quad \forall \ n \geq m \quad \text{and} \quad \forall \ x \in [0, k], k < 1.$$

$\Rightarrow \quad < f_n >$ is uniformly convergent on $[0, k]$, $k < 1$.

When $x \to 1$, the number $\dfrac{\log \frac{1}{\varepsilon}}{\log \frac{1}{x}} \to \infty$. Thus it is not possible to find a positive integer m such that

$$| f_n(x) - f(x) | < \varepsilon \ \forall \ n \geq m \quad \text{and} \quad \forall \ x \in [0, 1].$$

Hence, the sequence $<f_n>$ is not uniformly convergent on any interval containing 1 and in particular on $[0, 1]$.

Example 2. *Show that the sequence $< f_n >$ defined by $f_n(x) = x^n$, $x \in [0, 1]$ is not uniformly convergent.*

Sol. *Please try yourself.*

Example 3. *Show that the sequence of functions $< f_n >$, where $f_n(x) = \dfrac{1}{n + x}$, is uniformly convergent in any interval $[0, k]$, $k > 0$.*

Sol. Here $\qquad f(x) = \lim\limits_{n \to \infty} f_n(x) = 0 \quad \forall \ x \geq 0$

Let $\varepsilon > 0$ be given.

$$| f_n(x) - f(x) | = \left| \frac{1}{n + x} - 0 \right| = \frac{1}{n + x} < \varepsilon \text{ if } n + x > \frac{1}{\varepsilon}$$

Now $\forall \ x \in [0, k]$, $0 \leq x \leq k$

$$\therefore \qquad n + k \geq n + x > \frac{1}{\varepsilon} \ \Rightarrow \ n > \frac{1}{\varepsilon} - k$$

Choose a positive integer m just $\geq \dfrac{1}{\varepsilon} - k$.

Then $\qquad | f_n(x) - f(x) | < \varepsilon \ \forall \ n \geq m \quad \text{and} \quad \forall \ x \in [0, k]$.

$\Rightarrow \quad < f_n >$ is uniformly convergent on $[0, k]$.

Example 4. *Show that the sequence $<f_n>$ where $f_n(x) = \dfrac{n}{x + n}$, $x \geq 0$ is uniformly convergent in any finite interval.*

Sol. Here $\qquad f(x) = \lim\limits_{n \to \infty} f_n(x) = \lim\limits_{n \to \infty} \dfrac{n}{x + n} = \lim\limits_{n \to \infty} \dfrac{1}{\frac{x}{n} + 1} = 1 \ \forall \ x \in \mathrm{R}$

Let $\varepsilon > 0$ be given.

For $x = 0$, $\qquad | f_n(x) - f(x) | = | 1 - 1 | = 0 < \varepsilon \ \forall \ n \geq 1$

For $x > 0$, we have $\ | f_n(x) - f(x) | = \left| \dfrac{n}{x + n} - 1 \right| = \left| \dfrac{-x}{x + n} \right| = \dfrac{x}{x + n} < \varepsilon$

if $\qquad \dfrac{x + n}{x} > \dfrac{1}{\varepsilon} \quad$ or $\quad$ if $\ 1 + \dfrac{n}{x} > \dfrac{1}{\varepsilon}$ or if $\ n > x\left(\dfrac{1}{\varepsilon} - 1\right)$

Now $x\left(\dfrac{1}{\varepsilon} - 1\right)$ increases with x and tends to infinity as $x \to \infty$ so that it is not possible to choose a positive integer m such that

$$| f_n(x) - f(x) | < \varepsilon \quad \forall \ n \geq m \quad \text{and} \quad \forall \ x \geq 0.$$

However, if we consider a finite interval $[0, k]$ where k is any fixed positive number, however large, then the maximum value of $x\left(\dfrac{1}{\varepsilon} - 1\right)$ is $k\left(\dfrac{1}{\varepsilon} - 1\right)$.

If we choose a positive integer m just $\geq k\left(\dfrac{1}{\varepsilon} - 1\right)$, then

$$| f_n(x) - f(x) | < \varepsilon \ \forall \ n \geq m \quad \text{and} \quad \forall \ x \in [0, k].$$

Hence $< f_n >$ is uniformly convergent on any finite interval.

Example 5. *Test for uniform convergence the sequence $< f_n >$ where $f_n(x) = e^{-nx}$ for $x \geq 0$.*

Sol. Here $\qquad f(x) = \lim\limits_{n \to \infty} f_n(x) = \lim\limits_{n \to \infty} e^{-nx} = \begin{cases} 1, & \text{if } x = 0 \\ 0 & \text{if } x > 0 \end{cases}$

Let $\varepsilon > 0$ be given.

For $x > 0$, we have $\quad | f_n(x) - f(x) | = | e^{-nx} - 0 | = e^{-nx} \leq \varepsilon$

if $\quad e^{nx} > \dfrac{1}{\varepsilon}$ i.e., if $nx > \log \dfrac{1}{\varepsilon}$

or if $\quad n > \dfrac{\log \frac{1}{\varepsilon}}{x} \quad (\because x > 0)$

Now $\dfrac{\log \frac{1}{\varepsilon}}{x}$ decreases as x increases.

Choose a positive integer m just $\geq \dfrac{\log \frac{1}{\varepsilon}}{x}$, then

$$| f_n(x) - f(x) | < \varepsilon \ \forall\, n \geq m \quad \text{and} \quad x > 0.$$

$\Rightarrow$ $< f_n >$ is uniformly convergent on $[a, b]$, $a > 0$.

However, when $x \to 0$, $\dfrac{\log \frac{1}{\varepsilon}}{x} \to \infty$ so that it is not possible to choose a positive integer m such that

$$| f_n(x) - f(x) | < \varepsilon \quad \forall\, n \geq m \quad \text{and} \quad \forall\, x \geq 0.$$

Hence the sequence $< f_n >$ is not uniformly convergent on $[0, b]$.

Example 6. *Show that the sequence $< f_n >$ where $f_n(x) = e^{-nx}$ on $[0, k]$ is not uniformly convergent.*

Sol. *Please try yourself.*

Example 7. *Show that the sequence $< f_n >$ where $f_n(x) = e^{-nx}$ on $[a, b]$, $a > 0$ is uniformly convergent.*

Sol. *Please try yourself.*

$\left[\textbf{Hint.} \text{ Maximum value of } \dfrac{\log \frac{1}{\varepsilon}}{x} \text{ is } \dfrac{\log \frac{1}{\varepsilon}}{a}.\right]$

Example 8. *Show that the sequence of functions $< f_n >$ defined as $f_n(x) = \dfrac{x^n}{n}$ on $(-\infty, \infty)$ is not uniformly convergent.*

Sol. Here $\quad f(x) = \lim\limits_{n \to \infty} f_n(x) = \lim\limits_{n \to \infty} \dfrac{x^n}{n} = 0 \ \forall\, x$

Let $\varepsilon > 0$ be given.

$$| f_n(x) - f(x) | = \left| \frac{x^n}{n} - 0 \right| = \frac{|x|^n}{n} < \varepsilon \text{ if } n > \frac{|x|^n}{\varepsilon}$$

When $| x | > 1$, $\dfrac{|x|^n}{\varepsilon} \to \infty$ as $n \to \infty$ so that it is not possible to choose a positive integer m such that

$$| f_n(x) - f(x) | < \varepsilon \quad \forall\, n \geq m \quad \text{and} \quad \forall\, x \in \mathbb{R}.$$

$\Rightarrow$ $< f_n >$ is not uniformly convergent on $[a, b]$, where $a, b \in \mathbb{R}$ and $a < b$.

Example 9. *Show that the sequence $< f_n >$ where $f_n(x) = \dfrac{x^n}{n}$, $0 \leq x \leq 1$ converges uniformly to 0.*

Sol. Here $\quad f(x) = \lim\limits_{n \to \infty} f_n(x) = \lim\limits_{n \to \infty} \dfrac{x^n}{n} = 0 \ \forall\, x$

Let $\varepsilon > 0$ be given.

$$| f_n(x) - f(x) | = \left| \frac{x^n}{n} - 0 \right| = \frac{x^n}{n} < \varepsilon \text{ if } n > \frac{x^n}{\varepsilon}$$

Since $\quad 0 \leq x \leq 1 \ \Rightarrow \ 0 \leq x^n \leq 1$

$\therefore$ If we choose a positive integer m just $\geq \dfrac{x^n}{\varepsilon}$, then

$$| f_n(x) - f(x) | < \varepsilon \quad \forall\, n \geq m \quad \text{and} \quad \forall\, x \in [0, 1].$$

$\Rightarrow$ $< f_n >$ converges uniformly to 0.

Example 10. *Show that $x = 0$ is a point of non-uniform convergence of the sequence of functions $< f_n >$ where $f_n(x) = \dfrac{nx}{1 + n^2x^2}$.*

Sol. Here $\quad f(x) = \lim\limits_{n \to \infty} f_n(x) = \lim\limits_{n \to \infty} \dfrac{nx}{1 + n^2x^2} = \lim\limits_{n \to \infty} \dfrac{\frac{x}{n}}{\frac{1}{n^2} + x^2} = 0 \ \forall\, x \in \mathbb{R}$

Let $\varepsilon > 0$ be given. Then $| f_n(x) - f(x) | = \left| \dfrac{nx}{1 + n^2x^2} - 0 \right| = \dfrac{n|x|}{1 + n^2x^2} < \varepsilon$

if $\quad n\,|x| < \varepsilon + n^2x^2\varepsilon \quad$ i.e., if $\ \varepsilon x^2n^2 - |x|\,n + \varepsilon > 0$

i.e., if $\quad n > \dfrac{|x| + \sqrt{x^2 - 4\varepsilon^2x^2}}{2\varepsilon x^2} \quad$ i.e., if $\ n > \dfrac{1 + \sqrt{1 - 4\varepsilon^2}}{2\varepsilon |x|}$

If we choose a positive integer m just $\geq \dfrac{1 + \sqrt{1 - 4\varepsilon^2}}{2\varepsilon |x|}$, $x \neq 0$, then

$$| f_n(x) - f(x) | < \varepsilon \quad \forall\, n \geq m \quad \text{and} \quad x \neq 0$$

Thus the sequence $< f_n >$ is uniformly convergent in every interval which does not contain 0.

But, when $x \to 0$, $\dfrac{1 + \sqrt{1 - 4\varepsilon^2}}{2\varepsilon |x|} \to \infty$ so that it is not possible to choose a positive integer m such that

$$| f_n(x) - f(x) | < \varepsilon \quad \forall\, n \geq m \quad \text{and} \quad \forall\, x \in \mathbb{R}.$$

Hence $x = 0$ is a point of non-uniform convergence.

Example 11. *Show that the sequence of functions $< f_n >$ where $f_n(x) = \dfrac{n^2x}{1 + n^2x^2}$ is non-uniformly convergent on $[0, 1]$.*

Sol. When $\ x = 0$, $\ f_n(x) = 0 \ \forall\, n$

When $0 < x \leq 1$, $\quad \lim\limits_{n \to \infty} f_n(x) = \lim\limits_{n \to \infty} \dfrac{n^2x}{1 + n^2x^2} = \lim\limits_{n \to \infty} \dfrac{x}{\frac{1}{n^2} + x^2} = \dfrac{1}{x}$

$\therefore$ $$f(x) = \begin{cases} \frac{1}{x}, & \text{if } 0 < x \le 1 \\ 0, & \text{if } x = 0 \end{cases}$$

Let $\varepsilon > 0$ be given. Then for $0 < x \le 1$, we have

$$|f_n(x) - f(x)| = \left| \frac{n^2x}{1+n^2x^2} - \frac{1}{x} \right| = \left| \frac{-1}{x(1+n^2x^2)} \right| = \frac{-1}{x(1+n^2x^2)} < \varepsilon$$

if $$x(1+n^2x^2) > \frac{1}{\varepsilon} \quad i.e., \quad \text{if} \quad n > \frac{1}{x}\sqrt{\frac{1}{\varepsilon x} - 1}$$

Since $0 < x \le 1$, $\therefore$ $0 < \frac{1}{x} \le 1$

If we choose a positive integer m just $\ge \frac{1}{x}\sqrt{\frac{1}{\varepsilon x} - 1}$, then $|f_n(x) - f(x)| < \varepsilon \; \forall \; n \ge m$ and $0 < x \le 1$.

But, when $x \to 0$, $\frac{1}{x}\sqrt{\frac{1}{\varepsilon x} - 1} \to \infty$ so that it is not possible to choose a positive integer m such that

$$|f_n(x) - f(x)| < \varepsilon \quad \forall \; n \ge m \quad \text{and} \quad \forall \; x \in [0, 1].$$

Hence $< f_n >$ is non-uniformly convergent on $[0, 1]$ and $x = 0$ is a point of non-uniform convergence.

Example 12. *Show that the sequence $< \tan^{-1} nx >$, $x \ge 0$, is uniformly convergent on any interval $[a, b]$, $a > 0$ but is only pointwise convergent on $[0, b]$.*

Sol. Here $f_n(x) = \tan^{-1} nx$, $x \ge 0$

$\therefore$ $$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \tan^{-1} nx = \begin{cases} \frac{\pi}{2}, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Let $\varepsilon > 0$ be given.

For $x > 0$, $$|f_n(x) - f(x)| = \left| \tan^{-1} nx - \frac{\pi}{2} \right| = |\cot^{-1} nx| \quad \left[\because \tan^{-1} nx + \cot^{-1} nx = \frac{\pi}{2} \right]$$

$$= \cot^{-1} nx < \varepsilon$$

if $$nx > \cot \varepsilon \quad i.e., \quad \text{if} \quad n > \frac{\cot \varepsilon}{x}$$

Now $\frac{\cot \varepsilon}{x}$ decreases as x increases, the maximum value of $\frac{\cot \varepsilon}{x}$ being $\frac{\cot \varepsilon}{a}$ in $[a, b]$, $a > 0$.

If we choose a positive integer m just $\ge \frac{\cot \varepsilon}{a}$, then

$$|f_n(x) - f(x)| < \varepsilon \quad \forall \; n \ge m \quad \text{and} \quad \forall \; x \in [a, b], a > 0.$$

But as $x \to 0$, $\frac{\cot \varepsilon}{x} \to \infty$ so that it is not possible to choose a positive integer m such that

$$|f_n(x) - f(x)| < \varepsilon \quad \forall \; n \ge m \quad \text{and} \quad \forall \; x \ge 0.$$

Hence $< f_n >$ is not uniformly convergent on $[0, b]$ but is only pointwise convergent on $[0, b]$.

Example 13. *If $f_n(x) = \frac{\sin nx}{n}$, $0 \le x \le 1$, does there exist $m \in N$ such that*

$$|f_n(x) - f(x)| < \frac{1}{10} \; \forall \; n \ge m \text{ and } \forall \; x \in [0, 1] ?$$

Sol. *Please try yourself.* **[Ans. $m = 11$]**

[Hint. Here $f(x) = 0 \; \forall \; x \in [0, 1]$.

Example 14. *Show that the sequence $< f_n >$, where $f_n(x) = \frac{nx}{nx+1}$ is uniformly convergent on $[a, b]$, $a > 0$ but is only pointwise convergent on $[0, b]$.*

Sol. When $x = 0$, $f_n(x) = 0 \; \forall \; n$

When $x > 0$, $$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{nx+1} = \lim_{n\to\infty} \frac{x}{x + \frac{1}{n}} = 1$$

$\therefore$ $$f(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Let $\varepsilon > 0$ be given.

For $x > 0$, we have $$|f_n(x) - f(x)| = \left| \frac{nx}{nx+1} - 1 \right| = \left| \frac{-1}{nx+1} \right| = \frac{1}{nx+1} < \varepsilon$$

if $$nx + 1 > \frac{1}{\varepsilon} \quad i.e., \quad \text{if} \quad n > \frac{1}{x}\left(\frac{1}{\varepsilon} - 1\right)$$

Now $\frac{1}{x}\left(\frac{1}{\varepsilon} - 1\right)$ decreases as x increases. The maximum value of $\frac{1}{x}\left(\frac{1}{\varepsilon} - 1\right)$ on $[a, b]$, $a > 0$ is $\frac{1}{a}\left(\frac{1}{\varepsilon} - 1\right)$.

$\therefore$ If we choose a positive integer m just $\ge \frac{1}{a}\left(\frac{1}{\varepsilon} - 1\right)$, then

$$|f_n(x) - f(x)| < \varepsilon \quad \forall \; n \ge m \quad \text{and} \quad \forall \; x \in [a, b], a > 0.$$

However, when $x \to 0$, $\frac{1}{x}\left(\frac{1}{\varepsilon} - 1\right) \to \infty$ so that it is not possible to choose a positive integer m such that

$$|f_n(x) - f(x)| < \varepsilon \; \forall \; n \ge m \quad \text{and} \quad \forall \; x \in [0, b].$$

Hence the sequence $< f_n >$ is not uniformly convergent on $[0, b]$ but is only pointwise convergent on $[0, b]$.

Example 15. *Show that the sequence $< f_n >$ where $f_n(x) = \frac{n^2x}{1+n^4x^2}$ is non-uniformly convergent on $[0, 1]$.*

Sol. Here $$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{n^2x}{1+n^4x^2} = \lim_{n\to\infty} \frac{\frac{x}{n^2}}{\frac{1}{n^4} + x^2} = 0 \; \forall \; x \in [0, 1]$$

Let $\varepsilon > 0$ be given.

For $x > 0$, we have $| f_n(x) - f(x) | = \left| \dfrac{n^2 x}{1+n^4x^2} - 0 \right| = \dfrac{n^2 |x|}{1+n^4x^2} < \varepsilon$

if $n^2 |x| < \varepsilon + \varepsilon x^2 n^4$ i.e., if $\varepsilon x^2 n^4 - |x| n^2 + \varepsilon > 0$

i.e., if $n^2 > \dfrac{|x| + \sqrt{x^2 - 4\varepsilon^2 x^2}}{2\varepsilon x^2}$ i.e., if $n > \left[1 + \dfrac{\sqrt{1-4\varepsilon^2}}{2\varepsilon |x|}\right]^{1/2}$

If we choose a positive integer m just $\geq \left[1 + \dfrac{\sqrt{1-4\varepsilon^2}}{2\varepsilon |x|}\right]^{1/2}$,

then $| f_n(x) - f(x) | < \varepsilon \quad \forall\, n \geq m$ and $x \neq 0$.

$\Rightarrow$ $< f_n >$ is uniformly convergent on $[k, 1]$ where $0 < k < 1$.

As $x \to 0$, $\left[\dfrac{1 + \sqrt{1-4\varepsilon^2}}{2\varepsilon |x|}\right]^{1/2} \to \infty$ so that it is not possible to choose a positive integer m such that

$$| f_n(x) - f(x) | < \varepsilon \quad \forall\, n \geq m \text{ and } \forall\, x \in [0, 1].$$

Hence $< f_n >$ is non-uniformly convergent on $[0, 1]$ and $x = 0$ is a point of non-uniform convergence.

Example 16. *Show that $x = 0$ is a point of non-uniform convergence of the sequence $< f_n >$ where $f_n(x) = nxe^{-nx^2}$.*

Sol. Here $f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} nxe^{-nx^2} = \lim_{n\to\infty} \dfrac{nx}{e^{nx^2}} = 0$ for all x.

Thus the sequence $< f_n >$ converges pointwise to 0 on any interval $[0, k]$, $k > 0$.

Let us suppose, if possible, the sequence $< f_n >$ converges uniformly on $[0, k]$, so that for any $\varepsilon > 0$, there exists a positive integer m such that

$$| f_n(x) - f(x) | = nxe^{-nx^2} < \varepsilon \quad \forall\, n \geq m \text{ and } x \geq 0 \qquad ...(1)$$

Let m_0 be an integer greater than m and $e^2\varepsilon^2$, then for $x = \dfrac{1}{\sqrt{m_0}}$ and $n = m_0$, (1) gives

$$m_0 \cdot \frac{1}{\sqrt{m_0}} \cdot e^{-m_0 \cdot \frac{1}{m_0}} < \varepsilon \quad \text{or} \quad \frac{\sqrt{m_0}}{e} < \varepsilon \quad \text{or} \quad m_0 < e^2\varepsilon^2$$

Thus we arrive at a contradiction.

Hence the sequence $< f_n >$ is not uniformly convergent on $[0, k]$.

10.8. A TEST FOR UNIFORM CONVERGENCE OF SEQUENCES OF FUNCTIONS

To determine whether a given sequence $< f_n >$ is uniformly convergent or not in a given interval, we have been using the definition of uniform convergence. Thus, we find a positive integer m, independent of x which is not easy in most of the cases. The following test is more convenient in practice and does not involve the computation of m.

Theorem. (M_n Test)

Let $< f_n >$ be a sequence of functions on I such that

$$\lim_{n\to\infty} f_n(x) = f(x),\ \forall\, x \in I$$

and let $M_n = \sup \{ | f_n(x) - f(x) | : x \in I \}$

Then $< f_n >$ converges uniformly on I if and only if $\lim_{n\to\infty} M_n = 0$.

Proof. Necessary part

Let $< f_n >$ converge uniformly to f on I, so that for a given $\varepsilon > 0$, there exists a positive integer m such that

$$| f_n(x) - f(x) | < \varepsilon \quad \forall\, n \geq m \text{ and } \forall\, x \in I$$

$\Rightarrow$ $M_n = \sup \{ | f_n(x) - f(x) | : x \in I \} < \varepsilon \quad \forall\, n \geq m$

$\Rightarrow$ $M_n < \varepsilon \quad \forall\, n \geq m$

Since $\varepsilon > 0$ is arbitrary, $M_n \to 0$ as $n \to \infty$ i.e., $\lim_{n\to\infty} M_n = 0$.

Sufficiency part

Let $\lim_{n\to\infty} M_n = 0$ then for each $\varepsilon > 0$, there exists a positive integer m such that

$M_n < \varepsilon \quad \forall\, n \geq m$ and $\forall\, x \in I$

$\Rightarrow$ $\sup \{ | f_n(x) - f(x) | : x \in I \} < \varepsilon \quad \forall\, n \geq m$

$\Rightarrow$ $| f_n(x) - f(x) | < \varepsilon \quad \forall\, n \geq m$ and $\forall\, x \in I$

$\Rightarrow$ the sequence $\varepsilon < f_n >$ converges uniformly to f on I.

Note 1. M_n = the maximum value of $| f_n(x) - f(x) |$ for fixed n and $x \in I$.

Note 2. If M_n does not tend to 0, then the sequence $< f_n >$ is not uniformly convergent.

Note 3. $F(x)$ is maximum at $x = c \in I$ if

(*i*) $F'(c) = 0$ and (*ii*) $F''(c) \leq 0$.

ILLUSTRATIVE EXAMPLES—B

Example 1. *Show that the sequence of functions $< f_n >$, where $f_n(x) = \dfrac{x}{1+nx^2}$, $x \in R$ converges uniformly on any closed interval $[a, b]$.*

Sol. Here $f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \dfrac{x}{1+nx^2} = 0\ \forall\, x \in R$

$$| f_n(x) - f(x) | = \left| \frac{x}{1+nx^2} - 0 \right| = \left| \frac{x}{1+nx^2} \right|$$

Let $y = \dfrac{x}{1+nx^2}$ then $\dfrac{dy}{dx} = \dfrac{(1+nx)^2 \cdot 1 - x \cdot 2nx}{(1+nx^2)^2} = \dfrac{1-nx^2}{(1+nx^2)^2}$

For max. or min. $\dfrac{dy}{dx} = 0 \Rightarrow 1 - nx^2 = 0 \Rightarrow x = \dfrac{1}{\sqrt{n}}$

Also $\dfrac{d^2y}{dx^2} = \dfrac{(1+nx^2)^2(-2nx) - (1-nx^2) \cdot 2(1+nx^2) \cdot 2nx}{(1+nx^2)^4}$

$$= \frac{-2nx(1+nx^2) - 4nx(1-nx^2)}{(1+nx^2)^3}$$

$$\left.\frac{d^2y}{dx^2}\right]_{x=\frac{1}{\sqrt{n}}} = \frac{-2\sqrt{n}(1+1)}{(1+1)^3} = -\frac{\sqrt{n}}{2} < 0$$

$\Rightarrow$ y is maximum when $x = \frac{1}{\sqrt{n}}$ and maximum value of $y = \frac{\frac{1}{\sqrt{n}}}{1+1} = \frac{1}{2\sqrt{n}}$

$\therefore$ $$M_n = \max_{x \in [a, b]} |f_n(x) - f(x)| = \max_{x \in [a, b]} \left|\frac{x}{1+nx^2}\right| = \frac{1}{2\sqrt{n}} \to 0 \text{ as } n \to \infty$$

Hence $< f_n >$ converges uniformly to f on $[a, b]$.

Example 2. *Show that if $f_n(x) = \frac{n^2 x}{1+n^4 x^2}$, then $< f_n >$ converges non-uniformly on [0, 1].*

Sol. Here $$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{n^2 x}{1+n^4 x^2} = \lim_{n\to\infty} \frac{\frac{x}{n^2}}{\frac{1}{n^4}+x^2} = 0 \ \forall\ x \in [0, 1]$$

$$|f_n(x) - f(x)| = \left|\frac{n^2 x}{1+n^4 x^2} - 0\right| = \left|\frac{n^2 x}{1+n^4 x^2}\right|$$

Let $$y = \frac{n^2 x}{1+n^4 x^2}$$

then $$\frac{dy}{dx} = \frac{(1+n^4x^2).n^2 - n^2x.2n^4x}{(1+n^4x^2)^2} = \frac{n^2[1+n^4x^2 - 2n^4x^2]}{(1+n^4x^2)^2} = \frac{n^2(1-n^4x^2)}{(1+n^4x^2)^2}$$

For max. or min. $\frac{dy}{dx} = 0 \Rightarrow 1 - n^4x^2 = 0 \Rightarrow x = \frac{1}{n^2}$

Also $$\frac{d^2y}{dx^2} = n^2.\frac{(1+n^4x^2)^2(-2n^4x) - (1-n^4x^2).2(1+n^4x^2).2n^4x}{(1+n^4x^2)^4}$$

$$= \frac{n^2[-2n^4x(1+n^4x^2) - 4n^4x(1-n^4x^2)]}{(1+n^4x^2)^3}$$

$$= \frac{-2n^6x[(1+n^4x^2)+2(1-n^4x^2)]}{(1+n^4x^2)^3}$$

$$\left.\frac{d^2y}{dx^2}\right]_{x=\frac{1}{n^2}} = \frac{-2n^6.\frac{1}{n^2}\left(1+n^4.\frac{1}{n^4}\right)}{\left(1+n^4.\frac{1}{n^4}\right)^3} = \frac{-4n^4}{8} = -\frac{n^4}{2} < 0$$

$\Rightarrow$ y is maximum when $x = \frac{1}{n^2}$ and maximum value of y

$$= \frac{n^2.\frac{1}{n^2}}{1+n^4.\frac{1}{n^4}} = \frac{1}{2}. \text{ Also } x = \frac{1}{n^2} \to 0 \text{ as } n \to \infty.$$

$\therefore$ $$M_n = \max_{x \in [0, 1]} |f_n(x) - f(x)| = \max_{x \in [0, 1]} \left|\frac{n^2 x}{1+n^4 x^2}\right| = \frac{1}{2}$$

which does not tend to 0 as $n \to \infty$.

Hence $< f_n >$ converges non-uniformly on [0, 1].

Example 3. *Show that the sequence $< f_n >$, where $f_n(x) = \frac{nx}{1+n^2x^2}$, is not uniformly convergent on any interval containing zero.*

Sol. Here $$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{1+n^2x^2} = \lim_{n\to\infty} \frac{\frac{x}{n}}{\frac{1}{n^2}+x^2} = 0 \ \forall\ x$$

$$|f_n(x) - f(x)| = \left|\frac{nx}{1+n^2x^2} - 0\right| = \left|\frac{nx}{1+n^2x^2}\right|$$

Let $y = \frac{nx}{1+n^2x^2}$, then y is maximum when $x = \frac{1}{n}$ and maximum value of y is $\frac{1}{2}$.

[Prove it yourself]

Also $x = \frac{1}{n} \to 0$ as $n \to \infty$.

Let us take an interval $[a, b]$ containing 0.

$$M_n = \max_{x \in [a, b]} |f_n(x) - f(x)| = \max_{x \in [a, b]} \left|\frac{nx}{1+n^2x^2}\right| = \frac{1}{2}$$

which does not tend to zero as $n \to \infty$.

Hence the sequence $< f_n >$ is not uniformly convergent on any interval containing zero.

Example 4. *Show that the sequence $< f_n >$, where $f_n(x) = \frac{n^2x}{1+n^3x^2}$, is not uniformly convergent on [0, 1].*

Sol. *Please try yourself.*

[**Hint.** Here $y = \frac{n^2x}{1+n^3x^2}$ is maximum when $x = \frac{1}{n^{3/2}}$ and max. value of y is $\frac{\sqrt{n}}{2}$.

Also $x = \frac{1}{n^{3/2}} \to 0$ as $n \to \infty$.

M_n does not tend to 0 as $n \to \infty$].

Example 5. *Show that the sequence of functions $< f_n >$ where $f_n(x) = \frac{nx}{1+n^3x^2}$, $x \in R$ converges uniformly on any closed interval [a, b].*

Sol. *Please try yourself.*

Example 6. *Show that the sequence of functions* $<f_n>$, *where* $f_n(x) = \dfrac{n^2 x}{1+n^2x^2}$ *is non-uniformly convergent on [0, 1].*

Sol. When $x = 0$, $f_n(x) = 0 \quad \forall\, n$

When $0 < x \le 1$, $\quad \lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} \dfrac{n^2x}{1+n^2x^2} = \lim\limits_{n\to\infty} \dfrac{x}{\dfrac{1}{n^2}+x^2} = \dfrac{1}{x}$

$$\therefore \qquad f(x) = \begin{cases} \dfrac{1}{x}, & \text{if } 0 < x \le 1 \\ 0, & \text{if } x = 0 \end{cases}$$

When $0 < x \le 1$, $\quad |f_n(x) - f(x)| = \left| \dfrac{n^2x}{1+n^2x^2} - \dfrac{1}{x} \right| = \dfrac{1}{x(1+n^2x^2)}$

Let $y = \dfrac{1}{x(1+n^2x^2)}$, then y is maximum when $x = \dfrac{1}{n}$ and maximum value of y is $\dfrac{n}{2}$.

[Prove it yourself]

Also $x = \dfrac{1}{n} \to 0$ as $n \to \infty$., $M_n = \max\limits_{x\in[0,1]} |f_n(x) - f(x)| = \max\limits_{x\in[0,1]} \left[\dfrac{1}{x(1+n^2x^2)}\right] = \dfrac{n}{2}$

which does not tend to zero as $n \to \infty$.

Hence the sequence $<f_n>$ is not uniformly convergent on [0, 1].

Example 7. *Show that the sequence of functions* $<f_n>$, *where* $f_n(x) = nx(1-x)^n$ *is not uniformly convergent on [0, 1].*

Sol. For $0 < x < 1$, $f(x) = \lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} nx(1-x)^n$

$$= \lim_{n\to\infty} \frac{nx}{(1-x)^{-n}} \qquad \left| \text{Form } \frac{\infty}{\infty} \right.$$

$$= \lim_{n\to\infty} \frac{x}{-(1-x)^{-n}\log(1-x)} = \lim_{n\to\infty} \frac{-x(1-x)^n}{\log(1-x)}$$

$$= 0 \text{ since } (1-x)^n \to 0 \text{ as } n \to \infty$$

Also, when $x = 0$, $\quad f_n(x) = 0 \quad \forall\, n$; when $x = 1$, $f_n(x) = 0\ \forall\, n$

$\therefore \qquad f(x) = 0 \quad \forall\, x \in [0, 1]$

$$|f_n(x) - f(x)| = |nx(1-x)^n - 0| = nx(1-x)^n$$

Let $\qquad y = nx(1-x)^n$

then $\qquad \dfrac{dy}{dx} = n(1-x)^n - n^2x(1-x)^{n-1}$

$$= n(1-x)^{n-1}[(1-x) - nx] = n(1-x)^{n-1}[1-(n+1)x]$$

For max. or min. $\quad \dfrac{dy}{dx} = 0 \quad \Rightarrow \quad x = \dfrac{1}{n+1}$

Also $\qquad \dfrac{d^2y}{dx^2} = -n(n-1)(1-x)^{n-2}[1-(n+1)x] - n(n+1)(1-x)^{n-1}$

$$\left.\frac{d^2y}{dx^2}\right]_{x=\frac{1}{n+1}} = -n(n+1)\cdot\left(\frac{n}{n+1}\right)^{n-1} < 0$$

$\Rightarrow \quad y$ is maximum at $x = \dfrac{1}{n+1}$ and the maximum value of y is

$$\frac{n}{n+1}\left(1-\frac{1}{n+1}\right)^n = \left(\frac{n}{n+1}\right)^{n+1} = \left(1-\frac{1}{n+1}\right)^{n+1}$$

Also $\qquad x = \dfrac{1}{n+1} \to 0$ as $n \to \infty$.

$$\therefore \qquad M_n = \max_{x\in[0,1]} |f_n(x) - f(x)| = \left(1-\frac{1}{n+1}\right)^{n+1} \to \frac{1}{e} \text{ as } n \to \infty.$$

Since M_n does not tend to 0 as $n \to \infty$, the sequence $<f_n>$ is not uniformly convergent on [0, 1].

Here 0 is a point of non-uniform convergence since $x \to 0$ as $n \to \infty$.

Example 8. *Show that the sequence* $<x^{n-1}(1-x)>$ *is uniformly convergent on [0, 1].*

Sol. Here $\qquad f_n(x) = x^{n-1}(1-x)$

For $0 < x < 1$, $\quad f(x) = \lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} x^{n-1}(1-x) = 0$

Also when $x = 0$, $\quad f_n(x) = 0 \quad \forall\, n$; when $x = 1$, $f_n(x) = 0\ \forall\, n$

$\therefore \qquad f(x) = 0 \quad \forall\, x \in [0, 1]$

$$|f_n(x) - f(x)| = |x^{n-1}(1-x) - 0| = x^{n-1}(1-x)$$

Let $\qquad y = x^{n-1}(1-x)$

then $\qquad \dfrac{dy}{dx} = (n-1)x^{n-2}(1-x) - x^{n-1}$

$$= x^{n-2}[(n-1)(1-x) - x] = x^{n-2}[(n-1) - nx]$$

For max. or min. $\quad \dfrac{dy}{dx} = 0 \Rightarrow \quad x = \dfrac{n-1}{n}$

Also $\qquad \dfrac{d^2y}{dx^2} = (n-2)x^{n-3}[(n-1) - nx] - nx^{n-2}$

$$\left.\frac{d^2y}{dx^2}\right]_{x=\frac{n-1}{n}} = -n\left(\frac{n-1}{n}\right)^{n-2} < 0$$

$\Rightarrow \quad y$ is maximum when $x = \dfrac{n-1}{n}$ and the maximum value of y is

$$\left(\frac{n-1}{n}\right)^{n-1}\left(1-\frac{n-1}{n}\right) = \frac{1}{n}\left(1-\frac{1}{n}\right)^{n-1}$$

$$\therefore \qquad M_n = \max_{x\in[0,1]} |f_n(x) - f(x)| = \frac{1}{n}\left(1-\frac{1}{n}\right)^{n-1}$$

$$= \frac{1}{n}\left(1-\frac{1}{n}\right)^n\left(1-\frac{1}{n}\right)^{-1} \to 0 \times \frac{1}{e} \times 1 = 0 \text{ as } n \to \infty$$

Hence the sequence $<f_n>$ is uniformly convergent on [0, 1].

Example 9. *Show that the sequence* $<f_n>$, *where* $f_n(x) = nx\, e^{-nx^2}$, $x \ge 0$ *is not uniformly convergent on* $[0, k]$, $k > 0$.

Sol. Here $f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} nxe^{-nx^2} = \lim_{n\to\infty} \frac{nx}{e^{nx^2}}$ $\left| \text{Form } \frac{\infty}{\infty} \right.$

$$= \lim_{n\to\infty} \frac{x}{x^2 e^{nx^2}} = 0 \ \forall \ x \in [0, k]$$

$$| f_n(x) - f(x) | = | nxe^{-nx^2} - 0 | = nxe^{-nx^2}$$

Let $y = nxe^{-nx^2}$

then $\frac{dy}{dx} = ne^{-nx^2} + nx \cdot e^{-nx^2} \cdot (-2nx) = n e^{-nx^2} (1 - 2nx^2)$

For max. or min. $\frac{dy}{dx} = 0 \Rightarrow x = \frac{1}{\sqrt{2n}}$

Also $\frac{d^2y}{dx^2} = ne^{-nx^2}(-2nx)(1-2nx^2) + n e^{-nx^2}(-4nx)$

$$= -2n^2xe^{-nx^2} [(1-2nx^2) + 2]$$

$$\left.\frac{d^2y}{dx^2}\right]_{x=\frac{1}{\sqrt{2n}}} = -\frac{2n^2}{\sqrt{2n}} e^{-1/2} \cdot 2 < 0$$

$\Rightarrow$ y is maximum when $x = \frac{1}{\sqrt{2n}}$ and the maximum value of y is $n \cdot \frac{1}{\sqrt{2n}} \cdot e^{-1/2} = \sqrt{\frac{n}{2e}}$.

Also $x = \frac{1}{\sqrt{2n}} \to 0$ as $n \to \infty$

$\therefore$ $M_n = \max_{x \in [0,1]} | f_n(x) - f(x) | = \sqrt{\frac{n}{2e}} \to \infty$ as $n \to \infty$

Since M_n does not tend to zero as $n \to \infty$, the sequence $< f_n >$ is not uniformly convergent on $[0, k]$, $k > 0$. Here 0 is a point of non-uniform convergence.

Example 10. *Show that the sequence $< f_n >$, where $f_n(x) = \frac{\sin nx}{\sqrt{n}}$ is uniformly convergent on $[0, \pi]$.*

Sol. Here $f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{\sin nx}{\sqrt{n}} = \lim_{n\to\infty} \frac{1}{\sqrt{n}} \cdot \sin nx = 0 \ \forall \ x \in [0, \pi]$

$$| f_n(x) - f(x) | = \left| \frac{\sin nx}{\sqrt{n}} - 0 \right| = \left| \frac{\sin nx}{\sqrt{n}} \right|$$

Let $y = \frac{\sin nx}{\sqrt{n}}$ then $\frac{dy}{dx} = \sqrt{n} \cos nx$

For max. or min., $\frac{dy}{dx} = 0 \Rightarrow nx = \frac{\pi}{2}$ or $x = \frac{\pi}{2n}$

Also $\frac{d^2y}{dx^2} = -n^{3/2} \sin nx$

$$\left.\frac{d^2y}{dx^2}\right]_{x=\frac{\pi}{2n}} = -n^{3/2} \sin \frac{\pi}{2} = -n^{3/2} < 0$$

$\Rightarrow$ y is maximum when $x = \frac{\pi}{2n}$ and the maximum value of y is $\frac{\sin \frac{\pi}{2}}{\sqrt{n}} = \frac{1}{\sqrt{n}}$.

Also $x = \frac{\pi}{2n} \to 0$ as $n \to \infty$.

$\therefore$ $M_n = \max_{x \in [0, \pi]} | f_n(x) - f(x) | = \frac{1}{\sqrt{n}} \to 0$ as $n \to \infty$.

Hence the sequence $< f_n >$ converges uniformly to 0 on $[0, \pi]$.

Example 11. *Show that the sequence $< f_n >$, where $f_n(x) = \frac{x}{(n + x^2)^2}$ is uniformly convergent for all $x \geq 0$.*

Sol. *Please try yourself.*

$\left[\textbf{Hint.} \text{ Here } f(x) = 0 \text{ and } M_n = \frac{3\sqrt{3}}{16n^{3/2}} \to 0 \text{ as } n \to \infty\right]$.

Example 12. *Show that the sequence $< f_n >$, where $f_n(x) = \frac{x}{n(1 + nx^2)}$ is uniformly convergent for all x.*

Sol. *Please try yourself.*

$\left[\textbf{Hint.} \text{ Here } f(x) = 0 \text{ and } M_n = \frac{1}{2n^{3/2}} \to 0 \text{ as } n \to \infty\right]$.

10.9. SERIES OF REAL-VALUED FUNCTIONS

Def. *If $< f_n >$ is a sequence of real-valued functions on an interval I, then $f_1 + f_2 + \ldots\ldots + f_n + \ldots$ is called a series of real-valued functions defined on I.*

This series is denoted by $\sum_{n=1}^{\infty} f_n$ or simply by Σf_n.

For example :

(*i*) If $f_n : [0, \infty) \to R$ is defined by $f_n(x) = \frac{-1}{n + x}$, then the series is

$$\Sigma f_n = f_1 + f_2 + f_3 + \ldots = \frac{1}{1+x} + \frac{1}{2+x} + \frac{1}{3+x} + \ldots\ldots$$

(*ii*) If $f_n : R \to R$ is defined by $f_n(x) = \frac{\sin nx}{\sqrt{n}}$, then the series is

$$\Sigma f_n = f_1 + f_2 + f_3 + \ldots\ldots = \sin x + \frac{\sin 2x}{\sqrt{2}} + \frac{\sin 3x}{\sqrt{3}} + \ldots\ldots$$

10.10. CONVERGENCE (OR POINTWISE CONVERGENCE) OF A SERIES OF FUNCTIONS

Let Σf_n be a series of a functions defined on an interval I.

Let $S_1 = f_1, S_2 = f_1 + f_2, \ldots\ldots$

$S_n = f_1 + f_2 + \ldots\ldots + f_n$

then the sequence $< S_n >$ is a sequence of partial sums of the series Σf_n.

If the sequence $< S_n >$ converges pointwise on I, then the series Σf_n is said to converge pointwise on I. The limit function f of $< S_n >$ is called the pointwise sum or simply the sum of the series Σf_n and we write

$$\sum_{n=1}^{\infty} f_n(x) = f(x) \quad \forall\, x \in I \quad \text{or simply} \quad \Sigma f_n = f.$$

For example, consider the series Σf_n defined by $f_n(x) = x^n, -1 < x < 1$

then $\Sigma f_n(x) = x + x^2 + x^3 + \ldots + x^n + \ldots$ where $-1 < x < 1$

$$S_n(x) = x + x^2 + \ldots + x^n = \frac{x(1-x^n)}{1-x} \to \frac{x}{1-x} \text{ as } n \to \infty$$

[since $-1 < x < 1, x^n \to 0$ as $n \to \infty$]

$\Rightarrow$ The sequence $< S_n >$ of partial sums converges pointwise to $\frac{x}{1-x}$ on $(-1, 1)$.

$\Rightarrow$ The series Σf_n converges pointwise to $f(x) = \frac{x}{1-x}$ on $(-1, 1)$.

$$\Rightarrow \qquad \Sigma f_n(x) = \frac{x}{1-x} \text{ on } (-1, 1).$$

10.11. UNIFORM CONVERGENCE OF SERIES OF FUNCTIONS

Def. Let Σf_n be a series of functions defined on an interval I and $S_n = f_1 + f_2 + \ldots + f_n$. If the sequence $< S_n >$ of partial sums converges uniformly on I, then the series Σf_n is said to be uniformly convergent on I.

Thus, *a series of functions Σf_n converges uniformly to a function f on an interval I if for each $\varepsilon > 0$ and for each $x \in I$, there exists a positive integer m (depending only on ε and not on x) such that*

$$| S_n(x) - f(x) | < \varepsilon \ \forall\, n \geq m.$$

The uniform limit function f of $< S_n >$ is called the sum of the series Σf_n and we write $\Sigma f_n = f \ \forall\, x \in I$.

10.12. THEOREM (Cauchy's Criterion for uniform convergence of a series of functions)

A series of functions Σf_n is uniformly convergent on an interval I if and only if for each $\varepsilon > 0$ and for all $x \in I$, there exists a positive integer m (depending only on ε) such that

$$| f_{n+1}(x) + f_{n+2}(x) + \ldots + f_{n+p}(x) | < \varepsilon \ \forall\, n \geq m, p \in N.$$

Proof. Σf_n is uniformly convergent on I.

$\Leftrightarrow$ The sequence $< S_n >$ of its partial sums is uniformly convergent on I.

$\Leftrightarrow$ By Cauchy's general principle of convergence of a sequence, for each $\varepsilon > 0$ and for all $x \in I$, there exists a positive integer m (depending only on ε and not on x) such that

$$| S_{n+p}(x) - S_n(x) | < \varepsilon \ \forall\, n \geq m, p \in N$$

i.e., $$| f_{n+1}(x) + f_{n+2}(x) + \ldots + f_{n+p}(x) | < \varepsilon \ \forall\, n \geq m, p \in N.$$

Note 1. By definition, uniform convergence of a series implies pointwise convergence.

Note 2. The method of testing the uniform convergence of a series Σf_n, by definition, involves finding S_n which is not always easy. The following test avoids S_n.

10.13. THEOREM (Weierstrass's M-Test)

A series of functions $\sum_{n=1}^{\infty} f_n$ converges uniformly (and absolutely) on an interval I if there exists a convergent series $\sum_{n=1}^{\infty} M_n$ of non-negative terms (i.e., $M_n \geq 0 \ \forall\, n \in N$) such that

$$| f_n(x) | \leq M_n \quad \forall\, n \in N \ \text{and} \ \forall\, x \in I.$$

Proof. Since $\sum_{n=1}^{\infty} M_n$ is convergent, by Cauchy's criterion, for each $\varepsilon > 0$, there exists a positive integer m such that

$$| M_{n+1} + M_{n+2} + \ldots + M_{n+p} | < \varepsilon \ \forall\, n \geq m, p \in N \qquad \ldots(1)$$

Now, for all $x \in I$, $| f_n(x) | \leq M_n$ ...(2)

$$\therefore \quad | f_{n+1}(x) + f_{n+2}(x) + \ldots + f_{n+p}(x) | \leq | f_{n+1}(x) | + | f_{n+2}(x) | + \ldots + | f_{n+p}(x) |$$
$$\leq M_{n+1} + M_{n+2} + \ldots + M_{n+p} \qquad \text{[by (2)]}$$
$$< \varepsilon \ \forall\, n \geq m, p \in N \qquad \text{[by (1)]}$$

$\Rightarrow$ By Cauchy's criterion, the series $\sum_{n=1}^{\infty} f_n$ is uniformly convergent on I.

Also, $$| f_{n+1}(x) | + | f_{n+2}(x) | + \ldots + | f_{n+p}(x) | < \varepsilon \ \forall\, n \geq m, p \in N \text{ and } x \in I$$

$\Rightarrow$ The series $\sum_{n=1}^{\infty} | f_n |$ is uniformly convergent on I.

Hence the series $\sum_{n=1}^{\infty} f_n$ converges uniformly and absolutely on I.

ILLUSTRATIVE EXAMPLES—C

Example 1. *Show that the series $\sum_{n=1}^{\infty} \frac{x}{n(n+1)}$ is uniformly convergent in $(0, b)$ $b > 0$ but is not so in $(0, \infty)$.*

Sol. The given series is $\sum_{n=1}^{\infty} \frac{x}{n(n+1)} = \sum_{n=1}^{\infty} f_n(x)$

so that $$f_n(x) = \frac{x}{n(n+1)} = x\left[\frac{1}{n} - \frac{1}{n+1}\right]$$

$$\Rightarrow \quad f_1(x) = x\left[1 - \tfrac{1}{2}\right]$$
$$f_2(x) = x\left[\tfrac{1}{2} - \tfrac{1}{3}\right]$$
$$f_3(x) = x\left[\tfrac{1}{3} - \tfrac{1}{4}\right]$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$f_n(x) = x\left[\frac{1}{n} - \frac{1}{n+1}\right]$$

$$\therefore \quad S_n(x) = f_1(x) + f_2(x) + \ldots\ldots + f_n(x) = x\left[1 - \frac{1}{n+1}\right] = \frac{nx}{n+1}$$

$$S(x) = \lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} \frac{nx}{n+1} = \begin{cases} 0, & \text{if } x = 0 \\ x, & \text{if } x > 0 \end{cases}$$

$\therefore$ For $x > 0$ and for a given $\varepsilon > 0$, we have

$$| S_n(x) - S(x) | = \left| \frac{nx}{n+1} - x \right| = \left| \frac{-x}{n+1} \right| = \frac{x}{n+1} < \varepsilon$$

if $\quad n + 1 > \dfrac{x}{\varepsilon} \quad$ or $\quad$ if $n > \dfrac{x}{\varepsilon} - 1$

If we choose a positive integer m just $\geq \dfrac{x}{\varepsilon} - 1$, then $| S_n(x) - S(x) | < \varepsilon \; \forall \; n \geq m$ and $x > 0$

Also if $x = 0$, $| S_n(x) - S(x) | = 0 < \varepsilon \quad \forall \; n \geq 1$ so that $m = 1$ works in this case.

But when $x \to \infty$, $n \to \infty$.

This shows that the same value of m cannot be found which serves uniformly for every x in $(0, \infty)$.

But if the interval is $(0, b)$ where b is any positive number, then the maximum value of $\dfrac{x}{\varepsilon} - 1$ is $\dfrac{b}{\varepsilon} - 1$ on $(0, b)$.

$\therefore$ If we choose a positive integer m just $\geq \dfrac{b}{\varepsilon} - 1$, then the same value of m serves equally for every value of x in $(0, b)$, $b > 0$.

Thus, the sequence $< S_n >$ converges uniformly in $(0, b)$ but not in $(0, \infty)$.

Hence the series $\displaystyle\sum_{n=1}^{\infty} \frac{x}{n(n+1)}$ is uniformly convergent in $(0, b)$, $b > 0$ but not so in $(0, \infty)$.

Example 2. *Show that $x = 0$ is a point of non-uniform convergence of the series*

$$x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \ldots\ldots$$

Sol. Here $\quad S_n(x) = x^2 + \dfrac{x^2}{1+x^2} + \dfrac{x^2}{(1+x^2)^2} + \ldots\ldots + \dfrac{x^2}{(1+x^2)^{n-1}} \qquad$ (which is a G.P.)

$$= \frac{x^2\left[1 - \frac{1}{(1+x^2)^n}\right]}{1 - \frac{1}{1+x^2}} = (1+x^2)\left[1 - \frac{1}{(1+x^2)^n}\right] = (1+x^2) - \frac{1}{(1+x^2)^{n-1}}$$

$$\therefore \quad S(x) = \lim_{n\to\infty} S_n(x) = \begin{cases} 1+x^2, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Now for $x \neq 0$ and for a given $\varepsilon > 0$, we have

$$| S_n(x) - S(x) | = \left| (1+x^2) - \frac{1}{(1+x^2)^{n-1}} - (1+x^2) \right| = \frac{1}{(1+x^2)^{n-1}} < \varepsilon$$

if $\quad (1+x^2)^{n-1} > \dfrac{1}{\varepsilon} \quad$ or if $\quad n - 1 > \dfrac{\log \frac{1}{\varepsilon}}{\log(1+x^2)} \quad$ or if $\quad n > 1 + \dfrac{\log \frac{1}{\varepsilon}}{\log(1+x^2)}$

This shows that if $x \to 0$, then $n \to \infty$ so that $x = 0$ is a point of non-uniform convergence of $< S_n >$ and hence of the given series.

Note. However, if we consider the interval $[a, \infty)$, $a > 0$, then the maximum value of

$$1 + \frac{\log \frac{1}{\varepsilon}}{\log(1+x^2)} \text{ is } 1 + \frac{\log \frac{1}{\varepsilon}}{\log(1+a^2)}.$$

If we choose a positive integer m just $\geq 1 + \dfrac{\log \frac{1}{\varepsilon}}{\log(1+a^2)}$, then $| S_n(x) - S(x) | < \varepsilon \; \forall \; n \geq m$ and $\forall \; x \in [a, \infty)$.

Thus the series is uniformly convergent in $[a, \infty)$, $a > 0$ and non-uniformly convergent in $[0, \infty)$.

Example 3. *Show that the series* $\dfrac{x}{x+1} + \dfrac{x}{(x+1)(2x+1)} + \dfrac{x}{(2x+1)(3x+1)} + \ldots\ldots$ *is uniformly convergent on $[a, \infty)$, $a > 0$. Show that the series is non-uniformly convergent near $x = 0$.*

Sol. The given series is $\displaystyle\sum_{n=1}^{\infty} \frac{x}{[(n-1)x+1](nx+1)} = \sum_{n=1}^{\infty} f_n(x)$

so that
$$f_n(x) = \frac{x}{[(n-1)x+1](nx+1)} = \frac{1}{(n-1)x+1} - \frac{1}{nx+1}$$

$$\Rightarrow \quad f_1(x) = 1 - \frac{1}{x+1}$$

$$f_2(x) = \frac{1}{x+1} - \frac{1}{2x+1}$$

$$f_3(x) = \frac{1}{2x+1} - \frac{1}{3x+1}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$f_n(x) = \frac{1}{(n-1)x+1} - \frac{1}{nx+1}$$

$$\therefore \quad S_n(x) = 1 - \frac{1}{nx+1} = \frac{nx}{nx+1}$$

$$S(x) = \lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} \frac{nx}{nx+1} = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \end{cases}$$

For $x > 0$ and for a given $\varepsilon > 0$, we have

$$| S_n(x) - S(x) | = \left| \frac{nx}{nx+1} - 1 \right| = \left| \frac{-1}{nx+1} \right| = \frac{1}{nx+1} < \varepsilon$$

if $\quad nx + 1 > \dfrac{1}{\varepsilon} \quad$ or $\quad$ if $\quad n > \dfrac{1}{x}\left(\dfrac{1}{\varepsilon} - 1\right)$

This shows that if $x \to 0$, $n \to \infty$ so that it is not possible to choose a positive integer m such that

$$| S_n(x) - S(x) | < \varepsilon \;\; \forall \; n \geq m \quad \text{and} \quad \forall \; x \in (0, \infty).$$

Thus the convergence is non-uniform near $x = 0$.

Since $\frac{1}{x}\left(\frac{1}{\varepsilon}-1\right)$ increases as x decreases, if we consider the interval $[a, \infty)$, $a > 0$, then the maximum value of $\frac{1}{x}\left(\frac{1}{\varepsilon}-1\right)$ is $\frac{1}{a}\left(\frac{1}{\varepsilon}-1\right)$. If we choose a positive integer m just $\geq \frac{1}{a}\left(\frac{1}{\varepsilon}-1\right)$ then

$$| S_n(x) - S(x) | < \varepsilon \quad \forall\ n \geq m \quad \text{and} \quad \forall\ x \in [a, \infty).$$

Hence the series is uniformly convergent on $[a, \infty)$.

Example 4. *Show that the series* $\sum_{n=1}^{\infty}\left(\frac{n}{x+n}-\frac{n-1}{x+n-1}\right)$ *is uniformly convergent on any finite interval.*

Sol. Here
$$f_n(x) = \frac{n}{x+n} - \frac{n-1}{x+n-1}$$

$\therefore$
$$f_1(x) = \frac{1}{x+1} - 0$$
$$f_2(x) = \frac{2}{x+2} - \frac{1}{x+1}$$
$$f_3(x) = \frac{3}{x+3} - \frac{2}{x+2}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$f_n(x) = \frac{n}{x+n} - \frac{n-1}{x+n-1}$$

$\Rightarrow$
$$S_n(x) = \frac{n}{x+n}$$

Now proceed as in Example 4, Illustrative Examples—A.

Example 5. *Show that $x = 0$ is a point of non-uniform convergence of the series*

$$\sum_{n=1}^{\infty}\left[\frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2x^2}\right].$$

Sol. Here
$$f_n(x) = \frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2x^2}$$

$\therefore$
$$f_1(x) = \frac{x}{1+x^2} - 0$$
$$f_2(x) = \frac{2x}{1+2^2x^2} - \frac{x}{1+x^2}$$
$$f_3(x) = \frac{3x}{1+3^2x^2} - \frac{2x}{1+2^2x^2}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$f_n(x) = \frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2x^2}$$

$\Rightarrow$
$$S_n(x) = \frac{nx}{1+n^2x^2}$$

Now proceed as in Example 10, Illustrative Examples—A.

Example 6. *Test the series* $\sum_{n=1}^{\infty} x\left[\frac{n}{1+n^2x^2} - \frac{n+1}{1+(n+1)^2x^2}\right]$ *for uniform convergence in* $[0, 1]$.

Sol. Here
$$f_n(x) = \frac{nx}{1+n^2x^2} - \frac{(n+1)x}{1+(n+1)^2x^2}$$

$\therefore$
$$f_1(x) = \frac{x}{1+x^2} - \frac{2x}{1+2^2x^2}$$
$$f_2(x) = \frac{2x}{1+2^2x^2} - \frac{3x}{1+3^2x^2}$$
$$f_3(x) = \frac{3x}{1+3^2x^2} - \frac{4x}{1+4^2x^2}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$f_n(x) = \frac{nx}{1+n^2x^2} = \frac{(n+1)x}{1+(n+1)^2x^2}$$

$\Rightarrow$
$$S_n(x) = \frac{x^2}{1+x^2} - \frac{(n+1)x}{1+(n+1)^2x^2}$$

$\therefore$
$$S(x) = \lim_{n\to\infty} S_n(x) = \begin{cases} \dfrac{x}{1+x^2}, & \text{if } 0 < x \leq 1 \\ 0, & \text{if } x = 0 \end{cases}$$

For $0 < x \leq 1$ and for a given $\varepsilon > 0$, we have

$$| S_n(x) - S(x) | = \left| \frac{x}{1+x^2} - \frac{(n+1)x}{1+(n+1)^2x^2} - \frac{x}{1+x^2} \right|$$
$$= \left| -\frac{(n+1)x}{1+(n+1)^2x^2} \right| = \frac{(n+1)x}{1+(n+1)^2x^2} < \varepsilon$$

if
$$(n+1)^2x^2\varepsilon - (n+1)x + \varepsilon > 0$$

or if
$$(n+1) > \frac{x+\sqrt{x^2-4x^2\varepsilon^2}}{2x^2\varepsilon} \text{ or if } n > -1 + \frac{1+\sqrt{1-4\varepsilon^2}}{2x\varepsilon}$$

Now if $x \to 0$, then $n \to \infty$ so that it is not possible to choose a positive integer m such that

$$| S_n(x) - S(x) | < \varepsilon \quad \forall\ n \geq m \quad \text{and} \quad x \quad \forall \in [0, 1]$$

So the convergence is non-uniform in $[0, 1]$. Here $x = 0$ is a point of non-uniform convergence.

Example 7. *Test the uniform convergence of the series* $\sum_{n=1}^{\infty}\left[\frac{2n^2x^2}{e^{n^2x^2}} - \frac{2(n-1)^2x^2}{e^{(n-1)^2x^2}}\right]$ *in* $[0, 1]$.

Sol. Here
$$f_n(x) = \frac{2n^2x^2}{e^{n^2x^2}} - \frac{2(n-1)^2x^2}{e^{(n-1)^2x^2}}$$

$\therefore$
$$f_1(x) = \frac{2x^2}{e^{n^2}} - 0$$

$$f_2(x) = \frac{2.2^2x^2}{e^{2^2x^2}} - \frac{2x^2}{e^{x^2}}$$

$$f_3(x) = \frac{2.3^2x^2}{e^{3^2x^2}} - \frac{2.2^2x^2}{e^{2^2x^2}}$$

....

$$f_n(x) = \frac{2n^2x^2}{e^{n^2x^2}} - \frac{2(n-1)^2x^2}{e^{(n-1)^2x^2}}$$

$$\Rightarrow \quad S_n(x) = \frac{2n^2x^2}{e^{n^2x^2}}$$

$$\therefore \quad S(x) = \lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} \frac{2n^2x^2}{e^{n^2x^2}} \qquad \left| \text{Form } \frac{\infty}{\infty} \right.$$

$$= \lim_{n\to\infty} \frac{4nx^2}{2nx^2e^{n^2x^2}} = \lim_{n\to\infty} \frac{2}{e^{n^2x^2}} = 0 \ \forall \ x$$

Now the series $\sum_{n=1}^{\infty} f_n(x)$ will be uniformly convergent in [0, 1] if for a given $\varepsilon > 0$, there is always a positive integer m such that

$$| S_n(x) - S(x) | < \varepsilon \quad \forall \ n \geq m \quad \text{and} \quad \forall \ x \in [0, 1]$$

i.e.,
$$\frac{2n^2x^2}{e^{n^2x^2}} < \varepsilon \quad \forall \ n \geq m \quad \text{and} \quad \forall \ x \in [0, 1] \qquad \text{...(1)}$$

But, in particular, if we take $x = \frac{1}{n}$ which is a point of [0, 1] for all $n \in N$, then the inequality (1) gives $\frac{2}{e} < \varepsilon$ which shows that we take $\varepsilon < \frac{2}{e}$, the above inequality will not hold.

Hence the given series is non-uniformly convergent in [0, 1].

Example 8. *The sum to n terms of a series is* $S_n(x) = \frac{n^2x}{1+n^4x^2}$. *Show that the series is non-uniformly convergent on [0, 1].*

Sol. *Please try yourself.*

[**Hint.** See Example 15, Illustrative Examples—A].

Example 9. *Show that the series* $\frac{x}{1+x^2} + \left(\frac{2^2x}{1+2^3x^2} - \frac{x}{1+x^2}\right) + \left(\frac{3^2x}{1+3^3x^2} - \frac{2^2x}{1+2^3x^2}\right)$ *+ ... does not converge uniformly on [0, 1].*

Sol. Here
$$f_1(x) = \frac{x}{1+x^2}$$

$$f_2(x) = \frac{2^2x}{1+2^3x^2} - \frac{x}{1+x^2}$$

$$f_3(x) = \frac{3^2x}{1+3^3x^2} - \frac{2^2x}{1+2^3x^2}$$

....

$$f_n(x) = \frac{n^2x}{1+n^3x^2} - \frac{(n-1)^2x}{1+(n-1)^3x^2}$$

$$\therefore \quad S_n(x) = \frac{n^2x}{1+n^3x^2}$$

$$S(x) = \lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} \frac{n^2x}{1+n^3x^2} = \lim_{n\to\infty} \frac{\frac{x}{n}}{\frac{1}{n^3}+x^2} = 0 \ \forall \ x \in [0, 1]$$

$$| S_n(x) - S(x) | = \left| \frac{n^2x}{1+n^3x^2} - 0 \right| = \frac{n^2x}{1+n^3x^2}$$

Let
$$y = \frac{n^2x}{1+n^3x^2}$$

then
$$\frac{dy}{dx} = \frac{(1+n^3x^2).n^2 - n^2x.2n^3x}{(1+n^3x^2)^2} = \frac{n^2(1-n^3x^2)}{(1+n^3x^2)^2}.$$

For max. or min. $\frac{dy}{dx} = 0 \Rightarrow 1 - n^3x^2 = 0 \Rightarrow x = \frac{1}{n^{3/2}}$

Also
$$\frac{d^2y}{dx^2} = \frac{n^2[(1+n^3x^2)^2(-2n^3x) - (1-n^3x^2).2(1+n^3x^2).2n^3x]}{(1+n^3x^2)^4}$$

$$= \frac{n^2[-2n^3x(1+n^3x^2) - 4n^3x(1-n^3x^2)]}{(1+n^3x^2)^3} = \frac{-2n^5x(1+n^3x^2) + 2(1-n^3x^2)]}{(1+n^3x^2)^3}$$

$$\left.\frac{d^2y}{dx^2}\right]_{x=\frac{1}{n^{3/2}}} = \frac{-2n^5 \cdot \frac{1}{n^{3/2}}(1+1)}{(1+1)^3} = -\frac{1}{2}n^{7/2} < 0$$

$\Rightarrow$ y is maximum when $x = \frac{1}{n^{3/2}}$ and maximum value of $y = \frac{n^2 \cdot \frac{1}{n^{3/2}}}{1+1} = \frac{1}{2}\sqrt{n}$

$$\therefore \quad M_n = \max_{x\in[0,1]} | S_n(x) - S(x) | = \frac{1}{2}\sqrt{n} \to \infty \text{ as } n \to \infty.$$

Since M_n does not tend to zero as $n \to \infty$, the sequence $< S_n >$ and hence the given series is non-uniformly convergent on [0, 1]. Here 0 is a point of non-uniform convergence.

Example 10. *Show that the series* $\frac{x^2}{1+x} + \left(\frac{2x^2}{1+2x} - \frac{x^2}{1+x}\right) + \left(\frac{3x^2}{1+3x} - \frac{2x^2}{1+2x}\right)$ + ... *converges uniformly on [0, 1].*

Sol. Here
$$f_1(x) = \frac{x^2}{1+x}$$

$$f_2(x) = \frac{2x^2}{1+2x} - \frac{x^2}{1+x}$$

$$f_3(x) = \frac{3x^2}{1+3x} - \frac{2x^2}{1+2x}$$

....

$$f_n(x) = \frac{nx^2}{1+nx} - \frac{(n-1)x^2}{1+(n-1)x}$$

$$\therefore \quad S_n(x) = \frac{nx^2}{1+nx}$$

$$S(x) = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{nx^2}{1+nx} = \begin{cases} x, & \text{if } 0 < x \le 1 \\ 0, & \text{if } x = 0 \end{cases}$$

Let $\varepsilon > 0$ be given, then for $0 < x \le 1$, we have

$$|S_n(x) - S(x)| = \left|\frac{nx^2}{1+nx} - x\right| = \left|\frac{-x}{1+nx}\right| = \frac{x}{1+nx} < \varepsilon$$

if $\quad 1 + nx > \frac{x}{\varepsilon}$ or if $nx > \frac{x}{\varepsilon} - 1$ or if $n > \frac{1}{\varepsilon} - \frac{1}{x}$

If we choose a positive integer m just $\ge \frac{1}{\varepsilon} - \frac{1}{x}$, then

$$|S_n(x) - S(x)| < \varepsilon \quad \forall\, n \ge m \quad \text{and} \quad 0 < x \le 1$$

For $x = 0$, $|S_n(x) - S(x)| = 0 < \varepsilon\ \forall\, n \ge 1$.

Hence the series converges uniformly on $[0, 1]$.

Example 11. *Test for uniform convergence the series* $\sum_{n=0}^{\infty} xe^{-nx}$ *in the closed interval [0, 1].*

Sol. Here $\quad S_n(x) = \sum_{n=0}^{n-1} xe^{-nx} = x + xe^{-x} + xe^{-2x} + \ldots\ldots + xe^{-(n-1)x}$

which is a geometric series

$$= \frac{x(1-e^{-nx})}{1-e^{-x}} = \frac{x\left(1-\frac{1}{e^{nx}}\right)}{1-\frac{1}{e^x}} = \frac{xe^x}{e^x-1}\left(1-\frac{1}{e^{nx}}\right)$$

$$\therefore \quad S(x) = \lim_{n\to\infty} S_n(x) = \begin{cases} \frac{xe^x}{e^x-1}, & \text{if } 0 < x \le 1 \\ 0, & \text{if } x = 0 \end{cases}$$

Now for $0 < x \le 1$ and for a given $\varepsilon > 0$, we have

$$|S_n(x) - S(x)| = \left|\frac{xe^x}{e^x-1}\left(1-\frac{1}{e^{nx}}\right) - \frac{xe^x}{e^x-1}\right| = \left|\frac{-xe^x}{(e^x-1)e^{nx}}\right| = \frac{xe^x}{(e^x-1)e^{nx}} < \varepsilon$$

if $\quad \frac{(e^x-1)e^{nx}}{xe^x} > \frac{1}{\varepsilon}$

or if $\quad \log(e^x - 1) + nx - \log x - x > \log\frac{1}{\varepsilon}$

or if $\quad \log\left\{x + \frac{x^2}{2!} + \ldots\ldots\right\} - \log x + nx - x > \log\frac{1}{\varepsilon}$

or if $\quad \log\left\{1 + \frac{x}{2!} + \ldots\ldots\right\} + nx - x > \log\frac{1}{\varepsilon}$

or if $\quad n > \dfrac{\log\frac{1}{\varepsilon} + x - \log\left\{1+\frac{x}{2!}+\ldots\ldots\right\}}{x}$

This shows that when $x \to 0$, $n \to \infty$ so that it is not possible to choose a positive integer m such that

$$|S_n(x) - S(x)| < \varepsilon \quad \forall\, n \ge m \quad \text{and} \quad \forall\, x \in [0, 1].$$

Hence the series is non-uniformly convergent in any interval containing 0.

Example 12. *Show that $x = 0$ is a point of non-uniformly convergence of the series*

$$\sum_{n=1}^{\infty} \frac{-2x(1+x)^{n-1}}{[1+(1+x)^{n-1}][1+(1+x)^n]}.$$

Sol. Here $\quad f_n(x) = \dfrac{-2x(1+x)^{n-1}}{[1+(1+x)^{n-1}][1+(1+x)^n]} = \dfrac{2}{1+(1+x)^n} - \dfrac{2}{1+(1+x)^{n-1}}$

$$\Rightarrow \quad f_1(x) = \frac{2}{1+(1+x)} - \frac{2}{1+1}$$

$$f_2(x) = \frac{2}{1+(1+x)^2} - \frac{2}{1+(1+x)}$$

$$f_3(x) = \frac{2}{1+(1+x)^3} - \frac{2}{1+(1+x)^2}$$

....

$$f_n(x) = \frac{2}{1+(1+x)^n} - \frac{2}{1+(1+x)^{n-1}}$$

$$\therefore \quad S_n(x) = \frac{2}{1+(1+x)^n} - 1$$

$$S(x) = \lim_{n\to\infty} S_n(x) = \lim_{n\to\infty}\left[\frac{2}{1+(1+x)^n} - 1\right] = \begin{cases} -1 & \text{when } x > 0 \\ 0 & \text{when } x = 0 \\ 1 & \text{when } x < 0 \end{cases}$$

Thus for $x > 0$ and for a given $\varepsilon > 0$, we have

$$|S_n(x) - S(x)| = \left|\frac{2}{1+(1+x)^n} - 1 + 1\right| = \frac{2}{1+(1+x)^n} < \varepsilon$$

if $\quad 1 + (1+x)^n > \frac{2}{\varepsilon}$ or if $n \log(1+x) > \log\left(\frac{2}{\varepsilon} - 1\right)$

of if $\quad n > \dfrac{\log\left(\frac{2}{\varepsilon}-1\right)}{\log(1+x)}$

This shows that if $x \to 0, n \to \infty$ so that $x = 0$ is a point of non-uniform convergence of the series since no value of m can be chosen such that

$$| S_n(x) - S(x) | < \varepsilon \quad \forall\, n \geq m \text{ and for every } x \text{ near } x = 0.$$

Example 13. *Prove that the series* $x^4 + \dfrac{x^4}{1+x^4} + \dfrac{x^4}{(1+x^4)^2} + \ldots\ldots$ *converges in the interval* $[0, k]$, $k > 0$ *but the series is not uniformly convergent in* $[0, k]$.

Sol. Here $S_n(x)$ = Sum to n terms of the series

$$= x^4 + \frac{x^4}{1+x^4} + \frac{x^4}{(1+x^4)^2} + \ldots\ldots + \frac{x^4}{(1+x^4)^{n-1}}$$

$$= \frac{x^4\left[1 - \dfrac{1}{(1+x^4)^n}\right]}{1 - \dfrac{1}{1+x^4}} = (1+x^4)\left[1 - \frac{1}{(1+x^4)^n}\right]$$

$$\therefore \quad S(x) = \lim_{n\to\infty} S_n(x) = \begin{cases} 1+x^4, & \text{when } 0 < x \leq k \\ 0, & \text{when } x = 0 \end{cases}$$

As $S(x)$ exists for all values of x in $[0, k]$, $k > 0$, the series is convergent in this interval.

To text for uniform convergence, we have for $0 < x \leq k$ and for a given $\varepsilon > 0$,

$$| S_n(x) - S(x) | = \left| (1+x^4)\left\{1 - \frac{1}{(1+x^4)^n}\right\} - (1+x^4) \right| = \frac{1+x^4}{(1+x^4)^n} = \frac{1}{(1+x^4)^{n-1}} < \varepsilon$$

if $\quad (1+x^4)^{n-1} > \dfrac{1}{\varepsilon} \quad$ or if $(n-1)\log(1+x^4) > \log\dfrac{1}{\varepsilon}$

or if $\quad n - 1 > \dfrac{\log\frac{1}{\varepsilon}}{\log(1+x^4)} \quad$ or if $n > 1 + \dfrac{\log\frac{1}{\varepsilon}}{\log(1+x^4)}$

This shows that if $x \to 0, n \to \infty$ so that $x = 0$ is a point of non-uniform convergence of the series.

However, the series is uniformly convergent in $[h, k]$ where $0 < h < k$ since in this case. $| S_n(x) - S(x) | < \varepsilon$ for all $n \geq m$ and for every x in $[h, k]$ where m is a positive integer just

$$\geq 1 + \frac{\log\left(\frac{1}{\varepsilon}\right)}{\log(1+h^4)}.$$

Example 14. *Discuss the uniform convergence of the series* $\sum_{n=1}^{\infty} x^n(1-x)$ *on* $[0, 1]$.

Sol. Here $\quad S_n(x) = x(1-x) + x^2(1-x) + x^3(1-x) \ldots\ldots + x^n(1-x)$

$$= \frac{x(1-x)(1-x^n)}{1-x} = x(1-x^n)$$

$$\therefore \quad S(x) = \lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} x(1-x^n) = \begin{cases} 0 & \text{when } x = 0 \text{ or } 1 \\ x & \text{when } 0 < x < 1 \end{cases}$$

Now if $0 < x < 1$, then for a given $\varepsilon > 0$, we have

$$| S_n(x) - S(x) | = | x(1-x^n) - x | = x^{n+1} < \varepsilon$$

if $\quad \left(\dfrac{1}{x}\right)^{n+1} > \dfrac{1}{\varepsilon} \quad$ or if $n + 1 > \dfrac{\log\frac{1}{\varepsilon}}{\log\frac{1}{x}} \quad$ or if $\quad n > \dfrac{\log\frac{1}{\varepsilon}}{\log\frac{1}{x}} - 1$

This shows that if $x \to 1, n \to \infty$. Thus it is not possible to find a positive integer m such that

$$| S_n(x) - S(x) | < \varepsilon \quad \forall\, n \geq m \quad \text{and} \quad \text{for every } x \text{ in } [0, 1].$$

Here $x = 1$ is the point of non-uniform convergence of the series.

However, the series is uniformly convergent in $[0, b]$ where $0 < b < 1$ since in this case, we can choose a positive integer m just $\geq \dfrac{\log\frac{1}{\varepsilon}}{\log\frac{1}{b}} - 1$ such that $| S_n(x) - S(x) | < \varepsilon\ \forall\, n \geq m$ and $\forall\, x$ in $[0, b]$.

Example 15. *Show that the series* $\sum_{n=1}^{\infty} x^{n-1}(1-x)^2$ *converges uniformly to* $1 - x$ *in* $[0, 1]$.

Sol. *Please try yourself.*

Example 16. *Show that the series* $\sum_{n=1}^{\infty} x^{n-1}$ *converges uniformly to* $\dfrac{1}{1-x}$ *in* $[0, b]$, $0 < b < 1$ *but does not converge uniformly on* $[0, 1]$.

Sol. *Please try yourself.*

Example 17. *Show that if* $0 < r < 1$, *then each of the following series is uniformly convergent on R :*

(*i*) $\sum_{n=1}^{\infty} r^n \cos nx \qquad$ (*ii*) $\sum_{n=1}^{\infty} r^n \sin nx$

(*iii*) $\sum_{n=1}^{\infty} r^n \cos n^2x \qquad$ (*iv*) $\sum_{n=1}^{\infty} r^n \sin a^n x.$

Sol. (*i*) Here $\quad f_n(x) = r^n \cos nx$

$$| f_n(x) | = | r^n \cos nx | = r^n \,| \cos nx | \qquad (\because\ r > 0)$$
$$\leq r^n = M_n \ \forall\, x \in R.$$

Since $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} r^n$ is a geometric series with $0 < r < 1$, it is convergent.

Hence by Weierstrass's M-test, the given series converges uniformly on R.

(*ii*) *Please try yourself.*

(*iii*) *Please try yourself.*

(*iv*) *Please try yourself.*

Example 18. *Show that the following series are uniformly convergent for all real x.*

(*i*) $\sum_{n=1}^{\infty} \dfrac{\sin(x^2 + n^2x)}{n(n+2)} \qquad$ (*ii*) $\sum_{n=1}^{\infty} \dfrac{\cos(x^2 + n^2x)}{n(n^2+2)}$

Sol. (*i*) Here $\quad f_n(x) = \dfrac{\sin(x^2 + n^2x)}{n(n+2)}$

$$|f_n(x)| = \left|\frac{\sin(x^2+n^2x)}{n(n+2)}\right| = \frac{|\sin(x^2+n^2x)|}{n(n+2)} \le \frac{1}{n(n+2)} \le \frac{1}{n^2} = M_n \ \forall\, x \in R$$

Since $\sum\limits_{n=1}^{\infty} M_n = \sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$ is convergent, by Weierstrass's M-test, the given series is uniformly convergent for all real x.

(*ii*) *Please try yourself.*

Example 19. *Show that the following series are uniformly and absolutely convergent for all real values of x and p > 1*

(*i*) $\sum\limits_{n=1}^{\infty} \dfrac{\sin nx}{n^p}$ $\qquad$ (*ii*) $\sum\limits_{n=1}^{\infty} \dfrac{\cos nx}{n^p}$.

Sol. (*i*) Here $\quad f_n(x) = \dfrac{\sin nx}{n^p}$

$$|f_n(x)| = \left|\frac{\sin nx}{n^p}\right| \frac{|\sin nx|}{n^p} \le \frac{1}{n^p} = M_n \ \forall\, x \in R$$

Since $\sum\limits_{n=1}^{\infty} M_n = \sum\limits_{n=1}^{\infty} \dfrac{1}{n^p}$ is convergent for $p > 1$, by Weierstrass's M-test, the given series converges uniformly and absolutely for all real values of x.

(*ii*) *Please try yourself.*

Example 20. *Show that the series* $\cos x + \dfrac{\cos 2x}{2^2} + \dfrac{\cos 3x}{3^2} + \ldots\ldots$ *converges uniformly on* $\bar{R}$.

Sol. *Please try yourself.*

$\left[\textbf{Hint. } f_n(x) = \dfrac{\cos nx}{n^2}\right]$

Example 21. *Test for uniform convergence the series*

(*i*) $\sum\limits_{n=1}^{\infty} \dfrac{x}{(n+x^2)^2}$ $\qquad$ (*ii*) $\sum\limits_{n=1}^{\infty} \dfrac{x}{n(1+nx^2)}$.

Sol. (*i*) Here $\quad f_n(x) = \dfrac{x}{(n+x^2)^2}$

$$\Rightarrow \quad \frac{df_n(x)}{dx} = \frac{(n+x^2)^2 . 1 - x . 2(n+x^2) . 2x}{(n+x^2)^4} = \frac{(n+x^2) - 4x^2}{(n+x^2)^3} = \frac{n - 3x^2}{(n+x^2)^3}$$

For max. or min. $\dfrac{df_n(x)}{dx} = 0 \Rightarrow n - 3x^2 = 0 \Rightarrow x = \sqrt{\dfrac{n}{3}}$

Also $\dfrac{d^2f_n(x)}{dx^2} = \dfrac{(n+x^2)^3 . (-6x) - (n-3x^2) . 3(n+x^2)^2 . 2x}{(n+x^2)^6} = \dfrac{-6x[(n+x^2) + (n-3x^2)]}{(n+x^2)^4}$

$$\left.\frac{d^2f_n(x)}{dx^2}\right]_{x=\sqrt{\frac{n}{3}}} = \frac{-6\sqrt{\frac{n}{3}}\left(n+\frac{n}{3}\right)}{\left(n+\frac{n}{3}\right)^4} = -\frac{27\sqrt{3}}{32n^{5/2}} < 0$$

$\Rightarrow \quad f_n(x)$ is maximum at $x = \sqrt{\dfrac{n}{3}}$ and the maximum value of $f_n(x)$ is $\dfrac{\sqrt{\frac{n}{3}}}{\left(n+\frac{n}{3}\right)^2} = \dfrac{3\sqrt{3}}{16n^{3/2}}$

$$\Rightarrow \quad |f_n(x)| \le \frac{3\sqrt{3}}{16n^{3/2}} < \frac{1}{n^{3/2}} = M_n$$

Since $\sum\limits_{n=1}^{\infty} M_n = \sum\limits_{n=1}^{\infty} \dfrac{1}{n^{3/2}}$ is convergent, by Weierstrass's M-test, the given series is uniformly convergent for all values of x.

(*ii*) Here $\quad f_n(x) = \dfrac{x}{n(1+nx^2)}$

$$\Rightarrow \quad \frac{df_n(x)}{dx} = \frac{1}{n} . \frac{(1+nx^2) . 1 - x . 2nx}{(1+nx^2)^2} = \frac{1-nx^2}{n(1+nx^2)^2}$$

For max. or min. $\dfrac{df_n(x)}{dx} = 0 \Rightarrow 1 - nx^2 = 0 \Rightarrow x = \dfrac{1}{\sqrt{n}}$

Also $\quad \dfrac{d^2f_n(x)}{dx^2} = \dfrac{1}{n} . \dfrac{(1+nx^2)^2 . (-2nx) - (1-nx^2) . 2(1+nx^2) . 2nx}{(1+nx^2)^4}$

$$= -\frac{2x[(1+nx^2) + 2(1-nx^2)]}{(1+nx^2)^3}$$

$$\left.\frac{d^2f_n(x)}{dx^2}\right]_{x=\frac{1}{\sqrt{n}}} = -\frac{\frac{2}{\sqrt{n}}[1+1]}{(1+1)^3} = -\frac{1}{2\sqrt{n}} < 0$$

$\Rightarrow \quad f_n(x)$ is maximum at $x = \dfrac{1}{\sqrt{n}}$ and the maximum value of $f_n(x)$ is $\dfrac{\frac{1}{\sqrt{n}}}{n(1+1)} = \dfrac{1}{2n^{3/2}}$.

$$\Rightarrow \quad |f_n(x)| \le \frac{1}{2n^{3/2}} < \frac{1}{n^{3/2}} = M_n$$

Since $\sum\limits_{n=1}^{\infty} M_n = \sum\limits_{n=1}^{\infty} \dfrac{1}{n^{3/2}}$ is convergent, by Weierstrass's M-test, the given series is uniformly convergent for all values of x.

Example 22. *Show that the series* $\sum\limits_{n=1}^{\infty} \dfrac{1}{1+n^2x}$ *converges in* $[1, \infty)$.

Sol. Here $\quad f_n(x) = \dfrac{1}{1+n^2x}$

$$\therefore \quad |f_n(x)| = \left|\frac{1}{1+n^2x}\right| \le \frac{1}{1+n^2} < \frac{1}{n^2} = M_n \ \forall\, x \in [1, \infty).$$

Since $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, by Weierstrass's M-test, the given series is uniformly convergent for all values of $x \in [1, \infty)$.

Example 23. *Show that the series* $\sum_{n=1}^{\infty} \frac{a_n x^{2n}}{1+x^{2n}}$ *is uniformly convergent for all real x if* $\sum_{n=1}^{\infty} a_n$ *is absolutely convergent.*

Sol. Here $\quad f_n(x) = \frac{a_n x^{2n}}{1+x^{2n}}$

Since $\quad \frac{x^{2n}}{1+x^{2n}} < 1 \quad \forall\, x \in \mathbb{R}$

$$\therefore \quad |f_n(x)| = \left|\frac{a_n x^{2n}}{1+x^{2n}}\right| = |a_n| \cdot \frac{x^{2n}}{1+x^{2n}} < |a_n| = M_n \text{ for all } x \in \mathbb{R}$$

Since $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, therefore, $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} |a_n|$ is convergent. Hence by Weierstrass's M-test, the given series is uniformly convergent for all real x.

Example 24. *Show that the series* $\sum_{n=1}^{\infty} \frac{a_n x^n}{1+x^{2n}}$ *is uniformly convergent for all real x if* $\sum_{n=1}^{\infty} a_n$ *is absolutely convergent.*

Sol. Here $\quad f_n(x) = \frac{a_n x^n}{1+x^{2n}}$

Let $\quad y = \frac{x^n}{1+x^{2n}}$

Then
$$\frac{dy}{dx} = \frac{(1+x^{2n}) \cdot nx^{n-1} - x^n \cdot 2n x^{2n-1}}{(1+x^{2n})^2} = \frac{nx^{n-1}(1+x^{2n} - 2x^{2n})}{(1+x^{2n})^2} = \frac{nx^{n-1}(1-x^{2n})}{(1+x^{2n})^2}$$

For max. or min. $\quad \frac{dy}{dx} = 0 \Rightarrow x = 1$

Also
$$\frac{d^2y}{dx^2} = \frac{(1+x^{2n})^2 \cdot [n(n-1)x^{n-2}(1-x^{2n}) + nx^{n-1}(-2nx^{2n-1})] - nx^{n-1}(1-x^{2n}) \cdot 2(1+x^{2n}) \cdot 2nx^{2n-1}}{(1+x^{2n})^4}$$

$$= \frac{(1+x^{2n})[n(n-1)x^{n-2}(1-x^{2n}) - 2n^2x^{3n-2}] - 4n^2x^{3n-2}(1-x^{2n})}{(1+x^{2n})^3}$$

$$\left.\frac{d^2y}{dx^2}\right]_{x=1} = \frac{2[0-2n^2]-0}{(2)^3} = -\frac{n^2}{2} < 0$$

$$\Rightarrow \quad y = \frac{x^n}{1+x^{2n}} \text{ is maximum at } x = 1 \text{ and the maximum value of } y \text{ is } \frac{1}{2}.$$

$$\therefore \quad |f_n(x)| = \left|\frac{a_n x^n}{1+x^{2n}}\right| = \left|\frac{x^n}{1+x^{2n}}\right| |a_n|$$

$$\le \frac{1}{2}|a_n| < |a_n| = M_n \text{ for all real values of } x.$$

Since $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, therefore, $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} |a_n|$ is convergent. Hence by Weierstrass's M-test, the given series is uniformly convergent for all real x.

Example 25. *If the series Σa_n converges absolutely then prove that $\Sigma a_n \cos nx$ and $\Sigma a_n \sin nx$ are uniformly convergent on R.*

Sol. Let us test $\Sigma a_n \cos nx$.

Here $\quad f_n(x) = a_n \cos nx$

$$|f_n(x)| = |a_n \cos nx| = |a_n| \, |\cos nx| \le |a_n| = M_n \text{ for all real values of } x$$

Since Σa_n is absolutely convergent, therefore, $\Sigma M_n = \Sigma |a_n|$ is convergent. Hence by Weierstrass's M-test, the series $\Sigma a_n \cos nx$ is uniformly convergent on R.

Similarly, $\Sigma a_n \sin nx$ is uniformly convergent on R.

Example 26. *Show that the series* $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p} \cdot \frac{x^{2n}}{1+x^{2n}}$ *is absolutely and uniformly convergent for all real x if p > 1.*

Sol. Here $\quad f_n(x) = \frac{(-1)^n}{n^p} \cdot \frac{x^{2n}}{1+x^{2n}}$

Since $\quad \frac{x^{2n}}{1+x^{2n}} < 1 \quad \forall\, x \in \mathbb{R}$

$$\therefore \quad |f_n(x)| = \left|\frac{(-1)^n}{n^p} \cdot \frac{x^{2n}}{1+x^{2n}}\right| < \frac{1}{n^p} = M_n \text{ for all } x \in \mathbb{R}.$$

Since $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$, therefore, by Weierstrass's M-test, the given series is absolutely and uniformly convergent for all real x if $p > 1$.

Example 27. *Show that* $\sum_{n=1}^{\infty} \frac{1}{n^p + n^q x^2}$ *is uniformly convergent for all real x and p > 1.*

Sol. Here $\quad f_n(x) = \frac{1}{n^p + n^q x^2}$

Since $x^2 \ge 0$ for all real x.

$$\therefore \quad n^q x^2 \ge 0 \Rightarrow n^p + n^q x^2 \ge n^p \Rightarrow \frac{1}{n^p + n^q x^2} \le \frac{1}{n^p}$$

$$\therefore \quad |f_n(x)| = \left|\frac{1}{n^p + n^q x^2}\right| \le \frac{1}{n^p} = M_n \text{ for all real } x.$$

Since $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent for $p > 1$, therefore, by Weierstrass's M-test, the given series is uniformly convergent for all real x and $p > 1$.

Example 28. *Show that the series* $\sum_{n=1}^{\infty} \frac{x}{n^p + n^q x^2}$ *is uniformly convergent for all real* x *if* $p + q > 2$.

Sol. Here $\quad f_n(x) = \frac{x}{n^p + n^q x^2}$

$$\Rightarrow \quad \frac{df_n(x)}{dx} = \frac{(n^p + n^q x^2).1 - x.2n^q x}{(n^p + n^q x^2)^2} = \frac{n^p - n^q x^2}{(n^p + n^q x^2)^2}$$

For max. or min. $\frac{df_n(x)}{dx} = 0$

$$\Rightarrow \quad n^p - n^q x^2 = 0 \Rightarrow x^2 = n^{p-q} \Rightarrow x = n^{\frac{p-q}{2}}$$

Also
$$\frac{d^2 f_n(x)}{dx^2} = \frac{(n^p + n^q x^2)^2.(-2n^q x) - (n^p - n^q x^2).2(n^p + n^q x^2).2n^q x}{(n^p + n^q x^2)^4}$$

$$= -\frac{2n^q x[(n^p + n^q x^2) + 2(n^p - n^q x^2)]}{(n^p + n^q x^2)^3}$$

$$\left.\frac{d^2 f_n(x)}{dx^2}\right]_{x = n^{\frac{p-q}{2}}} = -\frac{2n^q . n^{\frac{p-q}{2}} (n^p + n^p)}{(n^p + n^p)^3} = -\frac{1}{2} n^{\frac{q-3p}{2}} < 0$$

$\Rightarrow$ $f_n(x)$ is maximum at $x = n^{\frac{p-q}{2}}$ and the maximum value of $f_n(x)$ is $\frac{n^{\frac{p-q}{2}}}{n^p + n^p} = \frac{1}{2n^{\frac{p+q}{2}}}$

$$\Rightarrow \quad |f_n(x)| = \left|\frac{x}{n^p + n^q x^2}\right| \le \frac{1}{2n^{\frac{p+q}{2}}} < \frac{1}{n^{\frac{p+q}{2}}} = M_n.$$

Since $\sum M_n = \sum \frac{1}{n^{\frac{p+q}{2}}}$ is convergent if $\frac{p+q}{2} > 1$

i.e., if $p + q > 2$, therefore, by Weierstrass's M-test, the given series is convergent for all real x if $p + q > 2$.

Example 29. *Show that the series* $1 + \frac{e^{-2x}}{2^2 - 1} + \frac{e^{-4x}}{4^2 - 1} + \frac{e^{-6x}}{6^2 - 1} + \ldots\ldots$ *is uniformly convergent for* $x \ge 0$.

Sol. Neglecting the first term, we have $f_n(x) = \frac{e^{-2nx}}{(2n)^2 - 1} = \frac{e^{-2nx}}{4n^2 - 1}$

For all $x \ge 0$, we have $\quad e^{2nx} \ge 1 \quad \Rightarrow \quad e^{-2nx} \le 1$

Also $\quad 3n^2 > 1 \ \forall\ n \Rightarrow 4n^2 > n^2 + 1 \Rightarrow 4n^2 - 1 > n^2$

$$\therefore \quad |f_n(x)| = \left|\frac{e^{-2nx}}{4n^2 - 1}\right| < \frac{1}{n^2} = M_n.$$

Since $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, therefore, by Weierstrass's M-test, the given series is uniformly convergent for $x \ge 0$.

Example 30. *Test for uniform convergence the series*

$$\frac{1}{(1+x)^3} + \frac{2}{(2+x)^3} + \frac{3}{(3+x)^3} + \ldots\ldots, x \ge 0.$$

Sol. The given series is $\sum_{n=1}^{\infty} \frac{n}{(n+x)^3}$

Here $\quad f_n(x) = \frac{n}{(n+x)^3}$

$$\forall\ x \ge 0, \quad |f_n(x)| = \left|\frac{n}{(n+x)^3}\right| = \frac{n}{(n+x)^3} < \frac{n}{n^3} = \frac{1}{n^2} = M_n$$

Since $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, therefore, by Weierstrass's M-test, the given series is uniformly convergent for all $x \ge 0$.

Example 31. *Show that the series* $\frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8} + \ldots\ldots$ *is uniformly convergent in* $(-1, 1)$.

Sol. The given series is $\sum_{n=1}^{\infty} \frac{2^n x^{2^n - 1}}{1 + x^{2^n}} = \sum_{n=1}^{\infty} f_n(x)$

$$|f_n(x)| = \left|\frac{2^n . x^{2^n - 1}}{1 + x^{2^n}}\right| \le 2^n . k^{2^n - 1} = M_n \text{ for } |x| \le k < 1 \qquad \ldots(1)$$

Now $\quad M_n = 2^n k^{2^n - 1} \Rightarrow M_{n+1} = 2^{n+1} . k^{2^{n+1} - 1}$

$$\therefore \quad \lim_{n\to\infty} \frac{M_n}{M_{n+1}} = \lim_{n\to\infty} \frac{k^{2^n - 1}}{2 . k^{2^{n+1} - 1}} = \lim_{n\to\infty} \frac{1}{2k^{2^{n+1} - 2^n}} = \lim_{n\to\infty} \frac{1}{2k^{2n}} = \infty, \text{ since } k < 1$$

By ratio test, the series $\sum_{n=1}^{\infty} M_n$ is convergent. Hence, by Weierstrass's M-test, the given series is uniformly convergent in $(-1, 1)$.

Example 32. *Test the series* $\Sigma f_n(x)$ *for uniform convergence where*

$$f_n(x) = \frac{1}{(x^2 + n)(x^2 + n + 1)}.$$

Sol. $\quad |f_n(x)| = \left|\frac{1}{(x^2 + n)(x^2 + n + 1)}\right| < \frac{1}{n^2} = M_n \ \forall\ x \in R$

Since $\Sigma M_n = \sum \frac{1}{n^2}$ is convergent, therefore, by Weierstrass's M-test, the given series is uniformly convergent for all real values of x.

Example 33. *Test for uniform convergence the series*

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots + \frac{x^n}{n!} + \ldots\ldots \text{ in } [-1, 1].$$

Sol. Neglecting 1, we have $f_n(x) = \dfrac{x^n}{n!}$

$$|f_n(x)| = \left|\frac{x^n}{n!}\right| \le \frac{1}{n!} \text{ for } -1 \le x \le 1$$

Since $n! \ge 2^n$ for $n > 3$, therefore, we have $|f_n(x)| \le \dfrac{1}{2^n} = \left(\dfrac{1}{2}\right)^n = M_n$

But we know that $\Sigma M_n = \sum \left(\dfrac{1}{2}\right)^n$ is convergent. Hence by Weierstrass's M-test, the given series is uniformly convergent in $[-1, 1]$.

Example 34. *Prove that if* δ *is any fixed positive number less than unity, the series* $\sum_{n=1}^{\infty} (n+1)x^n$ *converges uniformly in* $(-\delta, \delta)$.

Sol. Here $\quad -\delta < x < \delta$ and $0 < \delta < 1 \Rightarrow \ |x| < \delta < 1$

Also $\quad f_n(x) = (n+1)x^n$

$\therefore \quad |f_n(x)| = |(n+1)x^n| < (n+1)\delta^n = M_n$

$$\lim_{n\to\infty} \frac{M_n}{M_{n+1}} = \lim_{n\to\infty} \frac{(n+1)\delta^n}{(n+2)\delta^{n+1}} = \frac{1}{\delta} > 1$$

$\therefore$ By ratio test, ΣM_n is convergent. Hence the given series is uniformly convergent in $(-\delta, \delta)$.

Example 35. *Show that each of the following series is uniformly convergent for all values of* x

(i) $\sum \dfrac{1}{n^4 + n^2x^2}$ $\qquad$ (ii) $\sum \dfrac{1}{n^2 + n^4x^2}$ $\qquad$ (iii) $\sum \dfrac{1}{n^3 + n^4x^2}$.

Sol. *Please try yourself.*

$\left[\textbf{Hint. } \forall x \in \mathbb{R}, n^4 + n^2x^2 \ge n^4 \text{ so that } |f_n(x)| \le \dfrac{1}{n^4}\right]$

Example 36. *Prove that if k is any fixed positive number less than unity, then each of the following series is uniformly convergent in* $[-k, k]$:

(i) $\sum x^n$ $\qquad$ (ii) $\sum \dfrac{x^n}{n+1}$.

Sol. *Please try yourself.*

Example 37. *Prove that if k is any fixed number greater than unity, then each of the following series is uniformly convergent for all* $x \ge k$.

(i) $\sum \dfrac{1}{x^n}$ $\qquad$ (ii) $\sum \dfrac{1}{1 + x^n}$.

Sol. *Please try yourself.*

10.14. ABEL'S TEST

If (*i*) $\Sigma f_n(x)$ is uniformly convergent on [a, b],

(*ii*) the sequence $< g_n(x) >$ is monotonic decreasing for all $x \in [a, b]$, and

(*iii*) there exists a positive real number k such that $|g_n(x)| < k \ \forall x \in [a, b]$ and $n \in N$

then the series $\Sigma f_n(x)\, g_n(x)$ is uniformly convergent on [a, b].

Proof. $\Sigma f_n(x)$ is uniformly convergent on $[a, b]$

$\Rightarrow$ By Cauchy's criterion, for each $\varepsilon > 0$ and $\forall x \in [a, b]$, there exists a positive integer m (depending only on ε) such that

$$|f_{n+1}(x) + f_{n+2}(x) + \ldots\ldots + f_{n+p}(x)| < \frac{\varepsilon}{k} \quad \forall n \ge m, p \in N$$

$$\Rightarrow \quad \left|\sum_{r=n+1}^{n+p} f_r(x)\right| < \frac{\varepsilon}{k} \quad \forall n \ge m, p \in N$$

Also, the sequence $< g_n(x) >$ is monotonic decreasing on $[a, b]$

and $\quad |g_n(x)| < k \quad \forall x \in x[a, b]$ and $n \in N$

$\therefore$ By Abel's lemma (see Chapter 8), we get

$$\Rightarrow \quad \left|\sum_{r=n+1}^{n+p} f_r(x)\, g_r(x)\right| < \frac{\varepsilon}{k} \cdot k = \varepsilon \quad \forall n \ge m, p \in N \text{ and } x \in [a, b]$$

$$\Rightarrow \quad |f_{n+1}(x)\, g_{n+1}(x) + f_{n+2}(x)\, g_{n+2}(x) + \ldots\ldots + f_{n+p}(x)\, g_{n+p}(x)| < \varepsilon$$
$$\forall n \ge m, p \in N \text{ and } x \in [a, b]$$

Hence by Cauchy's criterion, $\Sigma f_n(x)\, g_n(x)$ is uniformly convergent on $[a, b]$.

10.15. DIRICHLET'S TEST

If (*i*) there exists a positive real number k such that

$$|S_n(x)| = \left|\sum_{r=1}^{\infty} f_r(x)\right| < k \ \forall x \in [a, b], n \in N \text{ and}$$

(*ii*) $< g_n(x) >$ is a positive monotonic decreasing sequence converging uniformly to zero on [a, b] then the series $\Sigma f_n(x)\, g_n(x)$ is uniformly convergent on [a, b].

Proof. Since $|S_n(x)| < k \quad \forall x \in [a, b], n \in N$

$\therefore \quad \forall x \in [a, b], n \ge m_1, p \in N$, we have

$$|S_{n+p}(x) - S_n(x)| \le |S_{n+p}(x)| + |S_n(x)| < k + k = 2k$$

$$\Rightarrow \quad \left|\sum_{r=n+1}^{n+p} f_r(x)\right| < 2k \quad \forall n \ge m_1, p \in N, x \in [a, b]$$

Also, the sequence $< g_n(x) >$ is positive monotonic decreasing on $[a, b]$.

$\therefore$ By Abel's lemma, we get

$$\left|\sum_{r=n+1}^{n+p} f_r(x)\, g_r(x)\right| < 2k g_{n+1}(x) \ \forall n \ge m_1, p \in N \text{ and } x \in [a, b] \quad \ldots(1)$$

Since $<g_n(x)>$ converges uniformly to zero on $[a, b]$.

$\therefore$ given $\varepsilon > 0$, there exists a positive integer m_2 such that

$$|g_n(x)| < \frac{\varepsilon}{2k} \quad \forall\, n \ge m_2 \qquad \text{...(2)}$$

Let $m = \max\{m_1, m_2\}$, then both (1) and (2) hold for $n \ge m$.

From (1) and (2), we have

$$\left|\sum_{r=n+1}^{n+p} f_r(x)\, g_r(x)\right| < 2k \cdot \frac{\varepsilon}{2k} = \varepsilon \quad \forall\, n \ge m,\, p \in \mathbb{N},\, x \in [a, b]$$

Hence $\Sigma f_n(x) g_n(x)$ is uniformly convergent on $[a, b]$.

ILLUSTRATIVE EXAMPLES

Example 1. *Prove that the series* $\sum \frac{(-1)^{n-1}}{n} x^n$ *is uniformly convergent on* $[0, 1]$.

Sol. Let $\qquad f_n(x) = \frac{(-1)^{n-1}}{n}$ and $g_n(x) = x^n$

The series $\Sigma f_n(x)$ is convergent by Leibnitz's test. Since it is independent of x, it is uniformly convergent on $[0, 1]$.

Also, for $0 \le x \le 1$, $x^n > x^{n+1}$ $\forall\, n \in \mathbb{N}$ and $|g_n(x)| = |x^n| = |x|^n \le 1$

$\therefore$ $<g_n(x)>$ is monotonic decreasing and bounded on $[0, 1]$ for all $n \in \mathbb{N}$.

Hence by Abel's test, the series $\sum f_n(x) g_n(x) = \sum \frac{(-1)^{n-1}}{n} x^n$ is uniformly convergent on $[0, 1]$.

Example 2. *If* Σa_n *is convergent, then show that* $\sum \frac{a_n}{n^x}$ *is uniformly convergent on* $[0, 1]$.

Sol. Let $\qquad f_n(x) = a_n$ and $g_n(x) = \frac{1}{n^x}$

The series $\Sigma f_n(x) = \Sigma a_n$ is given to be convergent. Since it is independent of x, it is uniformly convergent on $[0, 1]$.

Since $<n^x>$ increases on $[0, 1]$, $<\frac{1}{n^x}>$ decreases on $[0, 1]$.

Also $\qquad |g_n(x)| = \frac{1}{n^x} \le \frac{1}{n^0} = 1.$

$\therefore$ The sequence $<g_n(x)>$ is monotonic decreasing and bounded on $[0, 1]$ for all $n \in \mathbb{N}$.

Hence by Abel's test, the series $\sum f_n(x) g_n(x) = \sum \frac{a_n}{n^x}$ is uniformly convergent on $[0, 1]$.

Example 3. *Show that the series* $\sum \frac{(-1)^{n-1}}{n + x^2}$ *is uniformly convergent for all values of* x.

Sol. Let $\qquad f_n(x) = (-1)^{n-1}$ and $g_n(x) = \frac{1}{n + x^2}$

Now $\qquad S_n(x) = \sum_{r=1}^{\infty} f_r(x) = 1 - 1 + 1 - 1 + \ldots\ldots + (-1)^{n-1} = \begin{cases} 1, & \text{when } n \text{ is odd} \\ 0, & \text{when } n \text{ is even} \end{cases}$

$\Rightarrow$ $S_n(x)$ is bounded for all n and for all x.

Also $<g_n(x)> = <\frac{1}{n + x^2}>$ is a positive monotonic decreasing sequence converging to 0 for all values of x.

Hence by Dirichlet's test, the series $\sum f_n(x) g_n(x) = \sum \frac{(-1)^{n-1}}{n + x^2}$ is uniformly convergent for all x.

Example 4. *Prove that the series* $\frac{1}{1+x^2} - \frac{1}{2+x^2} + \frac{1}{3+x^2} - \frac{1}{4+x^2} + \ldots\ldots$ *is uniformly convergent in any interval.*

Sol. Please try yourself. (It is the same as Example 3).

Example 5. *Show that the series* $\cos x + \frac{1}{2}\cos 2x + \frac{1}{3}\cos 3x + \ldots\ldots$ *converges uniformly in* $(0, 2\pi)$.

Sol. The given series is $\sum \frac{\cos nx}{n}$.

Let $\qquad f_n(x) = \cos nx$ and $g_n(x) = \frac{1}{n}$

Now $\qquad S_n(x) = \sum_{r=1}^{\infty} f_r(x) = \cos x + \cos 2x + \cos 3x + \ldots\ldots + \cos nx$

$$= \frac{\cos\left[x + \frac{n-1}{2}x\right]\sin\frac{nx}{2}}{\sin\frac{x}{2}} = \frac{\cos\frac{n+1}{2}x \sin\frac{nx}{2}}{\sin\frac{x}{2}}$$

$$\therefore \quad |S_n(x)| = \frac{\left|\cos\frac{n+1}{2}x\right|\left|\sin\frac{nx}{2}\right|}{\left|\sin\frac{x}{2}\right|} \le \frac{1}{\left|\sin\frac{x}{2}\right|} \quad \text{or} \quad |S_n(x)| \le \left|\operatorname{cosec}\frac{x}{2}\right|$$

But $\operatorname{cosec}\frac{x}{2}$ is bounded for all values of x in $(0, 2\pi)$. Let k be the least upper bound of $\operatorname{cosec}\frac{x}{2}$ in $(0, 2\pi)$, then $|S_n(x)| < k$ for all $x \in (0, 2\pi)$.

Also $<g_n(x)> = <\frac{1}{n}>$ is a positive monotonic decreasing sequence converging to 0.

Hence by Dirichlet's test, the series $\sum f_n(x)g_n(x) = \sum \frac{\cos nx}{n}$ converges uniformly in $(0, 2\pi)$.

Example 6. *Test the series* $\sum_{n=1}^{\infty} \frac{\sin nx}{n}$ *for uniform convergence on* $[0, 1]$.

Sol. Let $\quad f_n(x) = \sin nx \quad$ and $\quad g_n(x) = \frac{1}{n}$

Now
$$S_n(x) = \sum_{r=1}^{n} f_r(x) = \sin x + \sin 2x + \sin 3x + \dots + \sin nx$$
$$= \frac{\sin\left[x + \frac{n-1}{2}x\right]\sin\frac{nx}{2}}{\sin\frac{x}{2}} = \frac{\sin\frac{n+1}{2}x\sin\frac{nx}{2}}{\sin\frac{x}{2}}$$

$$\therefore \quad |S_n(x)| = \frac{\left|\sin\frac{n+1}{2}x\right|\left|\sin\frac{nx}{2}\right|}{\left|\sin\frac{x}{2}\right|} \le \frac{1}{\left|\sin\frac{x}{2}\right|}$$

But $\operatorname{cosec}\frac{x}{2}$ is bounded on $(0, 1]$ which is a subset of $\left(0, \frac{\pi}{2}\right)$.

Let k be the least upper bound of $\operatorname{cosec}\frac{x}{2}$ on $(0, 1]$

When $x = 0$, $S_n(x) = 0 + 0 + 0 + \dots + 0 = 0$

$$\therefore \quad |S_n(x)| < k \quad \forall x \in [0, 1] \quad \text{and} \quad n \in \mathbb{N}.$$

Also $<g_n(x)> = <\frac{1}{n}>$ is a positive monotonic decreasing sequence converging to 0.

Hence by Dirichlet's test, the series $\Sigma f_n(x)g_n(x) = \sum \frac{\sin nx}{n}$ converges uniformly on $[0, 1]$.

Example 7. *Prove that the series* $\sum (-1)^n \frac{x^2+n}{n^2}$ *converges uniformly in every bounded interval, but does not converge absolutely for any value of* x.

Sol. Let the bounded interval be $[a, b]$ so that there exists a positive number k such that for all
$$x \in [a, b], \ |x| < k.$$

Let $\quad f_n(x) = (-1)^n \quad$ and $\quad g_n(x) = \frac{x^2+n}{n^2}$

$$S_n(x) = \sum_{r=1}^{\infty} f_r(x) = -1 + 1 - 1 + 1 \dots + (-1)^n = \begin{cases} -1, & \text{when } n \text{ is odd} \\ 0, & \text{when } n \text{ is even} \end{cases}$$

$\Rightarrow$ $S_n(x)$ is bounded for all $x \in [a, b]$ and for all $n \in \mathbb{N}$.

Also $\quad g_n(x) = \frac{x^2+n}{n^2} < \frac{k^2+n}{n^2}$

$\therefore$ $<g_n(x)>$ is a positive monotonic decreasing sequence converging to 0 uniformly for $x \in [a, b]$.

Hence by Dirichlet's test, the series $\Sigma f_n(x)g_n(x)$
$$= \sum (-1)^n \cdot \frac{x^2+n}{n^2} \text{ converges uniformly on } [a, b].$$

Now $\sum \left|(-1)^2 \frac{x^2+n}{n^2}\right| = \sum \frac{x^2+n}{n^2} \simeq \sum \frac{1}{n^2}$

which diverges. Hence the given series is not absolutely ~~convergent for any~~ value of x.

10.16. UNIFORM CONVERGENCE AND CONTINUITY

Theorem 1. *If a sequence of continuous functions* $<f_n>$ *is uniformly convergent to a function* f *on* $[a, b]$, *then* f *is continuous on* $[a, b]$.

Proof. Let $\varepsilon > 0$ be given.

Since $<f_n>$ is uniformly convergent of f on $[a, b]$, there exists a positive integer m such that
$$|f_n(x) - f(x)| < \frac{\varepsilon}{3} \quad \forall n \ge m \quad \text{and} \quad x \in [a, b] \qquad \dots(1)$$

Let c be any point of $[a, b]$, then from (1), in particular, we have
$$|f_n(c) - f(c)| < \frac{\varepsilon}{3} \ \forall n \ge m \qquad \dots(2)$$

Since f_n is continuous on $[a, b]$ for each $n \in \mathbb{N}$.

$\therefore$ f_n is continuous at $c \in [a, b]$.

$\Rightarrow$ there exists $\delta > 0$ such that $|f_n(x) - f_n(c)| < \frac{\varepsilon}{3}$ whenever $|x - c| < \delta \qquad \dots(3)$

Now
$$\begin{aligned} |f(x) - f(c)| &= |f(x) - f_n(x) + f_n(x) - f_n(c) + f_n(c) - f(c)|. \\ &\le |f(x) - f_n(x)| + |f_n(x) - f_n(c)| + |f_n(c) - f(c)| \\ &= |f_n(x) - f(x)| + |f_n(x) - f_n(c)| + |f_n(c) - f(c)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \text{ whenever } |x - c| < \delta \quad \text{[by (1), (2) and (3)]} \end{aligned}$$

$\Rightarrow$ f is continuous at c.

Since $c \in [a, b]$ is arbitrary, f is continuous on $[a, b]$.

Theorem 2. *If a series* $\sum_{n=1}^{\infty} f_n$ *of continuous functions is uniformly convergent to a function* f *on* $[a, b]$ *then the sum function* f *is also continuous on* $[a, b]$.

Proof. Let $S_n(x) = \sum_{r=1}^{\infty} f_r(x)$ for all $n \in N$. Let $\varepsilon > 0$ be given.

Since Σf_n converges uniformly to f on $[a, b]$, there exists a positive integer m such that

$$| S_n(x) - f(x) | < \frac{\varepsilon}{3} \quad \forall\, n \geq m \quad \text{and} \quad x \in [a, b] \qquad ...(1)$$

Let c be any point of $[a, b]$ then from (1), in particular, we have

$$| S_n(c) - f(c) | < \frac{\varepsilon}{3} \quad \forall\, n \geq m \qquad ...(2)$$

Since f_n is continuous on $[a, b]$ $\forall\, n \in N$

$\therefore$ $S_n = f_1 + f_2 + \ldots + f_n$ is continuous on $[a, b]$ for all $n \in N$

$\Rightarrow$ S_n is continuous at $c \in [a, b]$

$\Rightarrow$ there exists $\delta > 0$ such that $| S_n(x) - S_n(c) | < \frac{\varepsilon}{3}$ whenever $| x - c | < \delta$ ...(3)

Now
$$\begin{aligned} | f(x) - f(c) | &= | f(x) - S_n(x) + S_n(x) - S_n(c) + S_n(c) - f(c) | \\ &\leq | f(x) - S_n(x) | + | S_n(x) - S_n(c) | + | S_n(c) - f(c) | \\ &= | S_n(x) - f(x) | + | S_n(x) - S_n(c) | + | S_n(c) - f(c) | \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \text{ whenever } | x - c | < \delta \end{aligned} \qquad \text{[by (1), (2) and (3)]}$$

$\Rightarrow$ f is continuous at c.

Since $c \in [a, b]$ is arbitrary, f is continuous on $[a, b]$.

Remark. Uniform convergence of the sequence $< f_n >$ is only a sufficient but not a necessary condition for the continuity of the limit function f, *i.e.*, if the limit function f is continuous on $[a, b]$, then it is not necessary that the sequence $< f_n >$ is uniformly convergent on $[a, b]$. Theorem 1 shows that if the limit function f is discontinuous then the sequence $< f_n >$ of continuous functions cannot be uniformly convergent on $[a, b]$. Thus the theorem provides a very good negative test for uniform convergence of a sequence. Similarly, if the sum function f is discontinuous then the series Σf_n of continuous functions cannot be uniformly convergent.

ILLUSTRATIVE EXAMPLES

Example 1. *Test for uniform convergence and continuity the sequence $< f_n >$ where $f_n(x) = x^n$, $0 \leq x \leq 1$.*

Sol. The limit function f is given by

$$f(x) = \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} x^n = 0 \text{ for } 0 \leq x < 1$$

When $x = 1$, the sequence $< f_n >$ converges to 1.

$$\therefore \qquad f(x) = \begin{cases} 0, & \text{if } 0 \leq x < 1 \\ 1, & \text{if } x = 1 \end{cases}$$

Clearly is discontinuous at $x = 1$ and hence f is discontinuous on $[0, 1]$.

Also $\quad f_n(x) = x^n$, $0 \leq x \leq 1$ is continuous on $[0, 1]$ $\forall\, n \in N$.

Since $< f_n >$ is a sequence of continuous functions and its limit function f is discontinuous on $[0, 1]$.

$\therefore$ The sequence $< f_n >$ cannot converge uniformly on $[0, 1]$.

Example 2. *Test the uniform convergence and continuity of $< f_n >$ where*

$$f_n(x) = \frac{1}{1 + nx}, \; 0 \leq x \leq 1.$$

Sol. The limit function f is given by $f(x) = \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \frac{1}{1 + nx} = 0$ for $0 < x \leq 1$

When $x = 0$, the sequence $< f_n >$ converges to 1.

$$\therefore \qquad f(x) = \begin{cases} 0, & \text{if } 0 < x \leq 1 \\ 1, & \text{if } x = 0 \end{cases}$$

Clearly, f is discontinuous at $x = 0$ and hence f is discontinuous on $[0, 1]$.

Also $\qquad f_n(x) = \frac{1}{1 + nx}$, $0 \leq x \leq 1$ is continuous on $[0, 1]$ $\forall\, n \in N$.

Since $< f_n >$ is a sequence of continuous functions and its limit function f is discontinuous on $[0, 1]$.

$\therefore$ The sequence $< f_n >$ cannot converge uniformly on $[0, 1]$.

Example 3. *Show that the sequence $< f_n >$ where $f_n(x) = \tan^{-1} nx$ is not uniformly convergent on $[0, 1]$.*

Sol. *Please try yourself.*

$$\left[\textbf{Hint.}\; f(x) = \begin{cases} \frac{\pi}{2}, & \text{if } 0 < x \leq 1 \\ 1, & \text{if } x = 0 \end{cases} \text{is discontinuous at } x = 0. \right]$$

Example 4. *If $f_n(x) = \frac{1}{x + n}$, $x \geq 0$ then show that $< f_n >$ converges uniformly to the continuous function 0.*

Sol. Here $f_n(x)$ is continuous $\forall\, n \in N$ and $x \geq 0$

$$f(x) = \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \frac{1}{x + n} = 0 \quad \forall\, x \geq 0.$$

f being a constant function is continuous for all $x \geq 0$.

But continuous of f is no guarantee for uniform convergence of $< f_n >$.

Now proceeding as in Example 3, Illustrative Examples—A, $< f_n >$ is uniformly convergent for $x \geq 0$.

Hence $< f_n >$ converges uniformly to the continuous function 0.

Example 5. *Examine for uniform convergence and continuity of the limit function of the sequence $< f_n >$, where $f_n(x) = \frac{nx}{1 + n^2 x^2}$, $0 \leq x \leq 1$.*

Sol. Here $f_n(x)$ is continuous $\forall\, n \in N$ and $x \in [0, 1]$

$$f(x) = \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \frac{nx}{1 + n^2 x^2} = \lim_{n \to \infty} \frac{\frac{x}{n}}{\frac{1}{n^2} + x^2} = 0 \quad \forall\, x \in [0, 1]$$

Clearly, f is continuous for all $x \in [0, 1]$.

Now proceeding as in Example 10, Illustrative Examples – A, 0 is a point of non-uniform convergence of $< f_n >$ on $[0, 1]$.

Hence $< f_n >$ is not uniformly convergent on $[0, 1]$.

Example 6. *If $f_n(x) = nx(1-x)^n$, $x \in [0, 1]$ then show that $< f_n >$ is not uniformly convergent on $[0, 1]$ though the limit function is continuous on $[0, 1]$.*

Sol. *Please try yourself.*

[**Hint.** See Example 7. Illustrative Examples—B].

Example 7. *Show that the series $\sum_{n=1}^{\infty} (1-x)\, x^n$ is not uniformly convergent on $[0, 1]$.*

Sol. The terms of the series are continuous functions and converge pointwise to $S(x)$, where

$$S(x) = \begin{cases} 1, & \text{if } 0 \le x < 1 \\ 0, & \text{if } x = 1 \end{cases}$$

Since the sum function $S(x)$ is discontinuous at $x = 0 \in [0, 1]$, the given series is not uniformly convergent on $[0, 1]$.

Example 8. *Show that the series $x^4 + \dfrac{x^4}{1+x^4} + \dfrac{x^4}{(1+x^4)^2} + \ldots$ is not uniformly convergent on $[0, 1]$*

Sol. The terms of the series are continuous functions.

Here $$S_n(x) = x^4 + \frac{x^4}{1+x^4} + \frac{x^4}{(1+x^4)^2} + \ldots + \frac{x^4}{(1+x^4)^{n-1}} \qquad \text{(which is a G.P.)}$$

$$= \frac{x^4\left[1 - \dfrac{1}{(1+x^4)^n}\right]}{1 - \dfrac{1}{1+x^4}} = (1+x^4)\left[1 - \frac{1}{(1+x^4)^n}\right] = (1+x^4) - \frac{1}{(1+x^4)^{n-1}}$$

$$\therefore \quad S(x) = \lim_{n\to\infty} S_n(x) = \begin{cases} 1+x^4, & \text{if } 0 < x \le 1 \\ 0 & \text{if } x = 0 \end{cases}$$

which is discontinuous at $x = 0 \in [0, 1]$.

Hence the given series is not uniformly convergent on $[0, 1]$.

Example 9. *Show that the series $\sum_{n=1}^{\infty} f_n$, where $f_n(x) = \dfrac{nx}{1+n^2x^2} - \dfrac{(n-1)x}{1+(n-1)^2x^2}$ is not uniformly convergent on $[0, 1]$ though the sum function is continuous on $[0, 1]$.*

Sol. Here $S_n(x) = \dfrac{nx}{1+n^2x^2}$ [See Example 5, Illustrative Examples—C]

$$\therefore \quad S(x) = \lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} \frac{nx}{1+n^2x^2} = \lim_{n\to\infty} \frac{\dfrac{x}{n}}{\dfrac{1}{n^2} + x^2} = 0 \;\forall\, x \in [0, 1]$$

Clearly, the sum function $S(x)$ is continuous on $[0, 1]$.

Since $x = 0 \in [0, 1]$ is a point of non-uniform convergence of the sequence of partial sums $< S_n >$, the series is not uniformly convergent on $[0, 1]$.

(See Example 10, Illustrative Examples—A or example 3, Illustrative Examples—B).

10.17. UNIFORM CONVERGENCE AND INTEGRATION

Theorem 1. *If a sequence $< f_n >$ converges uniformly to f on $[a, b]$ and each function f_n is integrable on $[a, b]$, then f is integrable on $[a, b]$ and the sequence $< \int_a^b f_n\, dx >$ converges uniformly to $\int_a^b f\, dx$.*

Proof. Let $\varepsilon > 0$ be given.

Since $< f_n >$ converges uniformly to f on $[a, b]$, there exists a positive integer m such that

$$| f_n(x) - f(x) | < \frac{\varepsilon}{2(b-a)} \quad \forall\, n \ge m \quad \text{and} \quad x \in [a, b], a \ne b$$

$$\Rightarrow \quad f(x) - \frac{\varepsilon}{2(b-a)} < f_n(x) < f(x) + \frac{\varepsilon}{2(b-a)} \quad \forall\, n \ge m, x \in [a, b]$$

$$\Rightarrow \quad \left.\begin{aligned} f(x) &< f_n(x) + \frac{\varepsilon}{2(b-a)} \\ \text{and} \quad f(x) &> f_n(x) - \frac{\varepsilon}{2(b-a)} \end{aligned}\right\} \forall\, n \ge m, x \in [a, b] \qquad \ldots(1)$$

Also f_n is integrable on $[a, b]$ for each $n \in N$

$$\Rightarrow \quad \underline{\int_a^b} f_n(x)\, dx = \overline{\int_a^b} f_n(x)\, dx = \int_a^b f_n(x)\, dx \qquad \ldots(2)$$

Now, from (1), we have

$$\overline{\int_a^b} f(x)\, dx < \overline{\int_a^b}\left(f_n(x) + \frac{\varepsilon}{2(b-a)}\right) dx = \overline{\int_a^b} f_n(x)\, dx + \frac{\varepsilon}{2(b-a)} \cdot (b-a)$$

$$= \int_a^b f(x)\, dx + \frac{\varepsilon}{2} \qquad \text{[using (2)]} \ldots(3)$$

Again, from (1), we have

$$\underline{\int_a^b} f(x)\, dx > \underline{\int_a^b}\left(f_n(x) - \frac{\varepsilon}{2(b-a)}\right) dx$$

$$= \underline{\int_a^b} f_n(x)\, dx - \frac{\varepsilon}{2(b-a)}(b-a) = \int_a^b f_n(x)\, dx - \frac{\varepsilon}{2}$$

$$\Rightarrow \quad -\underline{\int_a^b} f(x)\, dx < -\int_a^b f_n(x)\, dx + \frac{\varepsilon}{2} \qquad \ldots(4)$$

Adding (3) and (4), we get $\quad 0 \le \overline{\int_a^b} f(x)\, dx - \underline{\int_a^b} f(x)\, dx < \varepsilon$

Since ε is arbitrary, $\quad \overline{\int_a^b} f(x)\, dx - \underline{\int_a^b} f(x)\, dx = 0 \Rightarrow \overline{\int_a^b} f(x)\, dx = \underline{\int_a^b} f(x)\, dx$

$\Rightarrow$ f is integrable on $[a, b]$.

Again, from (2), (3) and (4), we have

$$\int_a^b f_n(x)\,dx - \frac{\varepsilon}{2} < \int_a^b f(x)\,dx < \int_a^b f_n(x)\,dx + \frac{\varepsilon}{2}\ \forall\ n \ge m$$

$$\Rightarrow \quad \int_a^b f(x)\,dx - \frac{\varepsilon}{2} < \int_a^b f_n(x)\,dx < \int_a^b f(x)\,dx + \frac{\varepsilon}{2}\ \forall\ n \ge m.$$

$$\Rightarrow \quad \left|\int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx\right| < \frac{\varepsilon}{2}\ \forall\ n \ge m \Rightarrow \lim_{n\to\infty}\int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx.$$

Theorem 2. *If a series of functions $\sum_{n=1}^{\infty} f_n$ converges uniformly to f on $[a, b]$ and each function f_n is integrable on $[a, b]$, then f is integrable on $[a, b]$ and $\sum_{n=1}^{\infty}\int_a^b f_n(x)\,dx$ converges uniformly to $\int_a^b f(x)\,dx$*

i.e., $\sum_{n=1}^{\infty}\int_a^b f(x)\,dx = \int_a^b f(x)\,dx$ *i.e., the series is term by term integrable.*

Proof. Let $<S_n>$ denote the sequence of partial sums of $\sum_{n=1}^{\infty} f_n$.

Since $\sum_{n=1}^{\infty} f_n$ converges uniformly to f on $[a, b]$.

$\therefore$ The sequence $<S_n>$ converges uniformly to f on $[a, b]$.

Also, S_n being the sum of n integrable functions is integrable for each n.

$\therefore$ By Theorem 1, f is integrable on $[a, b]$

and $$\lim_{n\to\infty}\int_a^b S_n(x)\,dx = \int_a^b f(x)\,dx \quad i.e., \quad \sum_{n=1}^{\infty}\int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx.$$

Remark. It should be noted that uniform convergence of the sequence $<f_n>$ (or series $\sum_{n=1}^{\infty} f_n$) is only sufficient but not a necessary condition for the validity of term by term integration.

Note. If $<f_n>$ is a sequence of integrable functions converging to f on $[a, b]$ and if $\lim_{n\to\infty}\int_a^b f_n(x)\,dx \ne \int_a^b f(x)\,dx$, then $<f_n>$ cannot converge uniformly to f.

ILLUSTRATIVE EXAMPLES

Example 1. *Show that the sequence $<f_n>$, where $f_n(x) = nx\,e^{-nx^2}$, $n \in N$ is not uniformly convergent on $[0, 1]$.*

Sol. Here $$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\frac{nx}{e^{nx^2}} = \lim_{n\to\infty}\frac{nx}{1 + \frac{nx^2}{2!} + \frac{n^2x^4}{2!} + \ldots\ldots} = 0 \text{ for } x \in [0, 1].$$

Also $$\int_0^1 f(x)\,dx = 0 \text{ and } \int_0^1 f_n(x)\,dx = \int_0^1 nx\,e^{-nx^2}\,dx = \int_0^n \frac{1}{2}e^{-t}\,dt \text{ where } t = nx^2$$

$$= \frac{1}{2}[1 - e^{-n}]$$

$$\lim_{n\to\infty}\int_0^1 f_n(x)\,dx = \lim_{n\to\infty}\frac{1}{2}(1 - e^{-n}) = \frac{1}{2} \ne \int_0^1 f(x)\,dx$$

$\Rightarrow$ the sequence $<f_n>$ is not uniformly convergent on $[0, 1]$.

In fact, $x = 0$ is a point of non-uniform convergence.

Example 2. *Examine for term by term integration the series $\sum_{n=1}^{\infty} f_n$, where*

$S_n(x) = \sum_{i=1}^{n} f_i = nx\,e^{-nx^2}$, *over the intervals*

(*i*) $[0, 1]$ (*ii*) $[a, 1]$, $0 < a < 1$.

Sol. Here $$f(x) = \lim_{n\to\infty} S_n(x) = \lim_{n\to\infty}\frac{nx}{e^{nx^2}} = \lim_{n\to\infty}\frac{nx}{1 + \frac{nx^2}{1!} + \frac{n^2x^4}{2!} + \ldots\ldots} = 0 \text{ for all } x$$

Consider the interval [0, 1]

$$\int_0^1 f(x)\,dx = \int_0^1 0\,dx = 0$$

and $$\int_0^1 S_n(x)\,dx = \int_0^1 nx\,e^{-nx^2}\,dx = \int_0^n \frac{1}{2}e^{-t}\,dt \text{ where } t = nx^2 = \frac{1}{2}(1 - e^{-n})$$

$$\lim_{n\to\infty}\int_0^1 S_n(x)\,dx = \lim_{n\to\infty}\frac{1}{2}(1 - e^{-n}) = \frac{1}{2}$$

Since $\lim_{n\to\infty}\int_0^1 S_n(x)\,dx \ne \int_0^1 f(x)\,dx$ i.e., $\lim_{n\to\infty}\int_0^1 S_n(x)\,dx \ne \int_0^1\left(\lim_{n\to\infty} S_n(x)\right)dx$

the series $\sum_{n=1}^{\infty} f_n$ does not admit of term by term integration over the interval $[0, 1]$.

Now consider the interval $[a, 1]$, $0 < a < 1$

$$\int_a^1 f(x)\,dx = \int_a^1 0\,dx = 0$$

and $$\int_a^1 S_n(x)\,dx = \int_a^1 nx\,e^{-nx^2}\,dx = \int_{na^2}^{n}\frac{1}{2}e^{-t}\,dt \text{ where } t = nx^2$$

$$= \frac{1}{2}\left(e^{-na^2} - e^{-n}\right)$$

$$\lim_{n \to \infty} \int_a^1 S_n(x)\,dx = \lim_{n \to \infty} \frac{1}{2}\left(e^{-na^2} - e^{-n}\right) = 0 = \int_a^1 f(x)\,dx$$

Hence term by term integration is justified over the interval $[a, 1]$ where $0 < a < 1$.

Example 3. *Prove that* $\int_0^1 \left(\sum_{n=1}^{\infty} \frac{x^n}{n^2}\right) dx = \sum_{n=1}^{\infty} \frac{1}{n^2(n+1)}$.

Sol. Let $\qquad f_n(x) = \dfrac{x^n}{n^2}$

$$| f_n(x) | = \left|\frac{x^n}{n^2}\right| \le \frac{1}{n^2} = M_n \text{ for } 0 \le x \le 1$$

Since $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, therefore, by Weierstrass's M-test, the series

$\sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$ is uniformly convergent for $0 \le x \le 1$.

$\therefore$ The series can be integrated term by term.

$$\Rightarrow \int_0^1 \left(\sum_{n=1}^{\infty} \frac{x^n}{n^2}\right) dx = \sum_{n=1}^{\infty} \int_0^1 \frac{x^n}{n^2}\,dx = \sum_{n=1}^{\infty} \left[\frac{x^{n+1}}{n^2(n+1)}\right]_0^1 = \sum_{n=1}^{\infty} \frac{1}{n^2(n+1)}$$

Example 4. *Show that the series $1 - x + x^2 - x^3 + \ldots$, $0 \le x \le 1$ admits of term by term integration on $[0, 1]$, though it is not uniformly convergent on $[0, 1]$.*

Sol. The given series is $1 - x + x^2 - x^3 + \ldots$

When $x = 1$, the series $1 - 1 + 1 - 1 + \ldots$ oscillates.

For $0 \le x < 1$, $\qquad 1 - x + x^2 - x^3 + \ldots = \dfrac{1}{1-(-x)} = \dfrac{1}{1+x}$

Thus the series is not uniformly convergent on $[0, 1]$, $x = 1$ being the point of non-uniform convergence.

However, integrating term by term over the interval $[0, 1]$ *i.e.*, including 1, we have

$$\int_0^1 1\,dx - \int_0^1 x\,dx + \int_0^1 x^2\,dx - \int_0^1 x^3\,dx + \ldots$$
$$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots = \log 2$$

Also $\qquad \int_0^1 \frac{1}{1+x}\,dx = \left[\log(1+x)\right]_0^1 = \log 2.$

Thus the two sides are equal. Hence term by term integration is possible over $[0, 1]$, even though the given series is not uniformly convergent on $[0, 1]$.

Example 5. *Examine for term by term integration the series the sum of whose first n terms is*

$$n^2x(1-x)^n,\ 0 \le x \le 1.$$

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y multiply n power then term is converge

Sol. Here $\qquad S_n(x) = n^2x(1-x)^n.$

When $x = 0$ or 1, $\quad S_n(x) = 0$

When $0 < x < 1$, $\quad \lim_{n \to \infty} S_n(x) = \lim_{n \to \infty} \dfrac{n^2x}{(1-x)^{-n}} \qquad \left| \text{Form } \dfrac{\infty}{\infty}\right.$

$$= \lim_{n \to \infty} \frac{2nx}{-(1-x)^{-n}\log(1-x)} \qquad \left| \text{Form } \frac{\infty}{\infty}\right.$$

$$= \lim_{n \to \infty} \frac{2x}{(1-x)^{-n}[\log(1-x)]^2} = 0$$

$\therefore \qquad f(x) = \lim_{n \to \infty} S_n(x) = 0$ for all $x \in [0, 1]$

Also $\qquad \int_0^1 f(x)\,dx = \int_0^1 0\,dx = 0$ and $\int_0^1 S_n(x)\,dx = \int_0^1 n^2x(1-x)^n\,dx$

Changing x to $1 - x$ $\qquad = \int_0^1 n^2(1-x)\,x^n\,dx$

$$= \int_0^1 n^2(x^n - x^{n+1})\,dx = n^2\left(\frac{1}{n+1} - \frac{1}{n+2}\right) = \frac{n^2}{(n+1)(n+2)}$$

$$\lim_{n \to \infty} \int_0^1 S_n(x)\,dx = \lim_{n \to \infty} \frac{n^2}{(n+1)(n+2)} = 1$$

Since $\lim_{n \to \infty} \int_0^1 S_n(x)\,dx \ne \int_0^1 \left(\lim_{n \to \infty} S_n(x)\right) dx$, term by term integration is not justified on $[0, 1]$.

Example 6. *Show that the series for which*

(i) $S_n(x) = \dfrac{1}{1+nx}$ $\qquad$ *(ii)* $S_n(x) = nx(1-x)^n$

can be integrated term by term on $[0, 1]$, though they are not uniformly convergent on $[0, 1]$.

Sol. *(i)* Here $\qquad S_n(x) = \dfrac{1}{1+nx}$

so that $\qquad f(x) = \lim_{n \to \infty} S_n(x) = \lim_{n \to \infty} \dfrac{1}{1+nx} = \begin{cases} 0, & \text{if } 0 < x \le 1 \\ 1, & \text{if } x = 0 \end{cases}$

For $0 < x \le 1$ and for a given $\varepsilon > 0$, we have

$$| S_n(x) - f(x) | = \left|\frac{1}{1+nx} - 0\right| = \frac{1}{1+nx} < \varepsilon \text{ if } 1 + nx > \frac{1}{\varepsilon} \quad \text{or if} \quad n > \frac{\frac{1}{\varepsilon} - 1}{x}$$

If $x \to 0$, $n \to \infty$ so that $x = 0$ is a point of non-uniform convergence of the series. Thus the series does not converge uniformly on $[0, 1]$.

Now $\qquad \int_0^1 f(x)\,dx = \int_0^1 0\,dx = 0$

and $\qquad \int_0^1 S_n(x)\,dx = \int_0^1 \frac{dx}{1+nx} = \left[\frac{\log(1+nx)}{n}\right]_0^1 = \frac{\log(1+n)}{n}$

$$\lim_{n\to\infty}\int_0^1 S_n(x)\,dx = \lim_{n\to\infty}\frac{\log(1+n)}{n} \qquad \left| \text{Form } \frac{\infty}{\infty} \right.$$

$$= \lim_{n\to\infty}\frac{\frac{1}{1+n}}{1} = 0$$

Since $\lim_{n\to\infty}\int_0^1 S_n(x)\,dx = \int_0^1 \lim_{n\to\infty}[S_n(x)]\,dx$, the series is integrable term by term on [0, 1] although $x = 0$ is a point of non-uniform convergence of the series.

(*ii*) Here $S_n(x) = nx(1-x)^n$

When $0 < x < 1$, we have $\lim_{n\to\infty} S_n(x) = \lim_{n\to\infty}\frac{nx}{(1-x)^{-n}}$ $\qquad \left| \text{Form } \frac{\infty}{\infty} \right.$

$$= \lim_{n\to\infty}\frac{x}{-(1-x)^{-n}\log(1-x)} = 0$$

Also $S_n(x) = 0$ for $x = 0$ or 1

$\therefore$ $f(x) = 0$ for every x in [0, 1].

Now $\int_0^1 f(x)\,dx = \int_0^1 0\,dx = 0$ and $\int_0^1 S_n(x)\,dx = \int_0^1 nx(1-x)^n\,dx$

Changing x to $1-x$ $\quad = \int_0^1 n(1-x)x^n\,dx = \int_0^1 n(x^n - x^{n+1})\,dx$

$$= n\left[\frac{1}{n+1} - \frac{1}{n+2}\right] = \frac{n}{(n+1)(n+2)}$$

$$\lim_{n\to\infty}\int_0^1 S_n(x)\,dx = \lim_{n\to\infty}\frac{n}{(n+1)(n+2)} = 0$$

Since $\lim_{n\to\infty}\int_0^1 S_n(x)\,dx = \int_0^1 \lim_{n\to\infty}[S_n(x)]\,dx$

the series is integrable term by term on [0, 1] although $x = 0$ is a point of non-uniform convergence of the series.

(See Example 7, Illustrative Examples—B)

Example 7. *Test for uniform convergence and term by term integration the series* $\sum_{n=1}^{\infty}\frac{x}{(n+x^2)^2}$. *Also show that* $\int_0^1\left(\sum_{n=1}^{\infty}\frac{x}{(n+x^2)^2}\right)dx = \frac{1}{2}$.

Sol. The series $\sum_{n=1}^{\infty}\frac{x}{(n+x^2)^2}$ is uniformly convergent.

(See Example 21 (*i*), Illustrative Examples—C)

Hence it is integrable term by term between any finite limits.

$$\Rightarrow \int_0^1\left(\sum_{n=1}^{\infty}\frac{x}{(n+x^2)^2}\right)dx = \lim_{n\to\infty}\int_0^1\sum_{n=1}^{\infty}\frac{x}{(n+x^2)^2}\,dx = \lim_{n\to\infty}\sum_{n=1}^{\infty}\int_0^1 x(n+x^2)^{-2}\,dx$$

$$= \lim_{n\to\infty}\sum_{n=1}^{\infty}\left[\frac{(n+x^2)^{-1}}{-2}\right]_0^1 = \lim_{n\to\infty}\sum_{n=1}^{\infty}\frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= \lim_{n\to\infty}\frac{1}{2}\left[\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\ldots+\left(\frac{1}{n}-\frac{1}{n+1}\right)\right]$$

$$= \lim_{n\to\infty}\frac{1}{2}\left(1-\frac{1}{n+1}\right) = \frac{1}{2}.$$

Example 8. *Show that the series* $\sum_{n=1}^{\infty}\left[\frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2x^2}\right]$ *can be integrated term by term on [0, 1] although it is not uniformly convergent on [0, 1].*

Sol. Proceeding as in Example 5, Illustrative Examples—C, we have

$$S_n(x) = \frac{nx}{1+n^2x^2}$$

$$f(x) = \lim_{n\to\infty} S_n(x) = 0 \;\forall\; x \in [0, 1]$$

$x = 0$ is a point of non-uniform convergence of the series.

Now $\int_0^1 f(x)\,dx = \int_0^1 0\,dx = 0$

and $\int_0^1 S_n(x)\,dx = \int_0^1 \frac{nx}{1+n^2x^2}\,dx = \frac{1}{2n}\int_0^1\frac{2n^2x}{1+n^2x^2}\,dx$

$$= \frac{1}{2n}\left[\log(1+n^2x^2)\right]_0^1 = \frac{1}{2n}\log(1+n^2)$$

$$\lim_{n\to\infty}\int_0^1 S_n(x)\,dx = \lim_{n\to\infty}\frac{\log(1+n^2)}{2n} \qquad \left| \text{Form } \frac{\infty}{\infty} \right.$$

$$= \lim_{n\to\infty}\frac{\frac{2n}{1+n^2}}{2} = \lim_{n\to\infty}\frac{n}{1+n^2} \qquad \left| \text{Form } \frac{\infty}{\infty} \right.$$

$$= \lim_{n\to\infty}\frac{1}{2n} = 0$$

Since $\lim_{n\to\infty}\int_0^1 S_n(x)\,dx = \int_0^1 \lim_{n\to\infty}[S_n(x)]\,dx$, the series is integrable term by term on [0, 1] although $x = 0$ is a point of non-uniform convergence of the series.

10.18. UNIFORM CONVERGENCE AND DIFFERENTIATION

Theorem 1. *If a sequence of functions* $<f_n>$ *is such that*

(*i*) *each* f_n *is differentiable on* [*a*, *b*] (*ii*) *each* f_n' *is continuous on* [*a*, *b*]

(*iii*) $<f_n>$ *converges to f on* [*a*, *b*] (*iv*) $<f_n'>$ *converges uniformly to g on* [*a*, *b*]

than f is differentiable and $f'(x) = g(x)\;\forall\; x \in [a, b]$.

Proof. Since each f_n' is continuous on $[a, b]$ and $<f_n'>$ converges uniformly to g on $[a, b]$, therefore, g is continuous and hence integrable on $[a, b]$.

Also, since $<f_n'>$ converges uniformly to g on $[a, y]$, where $a \le y \le b$.

$$\therefore \quad \lim_{n\to\infty} \int_a^y f_n'(x)\,dx = \int_a^y g(x)\,dx \qquad ...(1)$$

By Fundamental theorem of Integral Calculus, we know that

$$\int_a^y f_n'(x)\,dx = f_n(y) - f_n(a)$$

$$\therefore \text{ From (1)}, \quad \lim_{n\to\infty} [f_n(y) - f_n(a)] = \int_a^y g(x)\,dx \qquad ...(2)$$

Since $<f_n>$ converges to f on $[a, b]$

$$\therefore \quad \lim_{n\to\infty} f_n(y) = f(y) \text{ and } \lim_{n\to\infty} f_n(a) = f(a).$$

$$\therefore \text{ From (2)}, \quad f(y) - f(a) = \int_a^y g(x)\,dx \Rightarrow f'(x) = g(y),\ a \le y \le b.$$

Changing y to x, we have $f'(x) = g(x)\ \forall\, x \in [a, b]$.

Theorem 2. *If a series of functions* $\sum_{n=1}^{\infty} f_n$ *is such that*

(i) each f_n is differentiable on $[a, b]$
(ii) each f_n' is continuous on $[a, b]$
(iii) $\sum_{n=1}^{\infty} f_n$ converges to f on $[a, b]$
(iv) $\sum_{n=1}^{\infty} f_n'$ converges uniformly to g on $[a, b]$ then f is differentiable on $[a, b]$ for $f'(x) = g(x)$ $\forall\, x \in [a, b]$.

Proof. Let $S_n = f_1 + f_2 + ... + f_n$ so that $<S_n>$ is the sequence of partial sums of the series $\sum_{n=1}^{\infty} f_n$.

Since $\sum_{n=1}^{\infty} f_n$ converges to f on $[a, b]$, the sequence $<S_n>$ converges to f on $[a, b]$.

Also $S_n' = f_1' + f_2' + ... + f_n'$ the sequence $<S_n'>$ of the partial sums of the series $\sum_{n=1}^{\infty} f_n'$ converges uniformly to g on $[a, b]$, where $g = \sum_{n=1}^{\infty} f_n'$.

$\therefore$ By Theorem 1, $f'(x) = g(x)\ \forall\, x \in [a, b]$.

Note 1. In the above theorems the derived sequence (series) must be uniformly convergent and the original sequence (series) need only be convergent.

Note 2. The conclusion of Theorem 2 can be written as $f' = \sum_{n=1}^{\infty} f_n' = f_1' + f_2' + ... + f_n'$ so that the series is differentiable term by term.

ILLUSTRATIVE EXAMPLES

Example 1. *Show that the sequence $<f_n>$ where $f_n(x) = \dfrac{nx}{1 + n^2x^2}$, $0 \le x \le 1$ cannot be differentiated term by term at $x = 0$.*

Sol. Here $\quad f(x) = \lim_{n\to\infty} f_n(x) = 0\ \forall\, x \in [0, 1]$.

$\therefore \quad f'(0) = 0$

Also
$$f_n'(0) = \lim_{h\to\infty} \frac{f_n(0+h) - f_n(0)}{h} = \lim_{h\to 0} \frac{\frac{nh}{1+n^2h^2} - 0}{h}$$
$$= \lim_{h\to\infty} \frac{n}{1+n^2h^2} = n$$

$\therefore \quad f_n'(0) \to \infty$ as $n \to \infty$

$\Rightarrow \quad f'(0) \ne \lim_{n\to\infty} f_n'(0)$

Hence $<f_n>$ cannot be differentiated term by term at $x = 0$.

Example 2. *Show that for the sequence $<f_n>$ where $f_n(x) = \dfrac{x}{1+nx^2}$ the formula $\lim_{n\to\infty} f_n'(x) = f'(x)$ is true if $x \ne 0$ and false if $x = 0$. Why so?*

Sol. The sequence $<f_n>$ converges uniformly to zero for all real x. [See Example 1, Illustrative Examples—B]

$\Rightarrow \quad f(x) = 0\ \forall\, x \in \mathbb{R} \quad \Rightarrow \quad f'(x) = 0\ \forall\, x \in \mathbb{R}$

When $x \ne 0$,
$$f_n'(x) = \frac{(1+nx^2).1 - x.2nx}{(1+nx^2)^2} = \frac{1-nx^2}{(1+nx^2)^2}$$

$$\lim_{n\to\infty} f_n'(x) = \lim_{n\to\infty} \frac{1-nx^2}{(1+nx^2)^2} \qquad \left|\ \text{Form } \frac{\infty}{\infty}\right.$$
$$= \lim_{n\to\infty} \frac{-x^2}{2(1+nx^2).x^2} = 0 = f'(x)$$

so that if $x \ne 0$, the formula $\lim_{n\to\infty} f_n'(x) = f'(x)$ is true.

At $x = 0$

$$f_n'(0) = \lim_{h\to 0} \frac{f_n(0+h) - f_n(0)}{h} = \lim_{h\to 0} \frac{\frac{h}{1+nh^2} - 0}{h} = \lim_{h\to 0} \frac{1}{1+nh^2} = 1.$$

so that $\lim_{n \to \infty} f_n'(0) = 1 \neq f'(0)$.

Hence at $x = 0$, the formula $= \lim_{n \to \infty} f_n'(x) = f'(x)$ is false.

It is so because the sequence $<f_n'>$ is not uniformly convergent in any interval containing zero.

Example 3. *Show that the function represented by* $\sum_{n=1}^{\infty} \frac{\sin nx}{n^3}$ *is differentiable for every x and its derivative is* $\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$.

Sol. Here $$f_n(x) = \frac{\sin nx}{n^3}$$

$$\therefore \quad f_n'(x) = \frac{\cos nx}{n^2} \Rightarrow \sum_{n=1}^{\infty} f_n'(x) = \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$$

Since $\left|\frac{\cos nx}{n^2}\right| \leq \frac{1}{n^2}$ $\forall$ x and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, therefore, by Weierstrass's M-test, the series $\sum_{n=1}^{\infty} f_n'$ is uniformly convergent for all x and hence $\sum_{n=1}^{\infty} f_n$ can be differentiated term by term.

$$\therefore \quad \left(\sum_{n=1}^{\infty} f_n\right)' = \sum_{n=1}^{\infty} f_n'$$

$$\Rightarrow \quad \left(\sum_{n=1}^{\infty} \frac{\sin nx}{n^3}\right)' = \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$$

Example 4. *Show that the differential co-efficient of*

$$\sum_{n=1}^{\infty} \frac{1}{n^3 + n^4x^2} \text{ is } -2x \sum_{n=1}^{\infty} \frac{1}{n^2(1+nx^2)^2} \text{ for all real } x.$$

Sol. Here $$f_n(x) = \frac{1}{n^3 + n^4x^2} = \frac{1}{n^3(1+nx^2)}$$

$$\Rightarrow \quad f_n'(x) = \frac{1}{n^3}\left[\frac{-2nx}{(1+nx^2)^2}\right] = -\frac{2x}{n^2(1+nx^2)^2}$$

$$\Rightarrow \quad \sum_{n=1}^{\infty} f_n'(x) = -2x \sum_{n=1}^{\infty} \frac{1}{n^2(1+nx^2)^2}$$

Now $f_n'(x)$ is maximum when $\frac{d}{dx} f_n'(x) = 0$

i.e., when $$-\frac{2}{n^2} \cdot \frac{(1+nx^2)^2 \cdot 1 - x\,2(1+nx^2) \cdot 2nx}{(1+nx^2)^4} = 0$$

or when $1 - 3nx^2 = 0$ or when $x = \frac{1}{\sqrt{3n}}$

$$\therefore \quad \text{Maximum value of } |f_n'(x)| = \frac{2 \cdot \frac{1}{\sqrt{3n}}}{n^2(1+\frac{1}{3})^2} = \frac{3\sqrt{3}}{8n^{5/2}} \Rightarrow |f_n'(x)| \leq \frac{3\sqrt{3}}{8n^{5/2}} < \frac{1}{n^{5/2}} \ \forall\, x.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^{5/2}}$ is convergent, therefore, by Weierstrass's M-test, the series $\sum_{n=1}^{\infty} f_n'$ is uniformly convergent for all real x and hence $\sum_{n=1}^{\infty} f_n$ can be differentiated term by term.

$$\therefore \quad \left(\sum_{n=1}^{\infty} f_n\right)' = \sum_{n=1}^{\infty} f_n' \Rightarrow \left(\sum_{n=1}^{\infty} \frac{1}{n^3 + n^4x^2}\right)' = -2x \sum_{n=1}^{\infty} \frac{1}{n^2(1+nx^2)^2}.$$

Example 5. *Show that the series for which* $S_n(x) = \frac{nx}{1+n^2x^2}$, $0 \leq x \leq 1$ *cannot be differentiated term by term at* $x = 0$.

Sol. Here $$f(x) = \lim_{n \to \infty} S_n(x) = \lim_{n \to \infty} \frac{nx}{1+n^2x^2} = 0 \text{ for } 0 \leq x \leq 1$$

$$\therefore \quad f'(0) = 0$$

Also $$S_n'(0) = \lim_{h \to 0} \frac{S_n(0+h) - S_n(0)}{h} = \lim_{h \to 0} \frac{\frac{nh}{1+n^2h^2} - 0}{h}$$

$$= \lim_{h \to 0} \frac{n}{1+n^2h^2} = n$$

$$\Rightarrow \quad \lim_{n \to \infty} S_n'(0) = \infty$$

Thus $$f'(0) \neq \lim_{n \to \infty} S_n'(0)$$

Hence the given series cannot be differentiated term by term.

Example 6. *Given the series* $\sum_{n=1}^{\infty} f_n$ *for which* $S_n(x) = \frac{1}{2n^2} \log(1 + n^4x^2)$, $0 \leq x \leq 1$.

Show that the series $\sum_{n=1}^{\infty} f_n'$ *does not converge uniformly, but the given series can be differentiated term by term.*

Sol. Here $$f(x) = \lim_{n \to \infty} S_n(x) = \lim_{n \to \infty} \frac{\log(1+n^4x^2)}{2n^2} \qquad \left| \text{Form } \frac{\infty}{\infty} \right.$$

$$= \lim_{n\to\infty} \frac{\frac{4n^3x^2}{1+n^4x^2}}{4n} = \lim_{n\to\infty} \frac{n^2x^2}{1+n^4x^2} = 0 \text{ for } 0 \le x \le 1$$

$$\therefore \quad f'(x) = 0$$

Also $$\lim_{n\to\infty} S_n'(x) = \lim_{n\to\infty} \left(\frac{1}{2n^2} \cdot \frac{2n^4x}{1+n^4x^2}\right) = \lim_{n\to\infty} \frac{n^2x}{1+n^4x^2} = 0 \text{ for } 0 \le x \le 1$$

$$\therefore \quad f'(x) = \lim_{n\to\infty} S_n'(x)$$

Thus term by term differentiation holds.

However, the series $\sum_{n=1}^{\infty} f_n'$ is not uniformly convergent for $0 \le x \le 1$ since the sequence $<S_n'>$ *i.e.*, $<\frac{n^2x}{1+n^4x^2}>$ has $x = 0$ as a point of non-uniform convergence.

Example 7. *Examine whether the series for which* $S_n(x) = \frac{1}{n+n^3x^2}$ *is differentiable term by term.*

Sol. *Please try yourself.* **[Ans. Yes]**

11 Improper Integrals

11.1. FINITE AND INFINITE INTERVALS

An interval is said to be finite or infinite according as its length is finite or infinite. Thus the intervals $[a, b]$, $[a, b)$, $(a, b]$, (a, b), each with length $(b - a)$, are finite (or bounded) if both a and b are finite. The intervals $[a, \infty)$, (a, ∞), $(-\infty, b]$ $(-\infty, b)$ and $(-\infty, \infty)$ are infinite (or unbounded) intervals.

11.2. BOUNDED FUNCTION

A function f is said to be bounded if its range is bounded. Thus, $f : [a, b] \to \mathbb{R}$ is bounded if there exist two real numbers m and M, $(m \le \text{M})$ such that

$$m \le f(x) \le \text{M} \quad \forall\, x \in [a, b]$$

f is also bounded if there exists a positive real number K such that

$$|f(x)| \le \text{K} \quad \forall\, x \in [a, b].$$

11.3. PROPER INTEGRAL

The definite integral $\int_a^b f(x)\,dx$ is called a proper integral if

(*i*) the interval of integration $[a, b]$ is finite (or bounded)

(*ii*) the integrand f is bounded on $[a, b]$

If F(x) is an indefinite integral of $f(x)$, then $\int_a^b f(x)\,dx = \text{F}(b) - \text{F}(a)$.

11.4. IMPROPER INTEGRAL

The definite integral $\int_a^b f(x)\,dx$ is called an improper integral if either or both the above conditions are not satisfied. Thus $\int_a^b f(x)\,dx$ is an improper integral if either the interval of integration $[a, b]$ is not finite or f is not bounded on $[a, b]$ or neither the interval $[a, b]$ is finite nor f is bounded over it.

(i) In the definite integral $\int_a^b f(x)\,dx$, if either a or b or both a and b are infinite so that the interval of integration is unbounded but f is bounded, then $\int_a^b f(x)\,dx$ is called an **improper integral of the first kind.**

For example, $\int_1^{\infty} \frac{dx}{\sqrt{x}}, \int_{-\infty}^{0} e^{2x}\,dx, \int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2}$ are improper integrals of the first kind.

(ii) In the definite integral $\int_a^b f(x)\,dx$, if both a and b are finite so that the interval of integration is finite but f has one or more points of infinite discontinuity *i.e.*, f is not bounded on $[a, b]$, then $\int_a^b f(x)\,dx$ is called an **improper integral of the second kind.**

For example, $\int_0^1 \frac{dx}{x^2}, \int_1^2 \frac{dx}{2-x}, \int_1^4 \frac{dx}{(x-1)(4-x)}$ are improper integrals of the second kind.

(iii) In the definite integral $\int_a^b f(x)dx$, if the interval of integration is unbounded (so that a or b or both are infinite) and f is also unbounded, then $\int_a^b f(x)dx$ is called an **improper integral of the third kind.**

For example, $\int_0^{\infty} \frac{e^{-x}}{\sqrt{x}}dx$ is an improper integral of the third kind.

11.5. IMPROPER INTEGRAL AS THE LIMIT OF A PROPER INTEGRAL

(a) *When the improper integral is of the first kind, either a or b or both a and b are infinite but f is bounded. We define*

(i) $\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$, $(t > a)$

The improper integral $\int_a^{\infty} f(x)\,dx$ is said to be **convergent** if the limit on the right hand side exists finitely and the integral is said to be **divergent** if the limit is $+\infty$ or $-\infty$.

If the integral is neither convergent nor divergent, then it is said to be **oscillating.**

(ii) $\int_{-\infty}^{b} f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$, $(t < b)$

The improper integral $\int_{-\infty}^{b} f(x)\,dx$ is said to be convergent if the limit on the right hand side exists finitely and the integral is said to be divergent if the limit is $+\infty$ or $-\infty$.

(iii) $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)dx + \int_c^{\infty} f(x)dx$ where c is any real number

$$= \lim_{t_1\to-\infty} \int_{t_1}^{c} f(x)\,dx + \lim_{t_2\to\infty} \int_c^{t_2} f(x)\,dx$$

The improper integral $\int_{-\infty}^{\infty} f(x)\,dx$ is said to be convergent if both the limits on the right hand side exist finitely and independent of each other, otherwise it is said to be divergent.

Note. $\int_{-\infty}^{\infty} f(x)\,dx \neq \lim_{t\to\infty}\left[\int_{-t}^{c} f(x)\,dx + \int_c^t f(x)\,dx\right]$.

(b) *When the improper integral is of the second kind, both a and b are finite but f has one (or more) points of infinite discontinuity on $[a, b]$.*

(i) If $f(x)$ becomes infinite at $x = b$ only, we define $\int_a^b f(x)\,dx = \lim_{\varepsilon\to0+} \int_a^{b-\varepsilon} f(x)\,dx$.

The improper integral $\int_a^b f(x)\,dx$ is said to be convergent if the limit on the right hand side exists finitely and the integral is said to be divergent if the limit is $+\infty$ or $-\infty$.

(ii) If $f(x)$ becomes infinite at $x = a$ only, we define $\int_a^b f(x)\,dx = \lim_{\varepsilon\to0+} \int_{a+\varepsilon}^{c} f(x)\,dx$.

The improper integral $\int_a^b f(x)\,dx$ converges if the limit on the right hand side exists finitely, otherwise it is said to be divergent.

(iii) If $f(x)$ becomes infinite at $x = c$ only where $a < c < b$, we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \lim_{\varepsilon_1\to0+} \int_a^{c-\varepsilon_1} f(x)\,dx + \lim_{\varepsilon_2\to0+} \int_{c+\varepsilon_2}^{b} f(x)\,dx.$$

The improper integral $\int_a^b f(x)\,dx$ is said to be convergent if both the limits on the right hand side exist finitely and independent of each other, otherwise it is said to be divergent.

Note 1. If f has infinite discontinuity at an end point of the interval of integration, then the point of discontinuity is approached from within the interval.

Thus if the interval of integration is $[a, b]$ and

(i) f has infinite discontinuity at 'a', we consider $[a+\varepsilon, b]$ as $\varepsilon \to 0+$.

(ii) f has infinite discontinuity at 'b', we consider $[a, b-\varepsilon]$ as $\varepsilon \to 0+$.

Note 2. A proper integral is always convergent.

Note 3. If $\int_a^b f(x)\,dx$ is convergent, then

(i) $\int_a^b kf(x)\,dx$ is convergent, $k \in \mathbb{R}$,

(ii) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ where $a < c < b$ and each integral or right hand side is convergent.

Note 4. For any c between a and b, *i.e.*, $a < c < b$, we have

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

If $\int_c^b f(x)$ is a proper integral, then the two integrals $\int_a^b f(x)\,dx$ and $\int_a^c f(x)\,dx$ converge or diverge together. Thus while testing the integral $\int_a^b f(x)\,dx$ for convergence at a, it may be replaced by $\int_a^c f(x)\,dx$ for any convenient c such that $a < c < b$.

ILLUSTRATIVE EXMAPLES

Example 1. *Examine the convergence of the improper integrals :*

(*i*) $\int_1^\infty \frac{1}{x}\,dx$ (*ii*) $\int_1^\infty \frac{dx}{\sqrt{x}}$ (*iii*) $\int_1^\infty \frac{dx}{x^{3/2}}$ (*iv*) $\int_0^\infty \frac{dx}{1+x^2}$.

Sol. (*i*) By definition, $\int_1^\infty \frac{dx}{x} = \lim_{t\to\infty}\int_0^t \frac{dx}{x} = \lim_{t\to\infty}[\log x]_1^t = \lim_{t\to\infty} \log t = \infty$

$\Rightarrow$ $\int_0^\infty \frac{dx}{x}$ is divergent.

(*ii*) By definition, $\int_1^\infty \frac{dx}{\sqrt{x}} = \lim_{t\to\infty}\int_1^t \int \frac{dx}{\sqrt{x}}$

$$= \lim_{t\to\infty}\int_1^t x^{-/2}\,dx = \lim_{t\to\infty}\left[2\sqrt{x}\right]_1^t = \lim_{t\to\infty}\left(2\sqrt{t}-2\right) = \infty$$

$\Rightarrow$ $\int_1^\infty \frac{dx}{\sqrt{x}}$ is divergent.

(*iii*) By definition, $\int_1^\infty \frac{dx}{x^{3/2}} = \lim_{t\to\infty}\int_1^t x^{-3/2}\,dx = \lim_{t\to\infty}\left[\frac{x^{-1/2}}{-\frac{1}{2}}\right]_1^t$

$$= \lim_{t\to\infty}\left[\frac{-2}{\sqrt{x}}\right]_1^t = \lim_{t\to\infty}\left(\frac{-2}{\sqrt{t}}+2\right) = 0+2 = 2 \text{ which is finite.}$$

$\Rightarrow$ $\int_1^\infty \frac{dx}{x^{3/2}}$ is convergent and its value is 2.

(*iv*) By definition, $\int_0^\infty \frac{dx}{1+x^2} = \lim_{t\to\infty}\int_0^t \frac{dx}{1+x^2} = \lim_{t\to\infty}\left[\tan^{-1} x\right]_0^t$

$$= \lim_{t\to\infty}(\tan^{-1} t - \tan^{-1} 0) = \frac{\pi}{2} \text{ which is finite.}$$

$\Rightarrow$ $\int_0^\infty \frac{dx}{1+x^2}$ is convergent and its value is $\frac{\pi}{2}$.

Example 2. *Examine for convergence the improper integrals :*

(*i*) $\int_0^\infty e^{-mx}\,dx\ (m>0)$ (*ii*) $\int_a^\infty \frac{x}{1+x^2}\,dx$ (*iii*) $\int_0^\infty \sin x\,dx$

(*iv*) $\int_0^\infty \frac{dx}{(1+x)^3}$ (*v*) $\int_0^\infty \frac{dx}{x^2+4a^2}$ (*vi*) $\int_0^\infty e^{2x}\,dx$.

Sol. (*i*) By definition, $\int_0^\infty e^{-mx}\,dx = \lim_{t\to\infty}\int_0^t e^{-mx}\,dx = \lim_{t\to\infty}\left[\frac{e^{-mx}}{-m}\right]_0^t = \lim_{t\to\infty} -\frac{1}{m}(e^{-mt}-1)$

$$= -\frac{1}{m}(0-1) = \frac{1}{m} \text{ which is finite.}$$

$\Rightarrow$ $\int_0^\infty e^{-mx}\,dx$ is convergent and its value is $\frac{1}{m}$.

(*ii*) By definition, $\int_a^\infty \frac{x}{1+x^2} = \lim_{t\to\infty}\int_a^t \frac{x}{1+x^2}\,dx = \lim_{t\to\infty}\int_a^t \frac{1}{2}\left(\frac{2x}{1+x^2}\right)dx$

$$= \lim_{t\to\infty}\left[\frac{1}{2}\log(1+x^2)\right]_a^t = \lim_{t\to\infty} \tfrac{1}{2}[\log(1+t^2) - \log(1+a^2)] = \infty.$$

$\Rightarrow$ $\int_0^\infty \frac{x}{1+x^2}\,dx$ is divergent.

(*iii*) $\int_0^\infty \sin x\,dx = \lim_{t\to\infty}\int_0^t \sin x\,dx = \lim_{t\to\infty}[-\cos x]_0^t = \lim_{t\to\infty}(1-\cos t)$

which does not exist uniquely since $\cos t$ oscillates between -1 and $+1$ when $1 \to \infty$.

$\Rightarrow$ $\int_0^\infty \sin x\,dx$ oscillates.

(*iv*) $\int_0^\infty \frac{dx}{(1+x)^3} = \lim_{t\to\infty}\int_0^t (1+x)^{-3}\,dx = \lim_{t\to\infty}\left[\frac{(1+x)^{-2}}{-2}\right]_0^t$

$$= \lim_{t\to\infty} -\frac{1}{2}\left[\frac{1}{(1+t)^2}-1\right] = -\frac{1}{2}(0-1) = \frac{1}{2} \text{ which is finite.}$$

$\Rightarrow$ $\int_0^\infty \frac{dx}{(1+x)^3}$ is convergent and its value is $\frac{1}{2}$.

(*v*) $\int_0^\infty \frac{dx}{x^2+4a^2} = \lim_{t\to\infty}\int_0^t \frac{dx}{x^2+(2a)^2} = \lim_{t\to\infty}\left[\frac{1}{2a}\tan^{-1}\frac{x}{2a}\right]_0^t$

$$= \lim_{t\to\infty}\frac{1}{2a}\left[\tan^{-1}\frac{t}{2a} - \tan^{-1} 0\right] = \frac{1}{2a}\left[\frac{\pi}{2}\right] = \frac{\pi}{4a} \text{ which is finite.}$$

$\Rightarrow$ $\int_0^\infty \frac{dx}{x^2+4a^2}$ is convergent and its value is $\frac{\pi}{4a}$.

(*vi*) *Please try yourself.* **[Ans. Divergent]**

Example 3. *Examine for convergence the improper integrals :*

(*i*) $\int_3^\infty \frac{dx}{(x-2)^2}$ (*ii*) $\int_0^\infty \frac{dx}{(1+x)^{2/3}}$ (*iii*) $\int_{\sqrt{2}}^\infty \frac{dx}{x\sqrt{x^2-1}}$

(*iv*) $\int_2^\infty \frac{2x^2}{x^4-1}\,dx$ (*v*) $\int_1^\infty \frac{x}{(1+2x)^3}\,dx$ (*vi*) $\int_1^\infty \frac{x}{(1+x)^3}\,dx$.

Sol. (*i*) $\int_3^\infty \frac{dx}{(x-2)^2} = \lim_{t\to\infty}\int_3^t (x-2)^{-2}\,dx = \lim_{t\to\infty}\left[\frac{(x-2)^{-1}}{-1}\right]_3^t$

$$= \lim_{t\to\infty} -\left[\frac{1}{t-2}-1\right] = -(0-1) = 1 \text{ which is finite.}$$

$\Rightarrow \quad \int_3^\infty \frac{dx}{(x-2)^2}$ is convergent and its value is 1.

(*ii*) *Please try yourself.* [**Ans.** Divergent]

(*iii*) $\int_{\sqrt{2}}^\infty \frac{dx}{x\sqrt{x^2-1}} = \lim_{t\to\infty}\int_{\sqrt{2}}^t \frac{dx}{x\sqrt{x^2-1}} = \lim_{t\to\infty}\left[\sec^{-1} x\right]_{\sqrt{2}}^{t}$

$$= \lim_{t\to\infty}\left(\sec^{-1} t - \sec^{-1}\sqrt{2}\right) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} \text{ which is finite.}$$

$\Rightarrow \quad \int_{\sqrt{2}}^\infty \frac{dx}{x\sqrt{x^2-1}}$ is convergent and its value is $\frac{\pi}{4}$.

(*iv*) $\int_2^\infty \frac{2x^2}{x^4-1}dx = \lim_{t\to\infty}\int_2^t \frac{(x^2+1)+(x^2-1)}{(x^2+1)(x^2-1)}dx$

$$= \lim_{t\to\infty}\int_2^t\left(\frac{1}{x^2-1}+\frac{1}{x^2+1}\right)dx = \lim_{t\to\infty}\left[\frac{1}{2}\log\frac{x-1}{x+1}+\tan^{-1}x\right]_2^t$$

$$= \lim_{t\to\infty}\left[\frac{1}{2}\log\frac{t-1}{t+1}+\tan^{-1}t-\frac{1}{2}\log\frac{1}{3}-\tan^{-1}2\right]$$

$$= \frac{1}{2}\lim_{t\to\infty}\log\frac{1-\frac{1}{t}}{1+\frac{1}{t}}+\frac{\pi}{2}+\frac{1}{2}\log 3-\tan^{-1}2$$

$$= \frac{1}{2}\log 1+\frac{\pi}{2}+\frac{1}{2}\log 3-\tan^{-1}2 = \frac{\pi}{2}+\frac{1}{2}\log 3-\tan^{-1}2 \text{ which is finite.}$$

$\Rightarrow \quad \int_2^\infty \frac{2x^2}{x^4-1}dx$ is convergent and its value is $\frac{\pi}{2}+\frac{1}{2}\log 3-\tan^{-1}2$.

(*v*) $\int_1^\infty \frac{x}{(1+2x)^3}dx = \lim_{t\to\infty}\int_1^t \frac{x}{(1+2x)^3}dx$

$$= \lim_{t\to\infty}\int_1^t \frac{\frac{1}{2}(1+2x)-\frac{1}{2}}{(1+2x)^3}dx = \lim_{t\to\infty}\int_1^t\left[\frac{1}{2}(1+2x)^{-2}-\frac{1}{2}(1+2x)^{-3}\right]dx$$

$$= \lim_{t\to\infty}\left[\frac{1}{2}\cdot\frac{(1+2x)^{-1}}{-1\times 2}-\frac{1}{2}\cdot\frac{(1+2x)^{-2}}{-2\times 2}\right]_1^t = \lim_{t\to\infty}\left[\frac{-1}{4(1+2x)}+\frac{1}{8(1+2x)^2}\right]_1^t$$

$$= \lim_{t\to\infty}\left[\frac{-1}{4(1+2t)}+\frac{1}{8(1+2t)^2}+\frac{1}{12}-\frac{1}{72}\right]$$

$$= 0+0+\frac{1}{12}-\frac{1}{72}=\frac{5}{72} \text{ which is finite.}$$

$\Rightarrow \quad \int_1^\infty \frac{x}{(1+2x)^3}dx$ is convergent and its value is $\frac{5}{72}$.

(*vi*) *Please try yourself.* $\left[\textbf{Ans. } \text{Converges to } \frac{3}{8}\right]$

Example 4. *Examine for convergence the integrals :*

(*i*) $\int_1^\infty xe^{-x}\,dx$ (*ii*) $\int_0^\infty x^2 e^{-x}\,dx$ (*iii*) $\int_0^\infty xe^{-x^2}\,dx$

(*iv*) $\int_0^\infty x^3 e^{-x^2}\,dx$ (*v*) $\int_0^\infty x\sin x\,dx$.

Sol. (*i*) $\int_0^\infty xe^{-x}\,dx = \lim_{t\to\infty}\int_1^t xe^{-x}\,dx$ (Intergrating by parts)

$$= \lim_{t\to\infty}\left[-xe^{-x}-e^{-x}\right]_1^t = \lim_{t\to\infty}(-te^{-t}-e^{-t}+e^{-1}+e^{-1})$$

$$= \lim_{t\to\infty}\left(\frac{-t}{e^t}\right)-\lim_{t\to\infty}e^{-t}+\frac{2}{e} \quad \text{(Applying L' Hospital's Rule to first limit)}$$

$$= \lim_{t\to\infty}\left(\frac{-1}{e^t}\right)-0+\frac{2}{e}=0+\frac{2}{e}=\frac{2}{e} \text{ which is finite.}$$

$\Rightarrow \quad \int_1^\infty xe^{-x}\,dx$ is convergent and its value is $\frac{2}{e}$.

(*ii*) $\int_0^\infty x^2 e^{-x}\,dx = \lim_{t\to\infty}\int_0^t x^2e^{-x}\,dx$ (Integrating by parts)

$$= \lim_{t\to\infty}\left[-x^2e^{-x}-2xe^{-x}-2e^{-x}\right]_0^t = \lim_{t\to\infty}(-t^2e^{-t}-2te^{-t}-2e^{-t}+2)$$

$$= \lim_{t\to\infty}\left(\frac{-t^2}{e^t}\right)-2\lim_{t\to\infty}\left(\frac{t}{e^t}\right)-0+2 \quad \text{(Applying L' Hospital's rule)}$$

$$= \lim_{t\to\infty}\left(\frac{-2t}{e^t}\right)-2\lim_{t\to\infty}\left(\frac{t}{e^t}\right)+2$$

(Again applying L' Hospital's rule to first limit)

$$= \lim_{t\to\infty}\left(\frac{-2}{e^t}\right)-2\times 0+2=0+2=2 \text{ which is finite.}$$

$\Rightarrow \quad \int_0^\infty x^2e^{-x}\,dx$ is convergent and its value is 2.

(*iii*) $\int_0^\infty xe^{-x^2}dx = \lim_{t\to\infty}\int_0^t xe^{-x^2}\,dx$

Put $x^2 = z$ so that $2xdx = dz$ or $xdx = \frac{1}{2}dz$

When $x = 0, z = 0$; when $x = t, z = t^2$

$\therefore \quad \int_0^\infty xe^{-x^2}dx = \lim_{t\to\infty}\int_0^{t^2}\frac{1}{2}e^{-z}\,dz = \lim_{t\to\infty}\left[-\frac{1}{2}e^{-z}\right]_0^{t^2}$

$$= \lim_{t\to\infty}-\frac{1}{2}(e^{-t^2}-1) = -\frac{1}{2}(0-1)=\frac{1}{2} \text{ which is finite.}$$

$\Rightarrow \quad \int_0^\infty x\, e^{-x^2}\, dx$ is convergent and its value is $\frac{1}{2}$.

(*iv*) $\int_0^\infty x^3 e^{-x^2}\, dx = \lim_{t\to\infty} \int_0^t x \,.\, x^2 e^{-x^2}\, dx$

Put $x^2 = z$ so that $2x\, dx = dz$

When $x = 0, z = 0$; When $x = t, z = t^2$

$\therefore \quad \int_0^\infty x^3 e^{-x^2}\, dx = \lim_{t\to\infty} \int_0^{t^2} \frac{1}{2} z e^{-z}\, dz$ (Integrating by parts)

$$= \lim_{t\to\infty} \frac{1}{2}\left[-ze^{-z} - e^{-z}\right]_0^{t^2} = \lim_{t\to\infty} \frac{1}{2}\left[-t^2 e^{-t^2} - e^{-t^2} + 1\right] = -\frac{1}{2}\lim_{t\to\infty}\left(\frac{t^2}{e^{t^2}}\right) - 0 + \frac{1}{2}$$

(Applying L' Hospital's rule)

$$= -\frac{1}{2}\lim_{t\to\infty}\left(\frac{2t}{2t\, e^{t^2}}\right) + \frac{1}{2} = -\frac{1}{2}\lim_{t\to\infty}\left(\frac{1}{e^{t^2}}\right) + \frac{1}{2} = 0 + \frac{1}{2} = \frac{1}{2} \text{ which is finite.}$$

$\Rightarrow \quad \int_0^\infty x^3 e^{-x^2}\, dx$ is convergent and its value is $\frac{1}{2}$.

(*v*) $\int_0^\infty x \sin x\, dx = \lim_{t\to\infty} \int_0^t x \sin x\, dx$ (Integrating by parts)

$$= \lim_{t\to\infty} \left[-x \cos x + \sin x\right]_0^t = \lim_{t\to\infty} (-t \cos t + \sin t)$$

which oscillates between $-\infty$ and $+\infty$ since $\cos t$ oscillates between -1 and $+1$ as $t \to \infty$.

$\Rightarrow \quad \int_0^\infty x \sin x\, dx$ is not convergent. (In fact, it oscillates infinitely.)

Example 5. *Examine for convergence the integrals :*

(*i*) $\int_1^\infty \frac{dx}{(1+x)\sqrt{x}}$ (*ii*) $\int_2^\infty \frac{dx}{x \log x}$ (*iii*) $\int_0^\infty e^{-x} \sin x\, dx$ (*iv*) $\int_0^\infty e^{-ax} \cos bx\, dx$.

Sol. (*i*) $\int_1^\infty \frac{dx}{(1+x)\sqrt{x}} = \lim_{t\to\infty} \int_1^t \frac{dx}{(1+x)\sqrt{x}}$

Put $\sqrt{x} = z$ so that $\frac{1}{2\sqrt{x}}\, dx = dz$

When $x = 1, z = 1$; when $x = t, z = \sqrt{t}$

$$\therefore \quad \int_1^\infty \frac{dx}{(1+x)\sqrt{x}} = \lim_{t\to\infty} \int_1^{\sqrt{t}} \frac{2dz}{1+z^2} = \lim_{t\to\infty} \left[2 \tan^{-1} z\right]_1^{\sqrt{t}}$$

$$= \lim_{t\to\infty} 2\left[\tan^{-1}\sqrt{t} - \tan^{-1} 1\right] = 2\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{\pi}{2} \text{ which is finite.}$$

$\Rightarrow \quad \int_1^\infty \frac{dx}{(1+x)\sqrt{x}}$ is convergent and its value is $\frac{\pi}{2}$.

(*ii*) $\int_2^\infty \frac{dx}{x \log x} = \lim_{t\to\infty} \int_2^t \frac{1/x}{\log x}\, dx$

$$= \lim_{t\to\infty} \left[\log(\log x)\right]_2^t = \lim_{t\to\infty} \left[\log(\log t) - \log(\log 2)\right] = \infty$$

$\Rightarrow \quad \int_2^\infty \frac{dx}{x \log x}$ is divergent.

(*iii*) $\int_0^\infty e^{-x} \sin x\, dx = \lim_{t\to\infty} \int_0^t e^{-x} \sin x\, dx$

$$= \lim_{t\to\infty} \left[\frac{e^{-x}}{(-1)^2 + 1^2}(-1 \sin x - 1 \cos x)\right]_0^t$$

$$\left[\because \int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2 + b^2}(a \sin bx - b \cos bx)\right]$$

$$= \lim_{t\to\infty} \left[-\frac{1}{2} e^{-x} (\sin x + \cos x)\right]_0^t = \lim_{t\to\infty} -\frac{1}{2}\left[e^{-t}(\sin t + \cos t) - 1\right]$$

$$= -\frac{1}{2}\left[(0 \times a \text{ finite quantity}) - 1\right] = \frac{1}{2} \text{ which is finite.}$$

$\Rightarrow \quad \int_0^\infty e^{-x} \sin x$ is convergent and its value is $\frac{1}{2}$.

(*iv*) $\int_0^\infty e^{-ax} \cos bx\, dx = \lim_{t\to\infty} \int_0^t e^{-ax} \cos bx\, dx$

$$= \lim_{t\to\infty} \left[\frac{e^{-ax}}{(-a)^2 + b^2}(-a \cos bx + b \sin bx)\right]_0^t$$

$$\left[\because \int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2 + b^2}(a \cos bx + b \sin bx)\right]$$

$$= \lim_{t\to\infty} \frac{1}{a^2 + b^2}\left[e^{-at}(-a \cos bt + b \sin bt) + a\right]$$

$$= \frac{1}{a^2 + b^2}\left[(0 \times \text{a finite quantity}) + a\right] = \frac{a}{a^2 + b^2} \text{ which is finite.}$$

$\Rightarrow \quad \int_0^\infty e^{-ax} \cos bx\, dx$ is convergent and its value is $\frac{a}{a^2 + b^2}$.

Example 6. *Examine the convergence of the integrals :*

(*i*) $\int_1^\infty \frac{dx}{x(x+1)}$ (*ii*) $\int_1^\infty \frac{dx}{x^2(x+1)}$ (*iii*) $\int_1^\infty \frac{\tan^{-1} x}{x^2}\, dx$ (*iv*) $\int_0^\infty e^{-\sqrt{x}}\, dx$.

Sol. (*i*) $\int_1^\infty \frac{dx}{x(x+1)} = \lim_{t\to\infty} \int_1^t \frac{dx}{x(x+1)} = \lim_{t\to\infty} \int_1^t \left(\frac{1}{x} - \frac{1}{x+1}\right) dx$ [Partial Fractions]

$$= \lim_{t\to\infty} [\log x - \log (x+1)]_1^t = \lim_{t\to\infty}\left[\log \frac{x}{x+1}\right]_1^t$$

$$= \lim_{t\to\infty}\left[\log \frac{t}{t+1} - \log \frac{1}{2}\right] = \lim_{t\to\infty}\left(\log \frac{1}{1+\frac{1}{t}}\right) + \log 2$$

$= \log 1 + \log 2 = \log 2$ which is finite.

$\Rightarrow \quad \int_1^\infty \frac{dx}{x(x+1)}$ is convergent and its value is $\log 2$.

(ii) $$\int_1^\infty \frac{dx}{x^2(x+1)} = \lim_{t\to\infty}\int_1^t \frac{dx}{x^2(x+1)} = \lim_{t\to\infty}\int_1^t \left(-\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x+1}\right)dx$$ [Partial Fractions]

$$= \lim_{t\to\infty}\left[-\log x - \frac{1}{x} + \log (x+1)\right]_1^t = \lim_{t\to\infty}\left[\log \frac{x+1}{x} - \frac{1}{x}\right]_1^t$$

$$= \lim_{t\to\infty}\left[\log\left(1+\frac{1}{t}\right) - \frac{1}{t} - \log 2 + 1\right]$$

$= \log 1 - 0 - \log 2 + 1 = 1 - \log 2$ which is finite.

$\Rightarrow \quad \int_1^\infty \frac{dx}{x^2(x+1)}$ is convergent and its value is $1 - \log 2$.

(iii) $$\int_1^\infty \frac{\tan^{-1} x}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{\tan^{-1} x}{x^2}\,dx$$

Put $x = \tan\theta$, so that $dx = \sec^2\theta\, d\theta$

$$\int \frac{\tan^{-1} x}{x^2}\,dx = \int \frac{\theta}{\tan^2\theta}\sec^2\theta\, d\theta = \int \theta \operatorname{cosec}^2\theta\, d\theta = \theta(-\cot\theta) - \int 1(-\cot\theta)\, d\theta$$

$$= -\theta\cot\theta + \log\sin\theta = -\frac{\tan^{-1} x}{x} + \log \frac{x}{\sqrt{1+x^2}}$$

$\therefore$ $$\int_1^\infty \frac{\tan^{-1} x}{x^2}\,dx = \lim_{t\to\infty}\left[-\frac{\tan^{-1} x}{x} + \log \frac{x}{\sqrt{1+x^2}}\right]_1^t$$

$$= \lim_{t\to\infty}\left[-\frac{\tan^{-1} t}{t} + \log \frac{t}{\sqrt{1+t^2}} + \tan^{-1} 1 - \log \frac{1}{\sqrt{2}}\right]$$

$$= 0 + \lim_{t\to\infty} \log \frac{1}{\sqrt{\frac{1}{t^2}+1}} + \frac{\pi}{4} + \frac{1}{2}\log 2$$

$= \log 1 + \frac{\pi}{4} + \frac{1}{2}\log 2 = \frac{\pi}{4} + \frac{1}{2}\log 2$ which is finite.

$\Rightarrow \quad \int_0^\infty \frac{\tan^{-1} x}{x^2}\,dx$ is convergent and its value is $\frac{\pi}{4} + \frac{1}{2}\log 2$.

(iv) $$\int_0^\infty e^{-\sqrt{x}}\,dx = \lim_{t\to\infty}\int_0^t e^{-\sqrt{x}}\,dx$$

Put $\sqrt{x} = z$, *i.e.*, $x = z^2$ so that $dx = 2z\,dz$

When $x = 0, z = 0$; when $x = t, z = \sqrt{t}$.

$\therefore$ $$\int_0^\infty e^{-\sqrt{x}}\,dx = \lim_{t\to\infty}\int_0^{\sqrt{t}} 2ze^{-z}\,dz$$ [Integrating by parts]

$$= \lim_{t\to\infty} 2\left[-ze^{-z} - e^{-z}\right]_0^{\sqrt{t}} = \lim_{t\to\infty} -2\left[\sqrt{t}\,e^{-\sqrt{t}} + e^{-\sqrt{t}} - 1\right]$$

$$= \lim_{t\to\infty}\left(\frac{-2\sqrt{t}}{e^{\sqrt{t}}}\right) - 0 + 2$$ (Applying L' Hospital's Rule)

$$= \lim_{t\to\infty}\left(\frac{-\frac{1}{\sqrt{t}}}{e^{\sqrt{t}}\cdot\frac{1}{2\sqrt{t}}}\right) + 2 = \lim_{t\to\infty}\left(\frac{-2}{e^{\sqrt{t}}}\right) + 2 = 0 + 2 = 2$$ which is finite.

$\Rightarrow \quad \int_0^\infty e^{-\sqrt{x}}\,dx$ is convergent and its value is 2.

Example 7. *Examine the convergence of the integrals :*

(i) $\int_{-\infty}^0 e^{2x}\,dx$ (ii) $\int_{-\infty}^0 \frac{dx}{p^2+q^2x^2}$ (iii) $\int_{-\infty}^0 e^{-x}\,dx$

(iv) $\int_{-\infty}^0 \sinh x\,dx$ (v) $\int_{-\infty}^0 \cosh x\,dx$ (vi) $\int_{-\infty}^1 \frac{dx}{1+x^2}$.

Sol. (i) $$\int_{-\infty}^0 e^{2x}\,dx = \lim_{t\to-\infty}\int_t^0 e^{2x}\,dx$$

$$= \lim_{t\to-\infty}\left[\frac{e^{2x}}{2}\right]_t^0 = \lim_{t\to-\infty}\frac{1}{2}(1-e^{2t}) = \frac{1}{2}(1-0) = \frac{1}{2}$$ which is finite.

$\Rightarrow \quad \int_{-\infty}^0 e^{2x}\,dx$ is convergent and its value is $\frac{1}{2}$.

(ii) $$\int_{-\infty}^0 \frac{dx}{p^2+q^2x^2} = \lim_{t\to-\infty}\int_t^0 \frac{dx}{q^2\left(\frac{p^2}{q^2}+x^2\right)} = \lim_{t\to-\infty}\left[\frac{1}{q^2}\cdot\frac{1}{p/q}\tan^{-1}\frac{x}{p/q}\right]_t^0$$

$$= \lim_{t\to-\infty}\frac{1}{pq}\left[0 - \tan^{-1}\frac{qt}{p}\right] = -\frac{1}{pq}\left(-\frac{\pi}{2}\right) = \frac{\pi}{2pq}$$ which is finite.

$\Rightarrow \quad \int_{-\infty}^0 \frac{dx}{p^2+q^2x^2}$ is convergent and its value is $\frac{\pi}{2pq}$.

(*iii*) $\int_{-\infty}^{0} e^{-x} dx = \lim_{t \to -\infty} \int_{t}^{0} e^{-x}\, dx = \lim_{t \to -\infty} \left[-e^{-x}\right]_t^0 = \lim_{t \to -\infty} (-1 + e^{-t}) = -1 + \infty = \infty$

$\Rightarrow \quad \int_{-\infty}^{0} e^{-x}\, dx$ diverges to $+\infty$.

(*iv*) $\int_{-\infty}^{0} \sinh x\, dx = \lim_{t \to -\infty} \int_{t}^{0} \frac{e^x - e^{-x}}{2}\, dx$

$$= \lim_{t \to -\infty} \left[\frac{1}{2}(e^x + e^{-x})\right]_t^0 = \lim_{t \to -\infty} \left[1 - \frac{1}{2}(e^t + e^{-t})\right] = 1 - \frac{1}{2}(0 + \infty) = -\infty$$

$\Rightarrow \quad \int_{-\infty}^{0} \sinh x\, dx$ diverges to $-\infty$.

(*v*) *Please try yourself.* [**Ans.** Diverges to ∞]

(*vi*) *Please try yourself.* $\left[\textbf{Ans. } \text{Converges to } \frac{3\pi}{4}\right]$

Example 8. *Examine the convergence of the integrals :*

(*i*) $\int_{-\infty}^{\infty} e^{-x}\, dx$ (*ii*) $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$ (*iii*) $\int_{-\infty}^{\infty} \frac{dx}{e^x + e^{-x}}$

(*iv*) $\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2}$ (*v*) $\int_{-\infty}^{\infty} \frac{dx}{x^2 + 2x + 2}$.

Sol. (*i*) $\int_{-\infty}^{\infty} e^{-x}\, dx = \int_{-\infty}^{0} e^{-x}\, dx + \int_{0}^{\infty} e^{-x}\, dx = \lim_{t_1 \to -\infty} \int_{t_1}^{0} e^{-x}\, dx + \lim_{t_2 \to \infty} \int_{0}^{t_2} e^{-x}\, dx$

$$= \lim_{t_1 \to -\infty} \left[-e^{-x}\right]_{t_1}^0 + \lim_{t_2 \to \infty} \left[-e^{-x}\right]_0^{t_2}$$

$$= \lim_{t_1 \to -\infty} (-1 + e^{-t_1}) + \lim_{t_2 \to \infty} (-e^{-t_2} + 1) = (-1 + \infty) + (0 + 1) = \infty$$

$\Rightarrow \quad \int_{-\infty}^{\infty} e^{-x}\, dx$ diverges to ∞.

(*ii*) $\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2}$

$$= \lim_{t_1 \to -\infty} \int_{t_1}^{0} \frac{dx}{1+x^2} + \lim_{t_2 \to \infty} \int_{0}^{t_2} \frac{dx}{1+x^2}$$

$$= \lim_{t_1 \to -\infty} \left[\tan^{-1} x\right]_{t_1}^0 + \lim_{t_2 \to \infty} \left[\tan^{-1} x\right]_0^{t_2}$$

$$= \lim_{t_1 \to -\infty} \left[-\tan^{-1} t_1\right] + \lim_{t_2 \to \infty} \left[\tan^{-1} t_2\right] = -\left(-\frac{\pi}{2}\right) + \frac{\pi}{2} = \pi \text{ which is finite.}$$

$\Rightarrow \quad \int_{-\infty}^{\infty} \frac{dx}{1+x^2}$ is convergent and its value is π.

(*iii*) $\int_{-\infty}^{\infty} \frac{dx}{e^x + e^{-x}} = \int_{-\infty}^{0} \frac{dx}{e^x + e^{-x}} + \int_{0}^{\infty} \frac{dx}{e^x + e^{-x}}$

$$= \lim_{t_1 \to -\infty} \int_{t_1}^{0} \frac{e^x}{e^{2x} + 1}\, dx + \lim_{t_2 \to \infty} \int_{0}^{t_2} \frac{e^x}{e^{2x} + 1}\, dx$$

Now $\int \frac{e^x}{e^{2x}+1}\, dx = \int \frac{dz}{z^2+1}$ where $z = e^x$

$$= \tan^{-1} z = \tan^{-1} e^x$$

$\therefore \quad \int_{-\infty}^{\infty} \frac{dx}{e^x + e^{-x}} = \lim_{t_1 \to -\infty} \left[\tan^{-1} e^x\right]_{t_1}^0 + \lim_{t_2 \to \infty} \left[\tan^{-1} e^x\right]_0^{t_2}$

$$= \lim_{t_1 \to -\infty} \left[\tan^{-1} 1 - \tan^{-1} e^{t_1}\right] + \lim_{t_2 \to \infty} \left[\tan^{-1} e^{t_2} - \tan^{-1} 1\right]$$

$$= \left(\frac{\pi}{4} - \tan^{-1} 0\right) + \left(\tan^{-1} \infty - \frac{\pi}{4}\right) = \frac{\pi}{2} \text{ which is finite.}$$

$\Rightarrow \quad \int_{-\infty}^{\infty} \frac{dx}{e^x + e^{-x}}$ is convergent and its value is $\frac{\pi}{2}$.

(*iv*) $\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} = \int_{-\infty}^{0} \frac{dx}{(1+x^2)^2} + \int_{0}^{\infty} \frac{dx}{(1+x^2)^2}$

$$= \lim_{t_1 \to -\infty} \int_{t_1}^{0} \frac{dx}{(1+x^2)^2} + \lim_{t_2 \to \infty} \int_{0}^{t_2} \frac{dx}{(1+x^2)^2}$$

Now putting $x = \tan\theta$ so that $dx = \sec^2\theta\, d\theta$, we have

$$\int \frac{dx}{(1+x^2)^2} = \int \frac{\sec^2\theta\, d\theta}{(1+\tan^2\theta)^2} = \int \frac{\sec^2\theta\, d\theta}{\sec^4\theta} = \int \cos^2\theta\, d\theta = \int \frac{1+\cos 2\theta}{2}\, d\theta$$

$$= \frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) = \frac{1}{2}(\theta + \sin\theta\cos\theta) = \frac{1}{2}\left(\tan^{-1} x + \frac{x}{1+x^2}\right)$$

$\therefore \quad \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} = \lim_{t_1 \to -\infty} \left[\frac{1}{2}\left(\tan^{-1} x + \frac{x}{1+x^2}\right)\right]_{t_1}^0 + \lim_{t_2 \to \infty} \left[\frac{1}{2}\left(\tan^{-1} x + \frac{x}{1+x^2}\right)\right]_0^{t_2}$

$$= \lim_{t_1 \to -\infty} \frac{1}{2}\left[-\tan^{-1} t_1 - \frac{t_1}{1+t_1^2}\right] + \lim_{t_2 \to \infty} \frac{1}{2}\left[\tan^{-1} t_2 + \frac{t_2}{1+t_2^2}\right]$$

$$= \frac{1}{2}\cdot\frac{\pi}{2} - \lim_{t_1 \to -\infty} \frac{t_1}{2(1+t_1^2)} + \frac{1}{2}\cdot\frac{\pi}{2} + \lim_{t_2 \to \infty} \frac{t_2}{2(1+t_2^2)}$$

$$= \frac{\pi}{2} - \lim_{t_1 \to -\infty} \frac{1}{4t_1} + \lim_{t_2 \to \infty} \frac{1}{4t_2} \qquad \text{(By L' Hospital's rule)}$$

$$= \frac{\pi}{2} - 0 + 0 = \frac{\pi}{2} \text{ which is finite.}$$

$\Rightarrow \quad \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2}$ is convergent and its value is $\frac{\pi}{2}$.

(v) $$\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2} = \lim_{t_1 \to -\infty} \int_{t_1}^{0} \frac{dx}{(x+1)^2+1} + \lim_{t_2 \to \infty} \int_0^{t_2} \frac{dx}{(x+1)^2+1}$$

$$= \lim_{t_1 \to -\infty} \left[\tan^{-1}(x+1)\right]_{t_1}^{0} + \lim_{t_2 \to \infty} \left[\tan^{-1}(x+1)\right]_0^{t_2}$$

$$= \lim_{t_1 \to -\infty} \left[\frac{\pi}{4} - \tan^{-1}(t_1+1)\right] + \lim_{t_2 \to \infty} \left[\tan^{-1}(t_2+1) - \frac{\pi}{4}\right]$$

$$= \frac{\pi}{4} - \tan^{-1}(-\infty) + \tan^{-1}\infty - \frac{\pi}{4} = \frac{\pi}{2} + \frac{\pi}{2} = \pi \text{ which is finite.}$$

$\Rightarrow$ $\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2}$ is convergent and its value is π.

Example 9. *Test the convergence of the integrals :*

(i) $\int_0^1 \frac{dx}{\sqrt{x}}$ (ii) $\int_0^1 \frac{dx}{x^2}$ (iii) $\int_1^2 \frac{x}{\sqrt{x-1}}\,dx$

(iv) $\int_1^2 \frac{dx}{\sqrt{x-1}}$ (v) $\int_1^2 \frac{dx}{x\sqrt{x^2-1}}$ (vi) $\int_2^3 \frac{x-1}{\sqrt{x-2}}\,dx.$

Sol. (i) 0 is the only point of infinite discontinuity of the integrand on [0, 1].

$\therefore$ $$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{\varepsilon \to 0+} \int_{0+\varepsilon}^{1} x^{-1/2}\,dx$$

$$= \lim_{\varepsilon \to 0+} \left[2\sqrt{x}\right]_{\varepsilon}^{1} = \lim_{\varepsilon \to 0+} 2\left(1-\sqrt{\varepsilon}\right) = 2 \text{ which is finite.}$$

$\Rightarrow$ $\int_0^1 \frac{dx}{\sqrt{x}}$ is convergent and its value is 2.

(ii) 0 is the only point of infinite discontinuity of the integrand on [0, 1].

$\therefore$ $$\int_0^1 \frac{dx}{x^2} = \lim_{\varepsilon \to 0+} \int_{0+\varepsilon}^{1} x^{-2}\,dx = \lim_{\varepsilon \to 0+} \left[-\frac{1}{x}\right]_{\varepsilon}^{1} = \lim_{\varepsilon \to 0+} \left(-1+\frac{1}{\varepsilon}\right) = \infty$$

$\Rightarrow$ $\int_0^1 \frac{dx}{x^2}$ diverges to ∞.

(iii) 1 is the only point of infinite discontinuity of the integrand on [1, 2].

$\therefore$ $$\int_1^2 \frac{x}{\sqrt{x-1}}\,dx = \lim_{\varepsilon \to 0+} \int_{1+\varepsilon}^{2} \frac{(x-1)+1}{\sqrt{x-1}}\,dx$$

$$= \lim_{\varepsilon \to 0+} \int_{1+\varepsilon}^{2} \left(\sqrt{x-1} + \frac{1}{\sqrt{x-1}}\right) dx = \lim_{\varepsilon \to 0+} \left[\frac{2}{3}(x-1)^{3/2} + 2\sqrt{x-1}\right]_{1+\varepsilon}^{2}$$

$$= \lim_{\varepsilon \to 0+} \left[\frac{2}{3} + 2 - \frac{2}{3}\varepsilon^{3/2} - 2\sqrt{\varepsilon}\right] = \frac{8}{3} \text{ which is finite.}$$

$\Rightarrow$ $\int_1^2 \frac{x}{\sqrt{x-1}}\,dx$ is convergent and its value is $\frac{8}{3}$.

(iv) *Please try yourself.* [**Ans.** Converges to 2]

(v) *Please try yourself.* $\left[\textbf{Ans. } \text{Converges to } \frac{\pi}{3}\right]$

(vi) *Please try yourself.* $\left[\textbf{Ans. } \frac{8}{3}\right]$

Example 10. *Examine the convergence of the integrals :*

(i) $\int_0^1 \log x\,dx$ (ii) $\int_0^{1/e} \frac{dx}{x(\log x)^2}$ (iii) $\int_0^{e} \frac{dx}{x(\log x)^3}$ (iv) $\int_1^2 \frac{dx}{x \log x}$.

Sol. (i) 0 is the only point of infinite discontinuity of the integrand on [0, 1].

$\therefore$ $$\int_0^1 \log x\,dx = \lim_{\varepsilon \to 0+} \int_{0+\varepsilon}^{1} (\log x)\,.\,1\,dx \qquad \text{(Integrating by parts)}$$

$$= \lim_{\varepsilon \to 0+} \left[x \log x - x\right]_{\varepsilon}^{1} = \lim_{\varepsilon \to 0+} (-1 - \varepsilon \log \varepsilon + \varepsilon)$$

$$= -1 \text{ which is finite.} \qquad \left[\because \lim_{x \to 0} x^n \log x = 0, n > 0\right]$$

$\Rightarrow$ $\int_0^1 \log x\,dx$ is convergent and its value is -1.

(ii) Since $\lim_{x \to 0} x(\log x)^n = 0, n > 0$, therefore, 0 is the only point of infinite discontinuity of the integrand on $\left[0, \frac{1}{e}\right]$.

$\therefore$ $$\int_0^{1/e} \frac{dx}{x(\log x)^2} = \lim_{\varepsilon \to 0+} \int_{0+\varepsilon}^{1/e} (\log x)^{-2}\,.\,\frac{1}{x}\,dx = \lim_{\varepsilon \to 0+} \left[\frac{(\log x)^{-1}}{-1}\right]_{\varepsilon}^{1/e}$$

$$= \lim_{\varepsilon \to 0+} \left[\frac{1}{\log \frac{1}{e}} - \frac{1}{\log \varepsilon}\right] = -[-1-0] = 1 \text{ which is finite.}$$

$\therefore$ $\int_0^{1/e} \frac{dx}{x(\log x)^2}$ is convergent and its value is 1.

(iii) *Please try yourself.* $\left[\textbf{Ans. } \text{Converges to } -\frac{1}{2}\right]$

(iv) 1 is the only point of infinite discontinuity of the integrand on [1, 2].

$\therefore$ $$\int_1^2 \frac{dx}{x \log x} = \lim_{\varepsilon \to 0+} \int_{1+\varepsilon}^{2} \frac{1/x}{\log x}\,dx$$

$$= \lim_{\varepsilon \to 0+} \Big[\log(\log x)\Big]_{1+\varepsilon}^{2} = \lim_{\varepsilon \to 0+} [\log\log 2 - \log\log(1+\varepsilon)]$$

$$= \log\log 2 - \log 0 = \log\log 2 - (-\infty) = \infty$$

$\Rightarrow$ $\int_1^2 \frac{dx}{x \log x}$ diverges to ∞.

Example 11. *Examine the convergence of the integrals :*

(*i*) $\int_0^a \frac{dx}{\sqrt{a-x}}$ (*ii*) $\int_0^2 \frac{dx}{\sqrt{4-x^2}}$ (*iii*) $\int_1^2 \frac{dx}{2-x}$ (*iv*) $\int_0^{\pi/2} \frac{\cos x}{\sqrt{1-\sin x}}dx$

(*v*) $\int_0^1 \frac{dx}{x^2-3x+2}$ (*vi*) $\int_0^1 \frac{dx}{x^2-1}$ (*vii*) $\int_0^{\pi/2} \tan\theta\, d\theta$.

Sol. (*i*) a is the only point of infinite discontinuity of the integrand on $[0, a]$.

$$\therefore \quad \int_0^a \frac{dx}{\sqrt{a-x}} = \lim_{\varepsilon\to 0+} \int_0^{a-\varepsilon} (a-x)^{-1/2}\, dx$$

$$= \lim_{\varepsilon\to 0+} \left[-2\sqrt{a-x}\right]_0^{a-\varepsilon} = \lim_{\varepsilon\to 0+} -2\left[\sqrt{\varepsilon}-\sqrt{a}\right] = 2\sqrt{a} \text{ which is finite.}$$

$$\therefore \quad \int_0^a \frac{dx}{\sqrt{a-x}} \text{ is convergent and its value is } 2\sqrt{a}.$$

(*ii*) 2 is the only point of infinite discontinuity of the integrand on $[0, 2]$.

$$\therefore \quad \int_0^2 \frac{dx}{\sqrt{4-x^2}} = \lim_{\varepsilon\to 0+} \int_0^{2-\varepsilon} \frac{dx}{\sqrt{4-x^2}} = \lim_{\varepsilon\to 0+} \left[\sin^{-1}\frac{x}{2}\right]_0^{2-\varepsilon}$$

$$= \lim_{\varepsilon\to 0+} \left[\sin^{-1}\frac{2-\varepsilon}{2} - \sin^{-1} 0\right] = \sin^{-1} 1 - 0 = \frac{\pi}{2} \text{ which is finite.}$$

$$\Rightarrow \quad \int_0^2 \frac{dx}{\sqrt{4-x^2}} \text{ converges to } \frac{\pi}{2}.$$

(*iii*) *Please try yourself.* **[Ans.** Diverges to ∞]

(*iv*) $\frac{\pi}{2}$ is the only point of infinite discontinuity of the integrand on $\left[0, \frac{\pi}{2}\right]$.

$$\therefore \quad \int_0^{\pi/2} \frac{\cos x}{\sqrt{1-\sin x}}dx = \lim_{\varepsilon\to 0+} \int_0^{\frac{\pi}{2}-\varepsilon} -(1-\sin x)^{-1/2}(-\cos x)\,dx = \lim_{\varepsilon\to 0+} \left[-2\sqrt{1-\sin x}\right]_0^{\frac{\pi}{2}-\varepsilon}$$

$$= \lim_{\varepsilon\to 0+} -2\left[\sqrt{1-\sin\left(\frac{\pi}{2}-\varepsilon\right)}-1\right] = -2\left[\sqrt{1-\sin\frac{\pi}{2}}-1\right] = 2$$

$$\Rightarrow \quad \int_0^{\pi/2} \frac{\cos x}{\sqrt{1-\sin x}}dx \text{ converges to 2.}$$

(*v*) 1 is the only point of infinite discontinuity of the integrand on $[0, 1]$.

$$\therefore \quad \int_0^1 \frac{dx}{x^2-3x+2} = \lim_{\varepsilon\to 0+} \int_0^{1-\varepsilon} \frac{dx}{(1-x)(2-x)}$$

$$= \lim_{\varepsilon\to 0+} \int_0^{1-\varepsilon} \left(\frac{1}{1-x}-\frac{1}{2-x}\right)dx = \lim_{\varepsilon\to 0+} \left[-\log(1-x)+\log(2-x)\right]_0^{1-\varepsilon}$$

$$= \lim_{\varepsilon\to 0+} \left[\log\frac{2-x}{1-x}\right]_0^{1-\varepsilon} = \lim_{\varepsilon\to 0+} \left[\log\frac{1+\varepsilon}{\varepsilon} - \log 2\right]$$

$$= \lim_{\varepsilon\to 0+} \log\left(1+\frac{1}{\varepsilon}\right) - \log 2 = \log\infty - \log 2 = \infty.$$

$$\Rightarrow \quad \int_0^1 \frac{dx}{x^2-3x+2} \text{ diverges to } \infty.$$

(*vi*) *Please try yourself.* **[Ans.** Diverges to $-\infty$]

(*vii*) $\frac{\pi}{2}$ is the only point of infinite discontinuity of the integrand on $\left[0, \frac{\pi}{2}\right]$.

$$\therefore \quad \int_0^{\pi/2} \tan\theta\, d\theta = \lim_{\varepsilon\to 0+} \int_0^{\pi/2-\varepsilon} \tan\theta\, d\theta$$

$$= \lim_{\varepsilon\to 0+} \left[\log\sec\theta\right]_0^{\pi/2-\varepsilon} = \lim_{\varepsilon\to 0+} \left[\log\sec\left(\frac{\pi}{2}-\varepsilon\right) - \log 1\right]$$

$$= \lim_{\varepsilon\to 0+} \log\operatorname{cosec}\varepsilon = \log\operatorname{cosec} 0 = \log\infty = \infty$$

$$\Rightarrow \quad \int_0^{\pi/2} \tan\theta\, d\theta \text{ diverges to } \infty.$$

Example 12. *Examine the convergence of the integrals :*

(*i*) $\int_{-1}^1 \frac{dx}{x^2}$ (*ii*) $\int_a^{3a} \frac{dx}{(x-2a)^2}$ (*iii*) $\int_0^{2a} \frac{dx}{(x-a)^2}$.

Sol. (*i*) The integrand becomes infinite at $x = 0$ and $-1 < 0 < 1$

$$\therefore \quad \int_{-1}^1 \frac{dx}{x^2} = \int_{-1}^0 \frac{dx}{x^2} + \int_0^1 \frac{dx}{x^2} = \lim_{\varepsilon_1\to 0+} \int_{-1}^{0-\varepsilon_1} \frac{dx}{x^2} + \lim_{\varepsilon_2\to 0+} \int_{0+\varepsilon_2}^1 \frac{dx}{x^2}$$

so that 0 enclosed within $(-\varepsilon_1, \varepsilon_2)$ is excluded.

$$= \lim_{\varepsilon_1\to 0+} \left[-\frac{1}{x}\right]_{-1}^{-\varepsilon_1} + \lim_{\varepsilon_2\to 0+} \left[-\frac{1}{x}\right]_{\varepsilon_2}^1$$

$$= \lim_{\varepsilon_1\to 0+} \left(\frac{1}{\varepsilon_1}-1\right) + \lim_{\varepsilon_2\to 0+} \left(-1+\frac{1}{\varepsilon_2}\right) = (\infty - 1) + (-1+\infty) = \infty$$

$$\Rightarrow \quad \int_{-1}^1 \frac{dx}{x^2} \text{ diverges to } +\infty.$$

(*ii*) The integrand becomes infinite at $x = 2a$ and $a < 2a < 3a$.

$$\therefore \quad \int_a^{3a} \frac{dx}{(x-2a)^2} = \int_a^{2a} \frac{dx}{(x-2a)^2} + \int_{2a}^{3a} \frac{dx}{(x-2a)^2}$$

$$= \lim_{\varepsilon_1 \to 0+} \int_a^{2a-\varepsilon_1} \frac{dx}{(x-2a)^2} + \lim_{\varepsilon_2 \to 0+} \int_{2a+\varepsilon_2}^{3a} \frac{dx}{(x-2a)^2}$$

$$= \lim_{\varepsilon_1 \to 0+} \left[\frac{-1}{x-2a}\right]_a^{2a-\varepsilon_1} + \lim_{\varepsilon_1 \to 0+} \left[\frac{-1}{x-2a}\right]_{2a+\varepsilon_2}^{3a}$$

$$= \lim_{\varepsilon_1 \to 0+} \left(\frac{1}{\varepsilon_1} - \frac{1}{a}\right) + \lim_{\varepsilon_2 \to 0+} \left(-\frac{1}{a} + \frac{1}{\varepsilon_2}\right) = \left(\infty - \frac{1}{2}\right) + \left(-\frac{1}{a} + \infty\right) = \infty$$

$\Rightarrow \displaystyle\int_a^{3a} \frac{dx}{(x-2a)^2}$ diverges to ∞.

(*iii*) *Please try yourself.* [**Ans.** Diverges to ∞]

Example 13. *Examine the convergence of the integrals :*

(*i*) $\displaystyle\int_0^4 \frac{dx}{x(4-x)}$ (*ii*) $\displaystyle\int_0^2 \frac{dx}{2x-x^2}$ (*iii*) $\displaystyle\int_{-a}^a \frac{x}{\sqrt{a^2-x^2}}\,dx$

(*iv*) $\displaystyle\int_0^\pi \frac{dx}{\sin x}$ (*v*) $\displaystyle\int_0^\pi \frac{dx}{1+\cos x}$

Sol. (*i*) Both the end points 0 and 4 are points of infinite discontinuity of the integrand on [0, 4].

$$\therefore \int_0^4 \frac{dx}{x(4-x)} = \int_0^1 \frac{dx}{x(4-x)} + \int_1^4 \frac{dx}{x(4-x)}$$

$$= \lim_{\varepsilon_1 \to 0+} \int_{0+\varepsilon_1}^1 \frac{1}{4}\left(\frac{1}{x} + \frac{1}{4-x}\right) dx + \lim_{\varepsilon_2 \to 0+} \int_1^{4-\varepsilon_2} \frac{1}{4}\left(\frac{1}{x} + \frac{1}{4-x}\right) dx$$

$$= \lim_{\varepsilon_1 \to 0+} \left[\frac{1}{4}\log\frac{x}{4-x}\right]_{\varepsilon_1}^1 + \lim_{\varepsilon_2 \to 0+} \left[\frac{1}{4}\log\frac{x}{4-x}\right]_1^{4-\varepsilon_2}$$

$$= \lim_{\varepsilon_1 \to 0+} \frac{1}{4}\left(\log\frac{1}{3} - \log\frac{\varepsilon_1}{4-\varepsilon_1}\right) + \lim_{\varepsilon_2 \to 0+} \frac{1}{4}\left(\log\frac{4-\varepsilon_2}{\varepsilon_2} - \log\frac{1}{3}\right)$$

$$= \frac{1}{4}\left[\log\frac{1}{3} - (-\infty)\right] + \frac{1}{4}\left[\infty - \log\frac{1}{3}\right] = \infty$$

$\Rightarrow \displaystyle\int_0^4 \frac{dx}{x(4-x)}$ diverges to ∞.

(*ii*) *Please try yourself.* [**Ans.** Diverges to ∞]

(*iii*) Both the end points $-a$ and a are points of infinite discontinuity of the integrand on $[-a, a]$.

$$\therefore \int_{-a}^a \frac{x}{\sqrt{a^2-x^2}}\,dx = \int_{-a}^0 \frac{x}{\sqrt{a^2-x^2}}\,dx + \int_0^a \frac{x}{\sqrt{a^2-x^2}}\,dx$$

$$= \lim_{\varepsilon_1 \to 0+} \int_{-a+\varepsilon_1}^0 -\frac{1}{2}(a^2-x^2)^{-1/2}(-2x)\,dx$$

$$+ \lim_{\varepsilon_2 \to 0+} \int_0^{a-\varepsilon_2} -\frac{1}{2}(a^2-z^2)^{-1/2}(-2x)\,dx$$

$$= \lim_{\varepsilon_1 \to 0+} \left[-\sqrt{a^2-x^2}\right]_{-a+\varepsilon_1}^0 + \lim_{\varepsilon_2 \to 0+} \left[-\sqrt{a^2-x^2}\right]_0^{a-\varepsilon_2}$$

$$= \lim_{\varepsilon_1 \to 0+} \left[-a + \sqrt{\varepsilon_1(2a-\varepsilon_1)}\right] + \lim_{\varepsilon_2 \to 0+} \left[-\sqrt{\varepsilon_2(2a-\varepsilon_2)} + a\right] = -a + a = 0$$

$\therefore \displaystyle\int_{-a}^a \frac{x}{\sqrt{a^2-x^2}}\,dx$ converges to 0.

(*iv*) Both the end points 0 and π are points of infinite discontinuity of the integrand on $[0, \pi]$.

$$\therefore \int_0^\pi \frac{dx}{\sin x} = \int_0^{\pi/2} \operatorname{cosec} x\,dx + \int_{\pi/2}^\pi \operatorname{cosec} x\,dx$$

$$= \lim_{\varepsilon_1 \to 0+} \int_{0+\varepsilon_1}^{\pi/2} \operatorname{cosec} x\,dx + \lim_{\varepsilon_2 \to 0+} \int_{\pi/2}^{\pi-\varepsilon_2} \operatorname{cosec} x\,dx$$

$$= \lim_{\varepsilon_1 \to 0+} \left[\log\tan\frac{x}{2}\right]_{\varepsilon_1}^{\pi/2} + \lim_{\varepsilon_2 \to 0+} \left[\log\tan\frac{x}{2}\right]_{\pi/2}^{\pi-\varepsilon_2}$$

$$= \lim_{\varepsilon_1 \to 0+} \left[\log\tan\frac{\pi}{4} - \log\tan\frac{\varepsilon_1}{2}\right] + \lim_{\varepsilon_2 \to 0+} \left[\log\tan\left(\frac{\pi}{2} - \frac{\varepsilon_2}{2}\right) - \log\tan\frac{\pi}{4}\right]$$

$$= 0 - (-\infty) + \infty - 0 = \infty$$

$\Rightarrow \displaystyle\int_0^\pi \frac{dx}{\sin x}$ diverges to ∞.

(*v*) π is the only point of infinite discontinuity of the integrand on $[0, \pi]$.

$$\therefore \int_0^\pi \frac{dx}{1+\cos x} = \lim_{\varepsilon \to 0+} \int_0^{\pi-\varepsilon} \frac{dx}{2\cos^2 x/2} = \lim_{\varepsilon \to 0+} \int_0^{\pi-\varepsilon} \frac{1}{2}\sec^2\frac{x}{2}\,dx$$

$$= \lim_{\varepsilon \to 0+} \left[\tan\frac{x}{2}\right]_0^{\pi-\varepsilon} = \lim_{\varepsilon \to 0+} \tan\left(\frac{\pi}{2} - \frac{\varepsilon}{2}\right) = \infty$$

$\Rightarrow \displaystyle\int_0^\pi \frac{dx}{1+\cos x}$ diverges to ∞.

Note. In the examples considered so far, the integrands admit of primitives in terms of elementary functions. In such cases it is easy to test the convergence of integrals. But every function does not posses a primitive in terms of elementary functions. Improper integrals of such functions cannot be examined for convergence by the procedure discussed so far. Thus arises the need for more advanced methods for testing the convergence of such integrals.

11.6. TESTS FOR CONVERGENCE OF $\int_a^b f(x)\,dx$ AT a

Let a be the only point of infinite discontinuity of f on $[a, b]$. The case when b is the only point of infinite discontinuity can be dealt with in the same way.

Without any loss of generality, we assume that f is positive (or non-negative) on $[a, b]$.

In case f is negative, we can replace it by $(-f)$ for testing the convergence of $\int_a^b f(x)\,dx$.

Theorem 1. *A necessary and sufficient condition for the convergence of the improper integral $\int_a^b f(x)\,dx$ at a, where f is positive on (a, b], is that there exists a positive number M, independent of $\varepsilon > 0$ such that*

$$\int_{a+\varepsilon}^b f(x)\,dx < M \ \forall\ \varepsilon \text{ in } (0, b-a).$$

Proof. Since a is the only point of infinite discontinuity of f on $[a, b]$, therefore, f is continuous on $(a, b]$.

Also f is positive on $(a, b]$.

$\Rightarrow$ For $a < a + \varepsilon < b$ *i.e.*, for $0 < \varepsilon < b - a$, f is positive and continuous on $[a + \varepsilon, b]$.

$\Rightarrow$ $\int_{a+\varepsilon}^b f(x)\,dx = A(\varepsilon)$ represents the area bounded by f on $[a + \varepsilon, b]$ and the x-axis.

$\Rightarrow$ As $\varepsilon \to 0+$, *i.e.*, as ε decreases, $A(\varepsilon)$ increases since the length of the interval increases.

$\Rightarrow$ $\lim_{\varepsilon \to 0+} A(\varepsilon) = \lim_{\varepsilon \to 0+} \int_{a+\varepsilon}^b f(x)\,dx$ will exist finitely iff $A(\varepsilon)$ is bounded above.

$\Rightarrow$ $\int_a^b f(x)\,dx$ will converge iff $\exists$ a real number $M > 0$ and independent of ε such that $A(\varepsilon) < M$

$\Rightarrow$ $\int_a^b f(x)\,dx$ converges iff $\int_{a+\varepsilon}^b f(x)\,dx < M \quad \forall\ \varepsilon$ in $(0, b-a)$

Note. If for every $M > 0$ and some ε in $(0, b-a)$. $A(\varepsilon) > M$, then $\int_{a+\varepsilon}^b f(x)\,dx$ is not bounded above.

$\therefore$ $\int_{a+\varepsilon}^b f(x)\,dx$ tends to $+\infty$ as $\varepsilon \to 0+$ and, hence, the improper integral $\int_a^b f(x)\,dx$ diverges to $+\infty$.

Theorem 2. Comparison Test I

If f and g are two positive functions with $f(x) \le g(x)$ for all x in (a, b] and a is the only point of infinite discontinuity on [a, b], then

(*i*) $\int_a^b g(x)\,dx$ *is convergent* $\Rightarrow \int_a^b f(x)\,dx$ *is convergent*

(*ii*) $\int_a^b f(x)\,dx$ *is divergent* $\Rightarrow \int_a^b g(x)\,dx$ *is divergent.*

Proof. Since f and g are positive and $f(x) \le g(x)\ \forall\ x \in (a, b]$

$$\therefore \quad \int_{a+\varepsilon}^b f(x)\,dx \le \int_{a+\varepsilon}^b g(x)\,dx \text{ for } 0 < \varepsilon < b - a$$

(*i*) Let $\int_a^b g(x)\,dx$ be convergent, then there exists a positive number M such that

$$\int_{a+\varepsilon}^b g(x)\,dx < M \text{ for } 0 < \varepsilon < b - a$$

$\therefore$ From (1), $\int_{a+\varepsilon}^b f(x)\,dx < M$ for $0 < \varepsilon < b - a$

Hence $\int_a^b f(x)\,dx$ is convergent.

(*ii*) Let $\int_a^b f(x)\,dx$ be divergent, then for every $M > 0$, there exists ε in $(0, b-a)$ such that

$$\Rightarrow \int_{a+\varepsilon}^b f(x)\,dx > M$$

$\therefore$ From (1), $\int_{a+\varepsilon}^b g(x)\,dx > M$

Hence $\int_a^b g(x)\,dx$ is divergent.

Theorem 3. Comparison Test II (*Limit Form*) —

If f and g be two positive functions on (a, b], a being the only point of infinite discontinuity, and $\lim_{x \to a+} \frac{f(x)}{g(x)} = l$ where l is non-zero finite number, then two integrals $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$ converge or diverge together.

Proof. Since f and g are positive on $(a, b]$

$$\therefore \quad \frac{f(x)}{g(x)} > 0 \text{ on } (a, b] \Rightarrow \lim_{x \to a+} \frac{f(x)}{g(x)} = l \ge 0$$

But $\quad l \ne 0$ (given)

$\therefore \quad l > 0$

Let ε be a positive real number such that $l - \varepsilon > 0$.

Since $\lim_{x \to a+} \frac{f(x)}{g(x)} = l$, therefore, there exists a neighbourhood (a, c), $a < c < b$, such that

$$\left|\frac{f(x)}{g(x)} - l\right| < \varepsilon\ \forall\ x \in (a, c)$$

$$\Rightarrow \quad l - \varepsilon < \frac{f(x)}{g(x)} < l + \varepsilon\ \forall\ x \in (a, c)$$

$$\Rightarrow \quad (l - \varepsilon)\,g(x) < f(x) < (l + \varepsilon)\,g(x) \qquad [\because\ g(x) > 0]$$

$$\Rightarrow \quad k g(x) < f(x) < K g(x) \text{ where } k, K > 0 \qquad \ldots(1)$$

Now,

(i) $\int_a^b f(x)\,dx$ converges at a $\Rightarrow$ $\int_a^c f(x)dx$ converges at a $\left[\because \int_c^b f(x)\,dx \text{ is proper}\right]$

Since $k\,g(x) < f(x)\ \forall\ x \in (a, c)$ [From (1)]

$\therefore$ $k\int_a^c g(x)\,dx$ converges at a [By comparison test I]

$\Rightarrow$ $\int_a^c g(x)\,dx$ converges at a

$\Rightarrow$ $\int_a^b g(x)\,dx$ converges at a. $\left[\because \int_c^b g(x)\,dx \text{ is proper}\right]$

(ii) $\int_a^b f(x)\,dx$ diverges at a $\Rightarrow$ $\int_a^c f(x)\,dx$ diverges at a $\left[\because \int_c^b f(x)\,dx \text{ is proper}\right]$

Since $K\,g(x) > f(x)\ \forall\ x \in (a, c)$ [From (1)]

$\therefore$ $K\int_a^c g(x)\,dx$ diverges at a $\Rightarrow$ $\int_a^c g(x)\,dx$ diverges at a

$\Rightarrow$ $\int_a^b g(x)\,dx$ diverges at a $\left[\because \int_c^b g(x)\,dx \text{ is proper}\right]$

It can similarly be shown that

$\int_a^b g(x)\,dx$ converges at a $\Rightarrow$ $\int_a^b f(x)\,dx$ diverges at a.

and $\int_a^b g(x)\,dx$ diverges at a $\Rightarrow$ $\int_a^b f(x)\,dx$ diverges at a.

Theorem 4. *Let f and g be two positive functions on (a, b], a being the only point of infinite discontinuity. Then*

(i) $\lim\limits_{x \to a+} \dfrac{f(x)}{g(x)} = 0$ *and* $\int_a^b g(x)\,dx$ *converges* $\Rightarrow$ $\int_a^b f(x)\,dx$ *converges*

(ii) $\lim\limits_{x \to a+} \dfrac{f(x)}{g(x)} = \infty$ *and* $\int_a^b g(x)\,dx$ *diverges* $\Rightarrow$ $\int_a^b f(x)\,dx$ *diverges.*

Proof. (i) $\lim\limits_{x \to a+} \dfrac{f(x)}{g(x)} = 0$

$\Rightarrow$ given $\varepsilon > 0$, $\exists$ *a nbd* (a, c), $a < c < b$ such that

$$\left|\frac{f(x)}{g(x)}\right| < \varepsilon\ \ \forall\ x \in (a, c) \quad \Rightarrow \quad -\varepsilon < \frac{f(x)}{g(x)} < \varepsilon\ \ \forall\ x \in (a, c)$$

$$\Rightarrow \quad 0 < \frac{f(x)}{g(x)} < \varepsilon\ \ \forall\ x \in (a, c) \Rightarrow f(x) < \varepsilon g(x)\ \forall\ x \in (a, c)$$

Now $\int_a^b g(x)\,dx$ converges at a $\Rightarrow$ $\int_a^c g(x)\,dx$ converges at a

$\Rightarrow$ $\int_a^c \varepsilon\, g(x)\,dx$ converges at a

Since $f(x) < \varepsilon\, g(x)\ \forall\ x \in (a, c)$

$\therefore$ $\int_a^c f(x)\,dx$ converges at a $\Rightarrow$ $\int_a^b f(x)$ converges at a.

(ii) $\lim\limits_{x \to a+} \dfrac{f(x)}{g(x)} = \infty$

$\Rightarrow$ Given a real number $K > 0$, $\exists$ a nbd (a, c), $a < c < b$ such that

$$\frac{f(x)}{g(x)} > K\ \ \forall\ x \in (a, c) \quad \Rightarrow \quad f(x) > Kg(x)\ \ \forall\ x \in (a, c)$$

Now $\int_a^b g(x)\,dx$ diverges at a $\Rightarrow$ $\int_a^c g(x)\,dx$ diverges at a

$\Rightarrow$ $\int_a^c Kg(x)\,dx$ diverges at a

Since $f(x) > Kg(x)\ \ \forall\ x \in (a, c)$ $\therefore$ $\int_a^c f(x)\,dx$ diverges at a

$\Rightarrow$ $\int_a^b f(x)\,dx$ diverges at a.

Theorem 5. Useful Comparison Integrals

(i) *The improper integral* $\int_a^b \dfrac{dx}{(x-a)^n}$ *is convergent if and only if* $n < 1$.

(ii) *The improper integral* $\int_a^b \dfrac{dx}{(b-x)^n}$ *is convergent if and only if* $n < 1$.

Proof. (i) If $n \le 0$, the integral $\int_a^b \dfrac{dx}{(x-a)^n}$ is proper.

If $n > 0$, the integral is improper and a is the only point of infinite discontinuity of the integrand on $[a, b]$.

Case I. When $n = 1$

$$\int_a^b \frac{dx}{(x-a)^n} = \int_a^b \frac{dx}{x-a} = \lim_{\varepsilon \to 0+} \int_{a+\varepsilon}^b \frac{dx}{x-a}$$

$$= \lim_{\varepsilon \to 0+} \left[\log(x-a)\right]_{a+\varepsilon}^b = \lim_{\varepsilon \to 0+} [\log(b-a) - \log \varepsilon]$$

$$= \log(b-a) - (-\infty) = \infty$$

$\Rightarrow$ $\int_a^b \dfrac{dx}{(x-a)^n}$ diverges if $n = 1$.

Case II. When $n \neq 1$.

$$\int_a^b \frac{dx}{(x-a)^n} = \lim_{\varepsilon \to 0+} \int_{a+\varepsilon}^b (x-a)^{-n}\,dx = \lim_{\varepsilon \to 0+} \left[\frac{(x-a)^{1-n}}{1-n}\right]_{a+\varepsilon}^b$$

$$= \lim_{\varepsilon \to 0+} \frac{1}{1-n}\left[(b-a)^{1-n} - \varepsilon^{1-n}\right].$$

Sub-Case 1. When $n > 1$ so that $n - 1 > 0$

$$\int_a^b \frac{dx}{(x-a)^n} = \lim_{\varepsilon \to 0+} \frac{1}{n-1}\left[\frac{1}{\varepsilon^{n-1}} - \frac{1}{(b-a)^{n-1}}\right] = \frac{1}{n-1}\left[\infty - \frac{1}{(b-a)^{n-1}}\right] = \infty$$

$\Rightarrow \quad \int_a^b \frac{dx}{(x-a)^n}$ diverges if $n > 1$.

Sub-Case 2. When $0 < n < 1$ so that $1 - n > 0$

$$\int_a^b \frac{dx}{(x-a)^n} = \lim_{\varepsilon \to 0+} \frac{1}{1-n}[(b-a)^{1-n} - \varepsilon^{1-n}] = \frac{(b-a)^{1-n}}{1-n} \text{ which is finite.}$$

$\Rightarrow \quad \int_a^b \frac{dx}{(x-a)^n}$ converges if $n < 1$.

Hence $\quad \int_a^b \frac{dx}{(x-a)^n}$ is convergent if and only if $n < 1$.

(*ii*) If $n \le 0$, the integral $\int_a^b \frac{dx}{(b-x)^n}$ is proper.

If $n > 0$, the integral is improper and b is the only point of infinite discontinuity of the integrand on $[a, b]$.

Case I. When $n = 1$

$$\int_a^b \frac{dx}{(b-x)^n} = \int_a^b \frac{dx}{b-x} = \lim_{\varepsilon \to 0+} \int_a^{b-\varepsilon} \frac{dx}{b-x} = \lim_{\varepsilon \to 0+} \left[\frac{\log(b-x)}{-1}\right]_a^{b-\varepsilon}$$

$$= \lim_{\varepsilon \to 0+} [-\log \varepsilon + \log(b-a)] = -(-\infty) + \log(b-a) = \infty$$

$\Rightarrow \quad \int_a^b \frac{dx}{(b-x)^n}$ diverges if $n = 1$.

Case II. When $n \ne 1$

$$\int_a^b \frac{dx}{(b-x)^n} = \lim_{\varepsilon \to 0+} \int_a^{b-\varepsilon} (b-x)^{-n}\, dx$$

$$= \lim_{\varepsilon \to 0+} \left[\frac{(b-x)^{1-n}}{(1-n)(-1)}\right]_a^{b-\varepsilon} = \lim_{\varepsilon \to 0+} \frac{1}{n-1}[\varepsilon^{1-n} - (b-a)^{1-n}]$$

Sub-Case 1. When $n > 1$ so that $n - 1 > 0$

$$\Rightarrow \quad \int_a^b \frac{dx}{(b-x)^n} = \lim_{\varepsilon \to 0+} \frac{1}{n-1}\left[\frac{1}{\varepsilon^{n-1}} - \frac{1}{(b-a)^{n-1}}\right] = \frac{1}{n-1}\left[\infty - \frac{1}{(b-a)^{n-1}}\right] = \infty$$

$\Rightarrow \quad \int_a^b \frac{dx}{(b-x)^n}$ diverges if $n > 1$.

Sub-Case 2. When $0 < n < 1$ so that $1 - n > 0$

$$\int_a^b \frac{dx}{(b-x)^n} = \lim_{\varepsilon \to 0+} \frac{1}{1-n}[(b-a)^{1-n} - \varepsilon^{1-n}] = \frac{(b-a)^{1-n}}{1-n} \text{ which is finite.}$$

$\Rightarrow \quad \int_a^b \frac{dx}{(b-x)^n}$ converges if $n < 1$.

Hence $\quad \int_a^b \frac{dx}{(b-x)^n}$ is convergent if and only if $n < 1$.

Theorem 6. (*i*) *If a is the only point of infinite discontinuity of f on $[a, b]$ and $\lim_{x \to a+} (x-a)^\mu f(x)$ exists and is non-zero finite, then $\int_a^b f(x)\, dx$ converges if and only if $\mu < 1$.*

(*ii*) *If b is the only point of infinite discontinuity of f on $[a, b]$ and $\lim_{x \to b+} (b-x)^\mu f(x)$ exists and is non-zero finite, then $\int_a^b f(x)\, dx$ converges if and only if $\mu < 1$.*

Proof. (*i*) Let $\quad g(x) = \frac{1}{(x-a)^\mu}$ then $\lim_{x \to a+} \frac{f(x)}{g(x)} = \lim_{x \to a+} (x-a)^\mu f(x)$

which exists and is non-zero finite. (given)

$\therefore$ By comparison test II, the two integrals $\int_a^b f(x)\, dx$ and $\int_a^b g(x)\, dx$ converge or diverge together.

But $\quad \int_a^b g(x)\, dx = \int_a^b \frac{dx}{(x-a)^\mu}$ converges iff $\mu < 1$.

$\therefore \quad \int_a^b f(x)\, dx$ converges iff $\mu < 1$.

(*ii*) *Please try yourself.*

ILLUSTRATIVE EXAMPLES

Example 1. *Examine the convergence of the integrals :*

(*i*) $\int_0^1 \frac{dx}{\sqrt{x^2 + x}}$ $\quad$ (*ii*) $\int_1^2 \frac{dx}{(1+x)\sqrt{2-x}}$ $\quad$ (*iii*) $\int_0^1 \frac{dx}{\sqrt{1-x^3}}$ $\quad$ (*iv*) $\int_0^1 \frac{dx}{x^{1/2}(1+x^2)}$

Sol. (*i*) Here $\quad f(x) = \frac{1}{\sqrt{x^2 + x}} = \frac{1}{\sqrt{x}\sqrt{x+1}}$

0 is the only point of infinite discontinuity of f on $[0, 1]$.

Take $g(x) = \frac{1}{\sqrt{x}}$, then $\lim_{x \to 0+} \frac{f(x)}{g(x)} = \lim_{x \to 0+} \frac{1}{\sqrt{x+1}} = 1$ which is non-zero and finite.

$\therefore$ By comparison test, $\int_0^1 f(x)\, dx$ and $\int_0^1 g(x)\, dx$ converge or diverge together.

But $\int_0^1 g(x)\,dx = \int_0^1 \frac{dx}{\sqrt{x}}$ $\left| \text{Form } \int_a^b \frac{dx}{(x-a)^n} \text{ with } a = 0 \right.$

converges $\left(\because\ n = \frac{1}{2} < 1 \right)$

$\therefore$ $\int_0^1 f(x)\,dx = \int_0^1 \frac{dx}{\sqrt{x^2+x}}$ is convergent.

(*ii*) Here $f(x) = \frac{1}{(1+x)\sqrt{2-x}}$

2 is the only point of infinite discontinuity of f on $[1, 2]$.

Take $g(x) = \frac{1}{\sqrt{2-x}}$, then $\lim_{x\to 2-} \frac{f(x)}{g(x)} = \lim_{x \to 2-} \frac{1}{1+x} = \frac{1}{3}$ which is non-zero and finite.

$\therefore$ By comparison test, $\int_1^2 f(x)\,dx$ and $\int_1^2 g(x)\,dx$ converge or diverge together.

But $\int_1^2 g(x)\,dx = \int_1^2 \frac{dx}{\sqrt{2-x}}$ $\left| \text{Form } \int_a^b \frac{dx}{(b-x)^n} \text{ with } b = 2 \right.$

converges. $\left(\because\ n = \frac{1}{2} < 1 \right)$

$\therefore$ $\int_1^2 f(x)\,dx = \int_1^2 \frac{dx}{(1+x)\sqrt{2-x}}$ is convergent.

(*iii*) Here $f(x) = \frac{1}{\sqrt{1-x^3}} = \frac{1}{\sqrt{1-x}\sqrt{1+x+x^2}}$

1 is the only point of infinite discontinuity of f on $[0, 1]$

Take $g(x) = \frac{1}{\sqrt{1-x}}$, then $\lim_{x\to 1-} \frac{f(x)}{g(x)} = \lim_{x\to 1-} \frac{1}{\sqrt{1+x+x^2}} = \frac{1}{\sqrt{3}}$ which is non-zero and finite.

$\therefore$ By comparison test, $\int_0^1 f(x)\,dx$ and $\int_0^1 g(x)\,dx$ converge or diverge together

But $\int_0^1 g(x)\,dx = \int_0^1 \frac{dx}{\sqrt{1-x}}$ $\left[\text{Form } \int_a^b \frac{dx}{(b-x)^n} \text{ with } b = 1 \right.$

converges.

$\therefore$ $\int_0^1 f(x)\,dx = \int_0^1 \frac{dx}{\sqrt{1-x^2}}$ is convergent. $\left(\because\ n = \frac{1}{2} < 1 \right)$

(*iv*) *Please try yourself.* **[Ans.** Convergent**]**

Example 2. *Examine the convergence of the integrals :*

(*i*) $\int_0^1 \frac{dx}{x^3(2+x^2)^5}$ (*ii*) $\int_0^1 \frac{dx}{\sqrt{x}(1+x)^2}$

(*iii*) $\int_0^1 \frac{dx}{(1+x)^2(1-x)^3}$ (*iv*) $\int_0^1 \frac{dx}{\sqrt{x(1-x)}}$.

Sol. (*i*) Here $f(x) = \frac{1}{x^3(2+x^2)^5}$

0 is the only point of infinite discontinuity of f on $[0, 1]$.

Take $g(x) = \frac{1}{x^3}$, then $\lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} \frac{1}{(2+x^2)^5} = \frac{1}{32}$ which is non-zero and finite.

$\therefore$ By comparison test, $\int_0^1 f(x)\,dx$ and $\int_0^1 g(x)\,dx$ converge or diverge together.

But $\int_0^1 g(x)\,dx = \int_0^1 \frac{dx}{x^3}$ $\left| \text{Form } \int_a^b \frac{dx}{(x-a)^n} \text{ with } a = 0 \text{ diverges } (\because\ n = 3 > 1) \right.$

$\therefore$ $\int_0^1 f(x)\,dx = \int_0^1 \frac{dx}{x^3(2+x^2)^5}$ is divergent.

(*ii*) *Please try yourself.* **[Ans.** Convergent**]**

(*iii*) Here $f(x) = \frac{1}{(1+x)^2(1-x)^3}$

1 is the only point of infinite discontinuity of f on $[0, 1]$.

Take $g(x) = \frac{1}{(1-x)^3}$, then $\lim_{x\to 1-} \frac{f(x)}{g(x)} = \lim_{x\to 1-} \frac{1}{(1+x^2)} = \frac{1}{2}$ which is non-zero and finite.

$\therefore$ By comparison test, $\int_0^1 f(x)\,dx$ and $\int_0^1 g(x)\,dx$ converge or diverge together.

But $\int_0^1 g(x)\,dx = \int_0^1 \frac{dx}{(1-x)^3}$ $\left[\text{Form } \int_a^b \frac{dx}{(b-x)^n} \text{ with } b = 1 \right]$ diverges. $(\because\ n = 3 > 1)$

$\therefore$ $\int_0^1 f(x)\,dx = \int_0^1 \frac{dx}{(1+x)^2(1-x)^2}$ is divergent.

(*iv*) Here $f(x) = \frac{1}{\sqrt{x(1-x)}}$

Both the end points 0 and 1 are the points of infinite discontinuity of f on $[0, 1]$.

We may write $\int_0^1 \frac{dx}{\sqrt{x(1-x)}} = \int_0^a \frac{dx}{\sqrt{x(1-x)}} + \int_a^1 \frac{dx}{\sqrt{x(1-x)}}$ ...(1)

where $0 < a < 1$.

To examine the convergence at x = 0

Let $I_1 = \int_0^a \frac{dx}{\sqrt{x(1-x)}}$

0 is the only point of infinite discontinuity of f on $[0, a]$.

Take $g(x) = \frac{1}{\sqrt{x}}$, then $\lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} \frac{1}{\sqrt{1-x}} = 1$ which is non-zero and finite.

$\therefore$ By comparison test, I_1 and $\int_0^a g(x)\,dx$ converge or diverge together.

But $\int_0^a g(x)\,dx = \int_0^a \frac{dx}{\sqrt{x}}$ is convergent. $\left(\because n = \frac{1}{2} < 1\right)$

$\therefore$ I_1 is convergent.

To examine the convergence at x = 1

Let $I_2 = \int_a^1 \frac{dx}{\sqrt{x(1-x)}}$

1 is the only point of infinite discontinuity of f on $[a, 1]$.

Take $g(x) = \frac{1}{\sqrt{1-x}}$, then $\lim_{x\to 1-} \frac{f(x)}{g(x)} = \lim_{x\to 1-} \frac{1}{\sqrt{x}} = 1$ which is non-zero and finite.

$\therefore$ By comparison text, I_2 and $\int_a^1 g(x)\,dx$ converge or diverge together.

But $\int_a^1 g(x)\,dx = \int_a^1 \frac{dx}{\sqrt{1-x}}$ is convergent. $\left(\because n = \frac{1}{2} < 1\right)$

$\therefore$ I_2 is convergent.

Since I_1 and I_2 are **BOTH** convergent, therefore, from (1), $\int_0^1 \frac{dx}{\sqrt{x(1-x)}}$ is convergent.

Note. If I_1 or I_2 is divergent, then $\int_0^1 f(x)\,dx$ is divergent.

Example 3. *Examine the convergence of the integrals :*

(*i*) $\int_2^3 \frac{dx}{(x-2)^{1/4}(3-x)^2}$ (*ii*) $\int_0^1 \frac{dx}{x^{1/2}(1-x)^{1/3}}$ (*iii*) $\int_a^b \frac{dx}{(x-a)\sqrt{b-x}}$.

Sol. (*i*) Here $f(x) = \frac{1}{(x-2)^{1/4}(3-x)^2}$

2 and 3 are the only points of infinite discontinuity of f on $[2, 3]$, we may write

$$\int_2^3 f(x)\,dx = \int_2^a f(x)\,dx + \int_a^3 f(x)\,dx, \text{ where } 2 < a < 3 \quad \ldots(1)$$

To test the convergence of $\int_2^a f(x)\,dx$ at x = 2

Take $g(x) = \frac{1}{(x-2)^{1/4}}$

$$\lim_{x\to 2+} \frac{f(x)}{g(x)} = \lim_{x\to 2+} \frac{1}{(3-x)^2} = 1 \text{ which is non-zero and finite.}$$

$\therefore$ By comparison test, the integrals $\int_2^a f(x)\,dx$ and $\int_2^a g(x)\,dx$ converge or diverge together.

But $\int_2^a g(x)\,dx = \int_2^a \frac{dx}{(x-2)^{1/4}}$ | Form $\int_a^b \frac{dx}{(x-a)^n}$ is convergent.

$\left(\because n = \frac{1}{4} < 1\right)$

$\therefore$ $\int_2^a f(x)\,dx$ is convergent.

To test the convergence of $\int_a^3 f(x)\,dx$ at x = 3

Take $g(x) = \frac{1}{(3-x)^2}$

$$\lim_{x\to 3-} \frac{f(x)}{g(x)} = \lim_{x\to 3-} \frac{1}{(x-2)^{1/4}} = 1 \text{ which is non-zero and finite.}$$

$\therefore$ By comparison test, the integrals $\int_a^3 f(x)\,dx$ and $\int_a^3 g(x)\,dx$ converge or diverge together.

But $\int_a^3 g(x)\,dx = \int_a^3 \frac{dx}{(3-x)^2}$ | Form $\int_a^b \frac{dx}{(b-x)^n}$ is divergent.

$\therefore$ $\int_a^3 f(x)\,dx$ is divergent. $(\because n = 2 > 1)$

Hence, from (1) $\int_2^3 f(x)\,dx$ is divergent.

(*ii*) *Please try yourself.* **[Ans.** Convergent]

(*iii*) *Please try yourself.* **[Ans.** Divergent]

Example 4. *Examine the convergence of*

(*i*) $\int_0^1 \frac{x^n}{1-x}\,dx$ (*ii*) $\int_0^1 \frac{x^n}{1+x}\,dx$ (*iii*) $\int_1^2 \frac{x^\lambda}{x-1}\,dx$ (*iv*) $\int_2^3 \frac{x^2+1}{x^2-4}\,dx$.

Sol. (*i*) Here $f(x) = \frac{x^n}{1-x}$.

If n ≥ 0, then 1 is the only point of infinite discontinuity of f on $[0, 1]$.

Take $g(x) = \frac{1}{1-x}$

$$\lim_{x\to 1-} \frac{f(x)}{g(x)} = \lim_{x\to 1-} x^n = 1 \text{ which is non-zero and finite.}$$

$\therefore$ By comparison test, the integrals $\int_0^1 f(x)\,dx$ and $\int_0^1 g(x)\,dx$ converge of diverge together.

But $\int_0^1 g(x)\,dx = \int_0^1 \frac{dx}{1-x}$ $\quad\Big|$ Form $\int_a^b \frac{dx}{(b-x)^n}$ is divergent. $(\because\ n = 1)$

$\therefore$ $\int_0^1 f(x)\,dx$ is divergent.

If n < 0, let $n = -m$ where $m > 0$.

Then $$f(x) = \frac{1}{x^m(1-x)}$$

0 and 1 both are the points of infinite discontinuity of f on [0, 1]. We may write

$$\int_0^1 f(x)dx = \int_0^a f(x)dx + \int_a^1 f(x)dx \text{ where } 0 < a < 1 \qquad ...(1)$$

To test the convergence of $\int_0^a f(x)\,dx$ at x = 0.

Take $$g(x) = \frac{1}{x^m}$$

$$\lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} \frac{1}{1-x} = 1 \text{ which is finite and non-zero.}$$

$\therefore$ By comparison test, the integrals $\int_0^a f(x)\,dx$ and $\int_0^a g(x)\,dx$ converge or diverge together.

But $\int_0^a g(x)\,dx = \int_0^a \frac{dx}{x^m}$ $\quad\Big|$ Form $\int_a^b \frac{dx}{(x-a)^n}$

is convergent if $0 < m < 1$ and divergent if $m \ge 1$.

$\therefore$ $\int_0^a f(x)\,dx$ is convergent if $-1 < n < 0$ and divergent if $n \le -1$.

To test the convergence of $\int_a^1 f(x)\,dx$ at x = 1.

Take $$g(x) = \frac{1}{1-x}$$

$$\lim_{x\to 1-} \frac{f(x)}{g(x)} = \lim_{x\to 1-} \frac{1}{x^m} = 1 \text{ which is finite and non zero.}$$

$\therefore$ By comparison test, the integrals $\int_a^1 f(x)\,dx$ and $\int_a^1 g(x)\,dx$ converge or diverge together.

But $\int_a^1 g(x)\,dx = \int_a^1 \frac{dx}{1-x}$ $\quad\Big|$ Form $\int_a^b \frac{dx}{(b-x)^n}$ is divergent. $(\because\ n = 1)$

$\therefore$ $\int_a^1 f(x)\,dx$ is divergent.

From (1), $\int_0^1 f(x)\,dx$ is divergent.

Hence $\int_0^1 f(x)\,dx$ is divergent for all $n \in$ R

Note. After a little practice, there is no need testing the convergence of $\int_0^a f(x)\,dx$ at $x = 0$, since divergence of $\int_a^1 f(x)\,dx$ is sufficient to imply divergence of $\int_0^1 f(x)\,dx$.

(*ii*) Here $$f(x) = \frac{x^n}{1+x}$$

If n ≥ 0, $\int_0^1 f(x)\,dx$ is proper and, hence, convergent.

If n < 0, let $n = -m$ where $m > 0$.

Then $$f(x) = \frac{1}{x^m(1+x)}$$

0 is the only point of infinite discontinuity of f on [0, 1].

Take $$g(x) = \frac{1}{x^m}$$

$$\lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} \frac{1}{1+x} = 1 \text{ which is non-zero and finite.}$$

$\therefore$ By comparison test, the integrals $\int_0^1 f(x)\,dx$ and $\int_0^1 g(x)\,dx$ converge or diverge together.

But $\int_0^1 g(x)\,dx = \int_0^1 \frac{dx}{x^m}$ $\Big|$ Form $\int_a^b \frac{dx}{(x-a)^n}$

is convergent if $0 < m < 1$ *i.e.*, $-1 < n < 0$ and divergent if $m \ge 1$ *i.e.*, $n \le -1$.

$\therefore$ $\int_0^1 f(x)\,dx$ is convergent if $-1 < n < 0$ and divergent if $n \le -1$.

Hence $\int_0^1 f(x)\,dx$ is convergent if $n > -1$ and divergent if $n \le -1$.

(*iii*) [**Hint.** $f(x) = \frac{x^\lambda}{x-1}$

For all values of $\lambda \in$ R, 1 is the only point of infinite discontinuity of f on [1, 2].

Take $g(x) = \frac{1}{x-1}$ etc.] **[Ans. Divergent]**

(*iv*) $$\int_2^3 \frac{x^2+1}{x^2-4}dx = \int_2^3 \frac{(x^2-4)+5}{x^2-4}\,dx$$

$$= \int_2^3 \left(1 + \frac{5}{x^2-4}\right)dx = \Big[\,x\,\Big]_2^3 + 5\int_2^3 \frac{dx}{x^2-4} = 1 + 5\int_2^3 \frac{dx}{x^2-4} \qquad ...(1)$$

Let $f(x) = \dfrac{1}{x^2-4} = \dfrac{1}{(x+2)(x-2)}$

2 is the only point of infinite discontinuity of f on [2, 3].

Take $g(x) = \dfrac{1}{x-2}$

$$\lim_{x\to 2+} \frac{f(x)}{g(x)} = \lim_{x\to 2+} \frac{1}{x+2} = \frac{1}{4} \text{ which is non-zero and finite.}$$

$\therefore$ By comparison test, $\int_2^3 f(x)\,dx$ and $\int_2^3 g(x)\,dx$ converge or diverge together.

But $\int_2^3 g(x)\,dx = \int_2^3 \dfrac{dx}{x-2}$ | Form $\int_a^b \dfrac{dx}{(x-a)^n}$ is divergent. $(\because\ n = 1)$

$\therefore$ $\int_2^3 f(x)\,dx$ is divergent.

Hence, from (1), $\int_2^3 \dfrac{x^2+1}{x^2-4}\,dx$ is divergent.

Example 5. *Examine the convergence of*

(i) $\int_0^2 \dfrac{\log x}{\sqrt{2-x}}\,dx$ (ii) $\int_0^1 \dfrac{\log x}{\sqrt{x}}\,dx$ (iii) $\int_1^2 \dfrac{\sqrt{x}}{\log x}\,dx$.

Sol. (i) Here $f(x) = \dfrac{\log x}{\sqrt{2-x}}$

Clearly both 0 and 2 are points of infinite discontinuity of f on [0, 2]. We may write

$$\int_0^2 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx \qquad ...(1)$$

To test the convergence of $\int_0^1 f(x)\,dx$ at $x = 0$

Since $f(x)$ is negative on (0, 1], we consider $-f(x)$.

Take $g(x) = \dfrac{1}{x^n}$

$$\lim_{x\to 0+} \frac{-f(x)}{g(x)} = \lim_{x\to 0+} \frac{-x^n \log x}{\sqrt{2-x}} = 0 \text{ if } n > 0 \qquad \left[\because \lim_{x\to 0+} x^n \log x = 0 \text{ if } n > 0\right]$$

$\therefore$ Taking n between 0 and 1, the integral $\int_0^1 g(x)\,dx$ is convergent.

$\therefore$ By comparison test, $\int_0^1 -f(x)\,dx$ is also convergent.

To test the convergence of $\int_1^2 f(x)\,dx$ at $x = 2$

Take $g(x) = \dfrac{1}{\sqrt{2-x}}$

$$\lim_{x\to 2-} \frac{f(x)}{g(x)} = \lim_{x\to 2-} \log x = \log 2 \text{ which is non-zero and finite.}$$

$\therefore$ By comparison test, $\int_1^2 f(x)\,dx$ and $\int_1^2 g(x)\,dx$ converge or diverge together.

But $\int_1^2 g(x)\,dx = \int_1^2 \dfrac{dx}{\sqrt{2-x}}$ | Form $\int_a^b \dfrac{dx}{(b-x)^n}$ is convergent. $\left(\because\ n = \dfrac{1}{2} < 1\right)$

$\therefore$ $\int_1^2 f(x)\,dx$ is also convergent.

Hence, from (1), $\int_0^2 f(x)\,dx$ is convergent.

(ii) Since $\dfrac{\log x}{\sqrt{x}}$ is negative on (0, 1], we take $f(x) = -\dfrac{\log x}{\sqrt{x}}$

Here 0 is the only point of infinite dicontinuity of f on [0, 1].

Take $g(x) = \dfrac{1}{x^n}$

$$\lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} -x^{n-1/2} \log x = 0 \text{ if } n - \frac{1}{2} > 0 \text{ i.e., if } n > \frac{1}{2} \left[\because \lim_{x\to 0+} x^n \log x = 0 \text{ if } n > 0\right]$$

Taking n between $\dfrac{1}{2}$ and 1, the integral $\int_0^1 g(x)\,dx$ is convergent.

$\therefore$ By comparison test, $\int_0^1 f(x)\,dx$ is also convergent.

Hence $\int_0^1 \dfrac{\log x}{\sqrt{x}}\,dx$ is convergent.

(iii) Here $f(x) = \dfrac{\sqrt{x}}{\log x}$

1 is the only point of infinite discontinuity of f on [1, 2].

Take $g(x) = \dfrac{1}{(x-1)^n}$

$$\lim_{x\to 1+} \frac{f(x)}{g(x)} = \lim_{x\to 1+} \frac{(x-1)^n \sqrt{x}}{\log x} \qquad \left|\text{Form } \frac{0}{0}\right.$$

$$= \lim_{x\to 1+} \frac{n(x-1)^{n-1}\sqrt{x} + \dfrac{(x-1)^n}{2\sqrt{x}}}{1/x}$$

$$= \lim_{x\to 1+} (x-1)^{n-1}\left[nx^{3/2} + \frac{(x-1)}{2}.\sqrt{x}\right] = 1 \text{ if } n = 1$$

Taking $n = 1$, $\int_1^2 g(x)\,dx = \int_1^2 \dfrac{dx}{x-1}$ | Form $\int_a^b \dfrac{dx}{(x-a)^n}$ is divergent. $(\because\ n = 1)$

Since $\lim\limits_{x \to 1+} \dfrac{f(x)}{g(x)} = 1$ which is non-zero and finite.

$\therefore$ By comparison test, $\int_1^2 f(x)\, dx$ is also divergent.

Example 6. *Examine the convergence of*

(*i*) $\displaystyle\int_0^1 \frac{\log x}{1+x}\, dx$ (*ii*) $\displaystyle\int_0^1 \frac{\log x}{1+x^2}\, dx$ (*iii*) $\displaystyle\int_0^2 \frac{\log x}{2-x}\, dx$

(*iv*) $\displaystyle\int_0^1 \frac{\log x}{1-x^2}\, dx$ (*v*) $\displaystyle\int_0^1 \frac{\log x}{\sqrt{1-x^2}}\, dx.$

Sol. (*i*) Since $\dfrac{\log x}{1+x}$ is negative on (0, 1], we take $f(x) = -\dfrac{\log x}{1+x}$

Here 0 is the only point of infinite discontinuity of f on [0, 1].

Take $g(x) = \dfrac{1}{x^n}$

$$\lim_{x \to 0+} \frac{f(x)}{g(x)} = \lim_{x \to 0+} -\frac{x^n \log x}{1+x} = 0 \text{ if } n > 0$$

Taking n between 0 and 1, the integral $\int_0^1 g(x)\, dx$ is convergent.

$\therefore$ By comparison test, $\int_0^1 f(x)\, dx$ is convergent.

Hence, $\displaystyle\int_0^1 \frac{\log x}{1+x}\, dx$ is convergent.

(*ii*) *Please try yourself.* **[Ans.** Convergent]

(*iii*) *Please try yourself.* **[Ans.** Divergent]

[Hint. See example 5 (*i*).

$\int_0^1 f(x)\, dx$ is convergent and $\int_1^2 f(x)\, dx$ is divergent.]

(*iv*) Since $\dfrac{\log x}{1-x^2}$ is negative on (0, 1], we take $f(x) = -\dfrac{\log x}{1-x^2}$

$$\lim_{x \to 1-} f(x) = \lim_{x \to 1-} -\frac{\log x}{1-x^2} \qquad \left| \text{Form } \frac{0}{0} \right.$$
$$= \lim_{x \to 1-} -\frac{1/x}{-2x} = \frac{1}{2}$$

$\therefore$ 0 is the only point of infinite discontinuity of f on [0, 1].

Take $g(x) = \dfrac{1}{x^n}$

$$\lim_{x \to 0+} \frac{f(x)}{g(x)} = \lim_{x \to 0+} -\frac{x^n \log x}{1-x^2} = 0 \text{ if } n > 0$$

Taking n between 0 and 1, the integral $\int_0^1 g(x)\, dx$ is convergent.

$\therefore$ By comparison test, $\int_0^1 f(x)\, dx$ is convergent.

Hence $\displaystyle\int_0^1 \frac{\log x}{1-x^2}\, dx$ is convergent.

(*v*) *Please try yourself.* **[Ans.** Convergent]

[Hint. 0 is the only point of infinite discontinuity since $\lim\limits_{x \to 1-} f(x) = 0$.]

Example 7. *Examine the convergence of*

(*i*) $\displaystyle\int_0^1 \frac{x^n \log x}{(1+x)^2}$ (*ii*) $\displaystyle\int_0^1 \frac{x^p \log x}{1+x^2}$

(*iii*) $\displaystyle\int_0^1 \frac{(x^p + x^{-p}) \log(1+x)}{x}\, dx$ (*iv*) $\displaystyle\int_0^1 x^{n-1} \log x\, dx.$

Sol. (*i*) $\displaystyle\lim_{x \to 0+} \frac{x^n \log x}{(1+x)^2} = 0 \text{ if } n > 0$

$\therefore$ $\displaystyle\int_0^1 \frac{x^n \log x}{(1+x)^2}\, dx$ *is proper and, hence, convergent so long as $n > 0$.*

If n = 0, let $f(x) = -\dfrac{\log x}{(1+x)^2}$

0 is the only point of infinite discontinuity.

Take $g(x) = \dfrac{1}{x^p}$

$$\lim_{x \to 0+} \frac{f(x)}{g(x)} = \lim_{x \to 0+} -\frac{x^p \log x}{(1+x)^2} = 0 \text{ if } p > 0$$

Taking p between 0 and 1, $\int_0^1 g(x)\, dx$ is convergent.

$\Rightarrow$ $\int_0^1 f(x)\, dx$ is convergent. $\Rightarrow$ $\displaystyle\int_0^1 \frac{x^n \log x}{(1+x)^2}\, dx$ is convergent.

If n < 0, let $n = -m$ where $m > 0$

Let $f(x) = -\dfrac{x^n \log x}{(1+x)^2} = -\dfrac{\log x}{x^m (1+x)^2}$

Take $g(x) = \dfrac{1}{x^q}$

$$\lim_{x \to 0+} \frac{f(x)}{g(x)} = \lim_{x \to 0+} \frac{x^{q-m} \log x}{(1+x)^2} = 0 \text{ if } q - m > 0$$

Taking $0 < q < 1$ and also $q - m > 0$ *i.e.*, $q > m$

$\Rightarrow \quad 0 < m < q < 1 \;\Rightarrow\; m < 1 \;\Rightarrow\; n > -1$

$\int_0^1 g(x)\,dx$ is convergent and hence $\int_0^1 f(x)\,dx$ is convergent.

$\therefore \quad \int_0^1 \frac{x^n \log x}{(1+x)^2}\,dx$ is convergent for all $n > -1$.

Note. $n > -1$ also covers the cases $n = 0$ and $n > 1$.

(ii) Please try yourself. **[Ans.** Convergent for $p > -1]$

(iii) Let p be positive and $f(x) = \left(x^p + \frac{1}{x^p}\right)\frac{\log(1+x)}{x}$

0 is the only point of infinite discontinuity.

Take $\quad g(x) = \frac{1}{x^p}$

$$\lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} (x^{2p}+1)\frac{\log(1+x)}{x} = 1$$

since $\lim_{x\to 0+} \frac{\log(1+x)}{x}$ $\quad \left| \text{Form } \frac{0}{0} \right.$

$$= \lim_{x\to 0+} \frac{\frac{1}{1+x}}{1} = 1$$

Since $\int_0^1 g(x)\,dx$ converges if $p < 1$

$\therefore \quad \int_0^1 f(x)\,dx$ *is convergent if* $0 < p < 1$.

If p = 0, $\quad f(x) = \frac{2\log(1+x)}{x}$

Since $\quad \lim_{x\to 0+} f(x) = \lim_{x\to 0+} \frac{2\log(1+x)}{x}$ $\quad \left| \text{Form } \frac{0}{0} \right.$

$$= \lim_{x\to 0+} \frac{\frac{2}{1+x}}{1} = 2$$

$\int_0^1 f(x)\,dx$ *is proper and, hence, convergent.*

If p < 0, let $\quad g(x) = \frac{1}{x^{-p}}$

$$\lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} \left(1+\frac{1}{x^{2p}}\right)\frac{\log(1+x)}{x} = \lim_{x\to 0+} \left(1+\frac{1}{x^{2p}}\right)$$

$$\left[\because \lim_{x\to 0+} \frac{\log(1+x)}{x} = 1\right]$$

$= 1$ since $p < 0$ which is non-zero and finite.

Since $\int_0^1 g(x)\,dx$ is convergent if $-p < 1$, *i.e.*, if $p > -1$, therefore, $\int_0^1 f(x)\,dx$ is convergent if $p > -1$.

Hence $\int_0^1 f(x)\,dx$ is convergent if $-1 < p < 1$.

(iv) We know that $\lim_{x\to 0} x^r \log x = 0$ when $r > 0$.

$\therefore$ *The given integral is a proper integral when* $n - 1 > 0$ *i.e., when* $n > 1$.

When n = 1, the given integral becomes

$$\int_0^1 \log x\,dx = \lim_{\varepsilon\to 0+} \int_\varepsilon^1 \log x\,dx \qquad \text{[Integrating by parts]}$$

$$= \lim_{\varepsilon\to 0+} [x\log x - x]_\varepsilon^1 = \lim_{\varepsilon\to 0+} (0 - 1 - \varepsilon\log\varepsilon + \varepsilon)$$

$$= -1 \qquad \left[\because \lim_{\varepsilon\to 0+} \varepsilon\log\varepsilon = 0\right]$$

$\Rightarrow$ *The given integral is convergent when* $n = 1$.

When n < 1, let $f(x) = -x^{n-1}\log x$ $\quad [\because\; x^{n-1}\log x$ is negative in $(0, 1)]$

Taking $g(x) = \frac{1}{x^\mu}$, we have $\lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} -x^{\mu+n+1}\log x = 0$ if $\mu + n - 1 > 0$

$= \infty$ if $\mu + n - 1 \le 0$

Taking $0 < \mu < 1$ and also $\mu > 1 - n$ so that $1 - n < \mu < 1$ or $1 - n < 1$ or $n > 0$

$\int_0^1 g(x)\,dx$ is convergent and hence $\int_0^1 f(x)\,dx$ is convergent.

$\therefore \quad \int_0^1 x^{n-1}\log(x)\,dx$ is convergent for all $n > 0$.

Also taking $\mu = 1$ and also $\mu \le 1 - n$ so that $n \le 0$.

$\int_0^1 g(x)\,dx$ is divergent and hence $\int_0^1 f(x)\,dx$ is divergent.

$\therefore \quad \int_0^1 x^{n-1}\log x\,dx$ is divergent for all $n \le 0$.

Example 8. *Discuss the convergence of*

(i) $\int_0^{\pi/2} \frac{\sin x}{x^p}\,dx$ $\quad$ *(ii)* $\int_0^{\pi/2} \frac{\cos x}{x^n}\,dx$ $\quad$ *(iii)* $\int_0^1 \frac{\operatorname{cosec} x}{x}\,dx$ $\quad$ *(iv)* $\int_0^1 \frac{\sec x}{x}\,dx$

Sol. *(i)* If p is negative or zero, the given integral is a proper integral and hence convergent when $p \le 0$.

When $p > 0$, the only point of infinite discontinuity is 0.

Let $\quad f(x) = \frac{\sin x}{x^p}$

Take $\quad g(x) = \frac{1}{x^\mu}$

$$\lim_{x \to 0+} \frac{f(x)}{g(x)} = \lim_{x \to 0+} x^{\mu - p} \sin x = \lim_{x \to 0+} x^{\mu - p + 1}\left(\frac{\sin x}{x}\right)$$
$$= 1 \text{ if } \mu - p + 1 = 0$$
$$= 0 \text{ if } \mu - p + 1 > 0$$
$$= \infty \text{ if } \mu - p + 1 < 0$$

By taking $0 < \mu < 1$ and also $\mu = p - 1$ so that $0 < p - 1 < 1$ *i.e.*, $1 < p < 2$.

$\int_0^{\pi/2} g(x)\, dx$ is convergent and hence $\int_0^{\pi/2} f(x)\, dx$ is convergent.

By taking $0 < \mu < 1$ and also $\mu > p - 1$ so that $-1 < p - 1 < \mu < 1$ *i.e.*, $0 < p < 2$.

$\int_0^{\pi/2} g(x)\, dx$ is convergent and hence $\int_0^{\pi/2} f(x)\, dx$ is convergent.

Hence $\int_0^{\pi/2} \frac{\sin x}{x^p}\, dx$ is convergent if $p < 2$ and divergent if $p \geq 2$.

(Second Method)

When $p > 0$, the only point of infinite discontinuity is 0.

Also $\quad \frac{\sin x}{x^p} = \frac{1}{x^{p-1}} \cdot \frac{\sin x}{x} \leq \frac{1}{x^{p-1}} \qquad \left[\because \frac{\sin x}{x} \leq 1\right]$

But $\int_0^{\pi/2} \frac{dx}{x^{p-1}}$ is convergent if $p - 1 < 1$ *i.e.*, if $p < 2$.

$\therefore$ By comparison test, $\int_0^{\pi/2} \frac{\sin x}{x^p}\, dx$ is convergent if $p < 2$ and divergent if $p \geq 2$.

(*ii*) If n is negative or zero, the given integral is a proper integral and hence convergent when $n \leq 0$.

When $n > 0$, the only point of infinite discontinuity is 0.

Let $\quad f(x) = \frac{\cos x}{x^n}$

Take $\quad g(x) = \frac{1}{x^\mu}$

$$\lim_{x \to 0+} \frac{f(x)}{g(x)} = \lim_{x \to 0+} x^{\mu - n} \cos x$$
$$= 1 \quad \text{if } \mu - n = 0$$
$$= 0 \quad \text{if } \mu - n > 0$$
$$= \infty \quad \text{if } \mu - n < 0$$

By taking $0 < \mu < 1$ and also $\mu = n$ so that $0 < n < 1$

$\int_0^{\pi/2} g(x)\, dx$ is convergent and hence $\int_0^{\pi/2} f(x)\, dx$ is convergent.

From the above discussion, it follows that the given integral is convergent if $n < 1$ and divergent if $n \geq 1$.

(*iii*) Since $|\operatorname{cosec} x| \geq 1$ for all values of x, we have

$$\left|\frac{\operatorname{cosec} x}{x}\right| \geq \frac{1}{|x|} = \frac{1}{x} \text{ for all } x \text{ in } (0, 1]$$

But $\int_0^1 \frac{1}{x}\, dx$ is divergent.

$\therefore \int_0^1 \frac{\operatorname{cosec} x}{x}\, dx$ is divergent.

(*iv*) *Please try yourself.* **[Ans. Divergent]**

Example 9. *Show that* $\int_0^{\pi/2} x^m \operatorname{cosec}^n x\, dx$ *exists if and only if* $n < m + 1$.

Sol. Here $\quad f(x) = x^m \operatorname{cosec}^n x = \frac{x^m}{\sin^n x} = \left(\frac{x}{\sin x}\right)^n . x^{m-n} = \left(\frac{x}{\sin x}\right)^n \cdot \frac{1}{x^{n-m}}$

$$\lim_{x \to 0+} f(x) = \begin{cases} 0 & \text{if } m - n > 0 \\ 1 & \text{if } m - n = 0 \\ \infty & \text{if } m - n < 0 \end{cases}$$

$\therefore$ The given integral is a proper integral if $m - n \geq 0$ *i.e.*, if $m \geq n$ and an improper integral if $m - n < 0$; 0 being the only point of infinite discontinuity of f on $\left[0, \frac{\pi}{2}\right]$.

When $m - n < 0$, *i.e.*, $n - m > 0$

$$f(x) = \left(\frac{x}{\sin x}\right)^n \cdot \frac{1}{x^{n-m}}$$

Take $\quad g(x) = \frac{1}{x^{n-m}}$

$$\lim_{x \to 0+} \frac{f(x)}{g(x)} = \lim_{x \to 0+} \left(\frac{x}{\sin x}\right)^n = 1 \text{ which is non-zero and finite.}$$

Also $\int_0^{\pi/2} g(x)\, dx = \int_0^{\pi/2} \frac{dx}{x^{n-m}}$ is convergent iff $n - m < 1$ *i.e.*, $n < m + 1$.

$\therefore$ By comparison test, the given integral is convergent iff $n < m + 1$, which also includes the case $n \leq m$ when the integral is proper.

Example 10. *Show that* $\int_0^{\pi/2} \frac{\sin^m x}{x^n}\, dx$ *exists if and only if* $n < m + 1$.

Sol. Here $\quad f(x) = \frac{\sin^m x}{x^n} = \left(\frac{\sin x}{x}\right)^n \cdot \frac{1}{x^{n-m}}$

$$\lim_{x \to 0+} f(x) = \begin{cases} 0 & \text{if } n - m < 0 \\ 1 & \text{if } n - m = 0 \\ \infty & \text{if } n - m > 0 \end{cases}$$

$\therefore$ The given integral is a proper integral if $n - m \le 0$ *i.e.*, if $n \le m$ and an improper integral if $n - m > 0$; 0 being the only point of infinite discontinuity of f on $\left[0, \frac{\pi}{2}\right]$.

When $n - m > 0$, take $\quad g(x) = \dfrac{1}{x^{n-m}}$

$$\lim_{x \to 0+} \frac{f(x)}{g(x)} = \lim_{x \to 0+} \left(\frac{\sin x}{x}\right)^m = 1 \text{ which is non-zero and finite.}$$

Also $\quad \displaystyle\int_0^{\pi/2} g(x)\, dx = \int_0^{\pi/2} \frac{dx}{x^{n-m}}$ is convergent iff $n - m < 1$. *i.e.*, $n < m + 1$.

$\therefore$ By comparison test, the given integral is convergent iff $n < m + 1$.

Example 11. *Examine the convergence of*

(*i*) $\displaystyle\int_0^1 \log x\, dx$ $\quad$ (*ii*) $\displaystyle\int_0^{\pi/4} \frac{1}{\sqrt{\tan x}}\, dx$ (*iii*) $\displaystyle\int_0^1 \frac{\sin x}{x}\, dx$ $\quad$ (*iv*) $\displaystyle\int_0^1 \left(\log \frac{1}{x}\right)^n dx.$

Sol. (*i*) 0 is the only point of infinite discontinuity and $\log x$ is negative on $(0, 1]$.

$$\int_0^1 \log x\, dx = \lim_{\varepsilon \to 0+} \int_{0+\varepsilon}^1 \log x\, dx = \lim_{\varepsilon \to 0+} [x \log x - x]_\varepsilon^1$$

$$= \lim_{\varepsilon \to 0+} [-1 - \varepsilon \log \varepsilon + \varepsilon] = -1 \qquad \left[\because \lim_{\varepsilon \to 0+} \varepsilon \log \varepsilon = 0\right]$$

$\Rightarrow$ The integral is convergent.

(Second Method)

Let $\quad f(x) = -\log x$

Take $\quad g(x) = \dfrac{1}{x^n}$

$$\lim_{x \to 0+} \frac{f(x)}{g(x)} = \lim_{x \to 0+} -x^n \log x = 0 \text{ if } n > 0$$

Taking n between 0 and 1, $\displaystyle\int_0^1 f(x)\, dx = \int_0^1 \frac{dx}{x^n}$ is convergent.

$\therefore$ By comparison test, $\displaystyle\int_0^1 f(x) dx$ is convergent.

Hence $\displaystyle\int_0^1 \log x\, dx$ is convergent.

(*ii*) 0 is the only point of infinite discontinuity of the integrand on $\left[0, \frac{\pi}{4}\right]$.

Let $\quad f(x) = \dfrac{1}{\sqrt{\tan x}} = \sqrt{\dfrac{\cos x}{\sin x}}$

Take $\quad g(x) = \dfrac{1}{\sqrt{x}}$

$$\lim_{x \to 0+} \frac{f(x)}{g(x)} = \lim_{x \to 0+} \sqrt{\frac{x}{\sin x}} \cdot \sqrt{\cos x} = 1 \text{ which is non-zero and finite.}$$

Since $\quad \displaystyle\int_0^{\pi/4} g(x)\, dx = \int_0^{\pi/4} \frac{dx}{\sqrt{x}} \qquad \left[\text{Form } \int_0^b \frac{dx}{x^n} \text{ is convergent } \left(\because\ n = \frac{1}{2} < 1\right)\right]$

$\therefore \quad \displaystyle\int_0^{\pi/4} f(x)\, dx$ is convergent.

(*iii*) Since $\displaystyle\lim_{x \to 0+} \frac{\sin x}{x} = 1$, the integral is proper and hence convergent.

(*iv*) $$\int_0^1 \left(\log \frac{1}{x}\right)^n dx = \int_0^a \left(\log \frac{1}{x}\right)^n dx + \int_a^1 \left(\log \frac{1}{x}\right)^n dx \qquad \ldots(1)$$

where $\quad 0 < a < 1$.

0 and 1 are the points of infinite discontinuity of the integrals on the right.

Let $\quad f(x) = \left(\log \dfrac{1}{x}\right)^n$

Convergence of $\displaystyle\int_0^a \left(\log \frac{1}{x}\right)^n dx$ at 0.

$$\lim_{x \to 0+} \left(\log \frac{1}{x}\right)^n = 1 \quad \text{if } n = 0$$
$$= 0 \quad \text{if } n < 0$$

$\therefore$ The integral is proper is $n \le 0$.

0 is the only point of infinite discontinuity if $n > 0$.

For $n > 0$, take $\quad g(x) = \dfrac{1}{x^p}$, $0 < p < 1$.

$$\lim_{x \to 0+} \frac{f(x)}{g(x)} = \lim_{x \to 0+} x^p \left(\log \frac{1}{x}\right)^n = 0$$

Also $\displaystyle\int_0^a g(x)\, dx$ converges, since $0 < p < 1$

$\therefore \quad \displaystyle\int_0^a f(x)\, dx = \int_0^a \left(\log \frac{1}{x}\right)^n dx$ converges.

Combining all cases, $\displaystyle\int_0^a \left(\log \frac{1}{x}\right)^n dx$ **converges for all** n.

Convergence of $\displaystyle\int_a^1 \left(\log \frac{1}{x}\right)^n dx$ at 1.

The integral is proper if $n \ge 0$ and 1 is the only point of infinite discontinuity if $n < 0$.

For $n < 0$, take $\quad g(x) = \dfrac{1}{(1-x)^{-n}}$

$$\lim_{x\to 1-}\frac{f(x)}{g(x)} = \lim_{x\to 1-}\left(\frac{\log\frac{1}{x}}{1-x}\right)^n = 1 \text{ which is non-zero and finite.}$$

But $\displaystyle\int_a^1 g(x)\,dx = \int_0^1 \frac{dx}{(1-x)^{-n}}$ is convergent if $-n < 1$ *i.e.,* if $n > -1$.

$\therefore$ By comparison test, $\displaystyle\int_a^1 f(x)\,dx = \int_a^1 \left(\log\frac{1}{x}\right)^n dx$ is convergent if $-1 < n < 0$.

Hence, from (1), $\displaystyle\int_0^1 \left(\log\frac{1}{x}\right)^n dx$ is convergent if $-1 < n < 0$.

Example 12. *Find the values of m and n for which the integral* $\displaystyle\int_0^1 e^{-mx} . x^n\, dx$ *converges.*

Sol. Irrespective of the values of m, when $n \geq 0$, the given integral is a proper integral and hence convergent.

When $n < 0$, whatever m may be, 0 is the only point of infinite discontinuity.

Let $\quad f(x) = e^{-mx} x^n$

Take $\quad g(x) = x^n = \dfrac{1}{x^{-n}}$

$$\lim_{x\to 0+}\frac{f(x)}{g(x)} = \lim_{x\to 0+} e^{-mx} = 1 \text{ which is non-zero and finite.}$$

$$\int_0^1 g(x)\,dx = \int_0^1 \frac{dx}{x^{-n}} \text{ converges if } -n < 1 \text{ i.e., if } n > -1.$$

$\therefore$ By comparison test, $\displaystyle\int_0^1 f(x)\,dx$ also converges if $n > -1$.

Hence $\displaystyle\int_0^1 e^{-mx} x^n\,dx$ converges only for $n > -1$, irrespective of the values of m.

Example 13. *Test the convergence of* $\displaystyle\int_0^1 x^{\alpha-1} e^{-x}\,dx$.

Sol. *Please try yourself.* [**Ans.** Convergent for $\alpha > 0$]

Example 14. *Show that* $\displaystyle\int_0^{\pi/2} \sin x \log \sin x\,dx$ *is convergent with value* $\log\left(\dfrac{2}{e}\right)$.

Sol. Here 0 is the only point of infinite discontinuity of the integrand on $\left[0, \dfrac{\pi}{2}\right]$

$$\lim_{x\to 0+} \sin x \log \sin x \qquad \text{Form } 0\times\infty$$

$$= \lim_{x\to 0+} \frac{\log \sin x}{\operatorname{cosec} x} \qquad \text{Form } \frac{\infty}{\infty}$$

$$= \lim_{x\to 0+} \frac{\cot x}{-\operatorname{cosec} x \cot x} = \lim_{x\to 0+} (-\sin x) = 0$$

$\Rightarrow \displaystyle\int_0^{\pi/2} \sin x \log \sin x\,dx$ is convergent.

To evaluate the integral, we have

$$\int_0^{\pi/2} \sin x \log \sin x\,dx = \lim_{\varepsilon\to 0+}\int_\varepsilon^{\pi/2} (\log \sin x) . \sin x\,dx \qquad \text{(Integrating by parts)}$$

$$= \lim_{\varepsilon\to 0+}\left[\{(\log \sin x)(-\cos x)\}_\varepsilon^{\pi/2} - \int_\varepsilon^{\pi/2} \cot x(-\cos x)\,dx\right]$$

$$= \lim_{\varepsilon\to 0+}\left[\cos\varepsilon \log \sin\varepsilon + \int_\varepsilon^{\pi/2}\frac{\cos^2 x}{\sin x}dx\right]$$

$$= \lim_{\varepsilon\to 0+}\left[\cos\varepsilon \log \sin\varepsilon + \int_\varepsilon^{\pi/2}\frac{1-\sin^2 x}{\sin x}dx\right]$$

$$= \lim_{\varepsilon\to 0+}\left[\cos\varepsilon \log \sin\varepsilon + \int_\varepsilon^{\pi/2}(\operatorname{cosec} x - \sin x)\,dx\right]$$

$$= \lim_{\varepsilon\to 0+}\left[\cos\varepsilon \log \sin\varepsilon + \left\{\log\tan\frac{x}{2} + \cos x\right\}_\varepsilon^{\pi/2}\right]$$

$$= \lim_{\varepsilon\to 0+}\left[\cos\varepsilon \log \sin\varepsilon - \log\tan\frac{\varepsilon}{2} - \cos\varepsilon\right]$$

$$= \lim_{\varepsilon\to 0+}\left[\cos\varepsilon \log\left(2\sin\frac{\varepsilon}{2}\cos\frac{\varepsilon}{2}\right) - \log\frac{\sin\frac{\varepsilon}{2}}{\cos\frac{\varepsilon}{2}} - \cos\varepsilon\right]$$

$$= \lim_{\varepsilon\to 0+}\left[\cos\varepsilon \log\left(2\cos\frac{\varepsilon}{2}\right) + \cos\varepsilon\log\sin\frac{\varepsilon}{2} - \log\sin\frac{\varepsilon}{2} + \log\cos\frac{\varepsilon}{2} - \cos\varepsilon\right]$$

$$= \lim_{\varepsilon\to 0+}\left[\cos\varepsilon \log\left(2\cos\frac{\varepsilon}{2}\right) + \log\cos\frac{\varepsilon}{2} - \cos\varepsilon - (1-\cos\varepsilon)\log\sin\frac{\varepsilon}{2}\right]$$

$$= \lim_{\varepsilon\to 0+}\left[\cos\varepsilon \log\left(2\cos\frac{\varepsilon}{2}\right) + \log\cos\frac{\varepsilon}{2} - \cos\varepsilon - 2\sin^2\frac{\varepsilon}{2}\log\sin\frac{\varepsilon}{2}\right]$$

$$= \log 2 + 0 - 1 - 0 \qquad \left[\because \lim_{\varepsilon\to 0+} 2\sin^2\frac{\varepsilon}{2}\log\sin\frac{\varepsilon}{2} = \lim_{t\to 0+} 2t^2\log t = 0 \text{ where } t = \sin\frac{\varepsilon}{2}\right]$$

$$= \log 2 - \log e$$

$$= \log\left(\frac{2}{e}\right).$$

Example 15. *Show that* $\int_0^{\pi/2} \log \sin x\, dx$ *is convergent and hence evaluate it.*

Sol. Let $\quad f(x) = \log \sin x.$

0 is the only point of infinite discontinuity of f on $\left[0, \frac{\pi}{2}\right]$.

Since f is negative on $\left[0, \frac{\pi}{2}\right]$, we consider $-f$ for testing convergence of the integral.

Take $\quad g(x) = \frac{1}{x^n}, n > 0$

$$\lim_{x \to 0+} \frac{-f(x)}{g(x)} = \lim_{x \to 0+} (-x^n \log \sin x) \qquad | \; 0 \times \infty$$

$$= \lim_{x \to 0+} -\frac{\log \sin x}{\frac{1}{x^n}} \qquad \left| \; \frac{\infty}{\infty} \right.$$

$$= \lim_{x \to 0+} \frac{\cot x}{\frac{n}{x^{n+1}}} = \lim_{x \to 0+} \frac{x^n}{n} \cdot \frac{x}{\tan x} = 0$$

Taking n between 0 and 1, $\int_0^{\pi/2} g(x)\, dx$ is convergent.

$\therefore$ By comparison test, $\int_0^{\pi/2} -f(x)\, dx$ is convergent.

Consequently, $\int_0^{\pi/2} f(x)\, dx$ is convergent.

To evaluate the integral, let $\quad I = \int_0^{\pi/2} \log \sin x\, dx$

Since $\quad \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$

$$\therefore \quad I = \int_0^{\pi/2} \log \sin\left(\frac{\pi}{2} - x\right) dx = \int_0^{\pi/2} \log \cos x\, dx$$

Adding,
$$2I = \int_0^{\pi/2} \log \sin x\, dx + \int_0^{\pi/2} \log \cos x\, dx$$

$$= \int_0^{\pi/2} (\log \sin x + \log \cos x)\, dx$$

$$= \int_0^{\pi/2} \log (\sin x \cos x)\, dx = \int_0^{\pi/2} \log\left(\frac{\sin 2x}{2}\right) dx$$

$$= \int_0^{\pi/2} \log \sin 2x\, dx - \int_0^{\pi/2} \log 2\, dx = \int_0^{\pi/2} \log \sin 2x\, dx - \frac{\pi}{2} \log 2 \qquad \ldots(1)$$

Putting $2x = t$ so that $dx = \frac{1}{2} dt$ when $x = 0, t = 0$; when $x = \frac{\pi}{2}, t = \pi$

$$\therefore \quad \int_0^{\pi/2} \log \sin 2x\, dx = \frac{1}{2} \int_0^{\pi} \log \sin t\, dt = \frac{1}{2} \times 2 \int_0^{\pi/2} \log \sin t\, dt$$

$$\left[\because \int_0^{2a} f(x)\, dx = 2\int_0^a f(x)\, dx \text{ if } f(2a-x) = f(x) \quad \text{Here } \sin(\pi - t) = \sin t\right]$$

$$= \int_0^{\pi/2} \log \sin x\, dx \qquad \left[\because \int_a^b f(t)\, dt = \int_a^b f(x)\, dx\right]$$

From (1), we have $\quad 2I = I - \frac{\pi}{2} \log 2$

Hence $\quad I = -\frac{\pi}{2} \log 2.$

Example 16. *Show that* $\int_0^1 \frac{\operatorname{cosec} x}{x^n} dx$ *is divergent if* $n \geq 1$.

Sol. We know that $|\operatorname{cosec} x| \geq 1$ for all x

$$\therefore \quad \left|\frac{\operatorname{cosec} x}{x^n}\right| \geq \frac{1}{x^n} \text{ for all } x \text{ in } (0, 1]$$

Also $\int_0^1 \frac{dx}{x^n}$ is divergent if $n \geq 1$.

$\therefore$ By comparison test, $\int_0^1 \frac{\operatorname{cosec} x}{x^n} dx$ is divergent if $n \geq 1$.

Example 17. *Test for convergence the integral* $\int_0^1 \frac{\sin x}{x^{3/2}} dx$.

Sol. Here $\quad f(x) = \frac{\sin x}{x^{3/2}} = \left(\frac{\sin x}{x}\right) \cdot \frac{1}{\sqrt{x}}$

0 is the only point of infinite discontinuity of f on $[0, 1]$.

Take $\quad g(x) = \frac{1}{\sqrt{x}}$

$$\lim_{x \to 0+} \frac{f(x)}{g(x)} = \lim_{x \to 0+} \frac{\sin x}{x} = 1 \text{ which is non-zero and finite.}$$

Since $\int_0^1 g(x)\, dx = \int_0^1 \frac{1}{\sqrt{x}} dx$ is convergent.

$\left(\because n = \frac{1}{2} < 1\right)$, therefore, by comparison test $\int_0^1 f(x)\, dx$ is also convergent.

11.7. GENERAL TEST FOR CONVERGENCE (*Integrand may change sign*)

This test for convergence of an improper integral (finite limits of integration but discontinuous integrand) holds whether or not the integrand keeps the same sign.

Cauchy's Test. *The improper integral $\int_a^b f(x)\,dx$, a being the only point of infinite discontinuity, converges at a if and only if to each $\varepsilon > 0$, there corresponds a $\delta > 0$ such that*

$$\left| \int_{a+\lambda_1}^{a+\lambda_2} f(x)\,dx \right| < \varepsilon \text{ for all } 0 < \lambda_1, \lambda_2 < \delta.$$

Proof. We know that the improper integral $\int_a^b f(x)\,dx$, a being the only point of infinite discontinuity, is said to exist (*i.e.*, coverage) if $\lim_{\lambda \to 0+} \int_{a+\lambda}^b f(x)\,dx$ exists finitely.

Let $\quad F(\lambda) = \int_{a+\lambda}^b f(x)\,dx$, so that $F(\lambda)$ is a function of λ.

Now, the necessary and sufficient condition for $\lim_{\lambda \to 0+} F(\lambda)$ to exist finitely is that to every $\varepsilon > 0$, there corresponds a $\delta > 0$ such that for all positive $\lambda_1, \lambda_2 < \delta$,

$$| F(\lambda_1) - F(\lambda_2) | < \varepsilon$$

$$\Rightarrow \quad \left| \int_{a+\lambda_1}^b f(x)\,dx - \int_{a+\lambda_2}^b f(x)\,dx \right| < \varepsilon$$

$$\Rightarrow \quad \left| \int_{a+\lambda_1}^b f(x)\,dx + \int_b^{a+\lambda_2} f(x)\,dx \right| < \varepsilon \Rightarrow \left| \int_{a+\lambda_1}^{a+\lambda_2} f(x)\,dx \right| < \varepsilon.$$

11.8. ABSOLUTE CONVERGENCE

Definition. The improper integral $\int_a^b f(x)\,dx$ is said to be absolutely convergent if $\int_a^b | f(x) |\,dx$ is convergent.

Theorem. *Every absolutely convergent integral is convergent.*

or $\qquad \int_a^b | f(x) |\,dx$ *exists* $\Rightarrow \int_a^b f(x)\,dx$ *exists.*

Proof. Since $\int_a^b | f(x) |\,dx$ exists, therefore, by Cauchy's test, for every $\varepsilon > 0$, there corresponds a $\delta > 0$ such that

$$\left| \int_{a+\lambda_1}^{a+\lambda_2} | f(x) |\,dx \right| < \varepsilon, \ \forall\, 0 < \lambda_1, \lambda_2, < \delta \qquad \ldots(1)$$

Also, we know that $\qquad \left| \int_{a+\lambda_1}^{a+\lambda_2} f(x)\,dx \right| \le \left| \int_{a+\lambda_1}^{a+\lambda_2} | f(x) |\,dx \right| \qquad \ldots(2)$

From (1) and (2), we have $\qquad \left| \int_{a+\lambda_1}^{a+\lambda_2} f(x)\,dx \right| < \varepsilon, \ \forall\, 0 < \lambda_1, \lambda_2 < \delta$

$\therefore$ By Cauchy's test, $\int_a^b f(x)\,dx$ exists.

Note 1. Since $| f(x) |$ is always positive, the comparison tests can be applied for examining the convergence of $\int_a^b | f(x) |\,dx$, *i.e.*, absolute convergence of $\int_a^b f(x)\,dx$.

Note 2. The converse of the above theorem is not true. Every convergent integral is not absolutely convergent. A convergent integral which is not absolutely convergent is called a **conditionally convergent integral.**

Example 1. *Test the convergence of* $\int_0^1 \frac{\sin \frac{1}{x}}{\sqrt{x}}\,dx$.

Sol. Let $\qquad f(x) = \dfrac{\sin \frac{1}{x}}{\sqrt{x}}$

Clearly, f does not keep the same sign in a neighbourhood of 0.

Now $\qquad | f(x) | = \left| \dfrac{\sin \frac{1}{x}}{\sqrt{x}} \right| = \dfrac{\left| \sin \frac{1}{x} \right|}{| \sqrt{x} |} \le \dfrac{1}{\sqrt{x}}, \ \forall\, x \in (0, 1]$

But $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ is convergent at 0. $\qquad \left(\because\ n = \frac{1}{2} < 1\right)$

$\therefore$ By comparison test, $\int_0^1 | f(x) |\,dx$ is convergent at 0.

Since absolute convergence $\Rightarrow$ convergence

$\therefore \quad \int_0^1 f(x)\,dx$ is convergent.

Example 2. *Show that* $\int_0^1 \frac{\sin \frac{1}{x}}{x^p}\,dx$, $p > 0$, *converges absolutely for* $p < 1$.

Sol. Let $\qquad f(x) = \dfrac{\sin \frac{1}{x}}{x^p}, p > 0$

Clearly, f does not keep the same sign in a neighbourhood of 0.

Now $\qquad | f(x) | = \left| \dfrac{\sin \frac{1}{x}}{x^p} \right| = \dfrac{\left| \sin \frac{1}{x} \right|}{| x^p |} \le \dfrac{1}{x^p}, \ \forall\, x \in (0, 1]$

Also $\int_0^1 \frac{dx}{x^p}$ converges iff $p < 1$.

$\therefore$ By convergent test, $\int_0^1 | f(x) |\,dx$ converges if $p < 1$

Hence $\int_0^1 f(x)\,dx$ converges absolutely for $p < 1$.

11.9. CONVERGENCE AT ∞, THE INTEGRAND BEING POSITIVE

Theorem. *A necessary and sufficient condition for the convergence of $\int_a^\infty f(x)\,dx$, where $f(x) > 0\ \forall\ x \in [x, t]$, is that there exists a positive number M, independent of t, such $\int_a^t f(x)\,dx < M\ \forall\ t \geq a$.*

Proof. Let $\quad F(t) = \int_a^t f(x)\,dx.$

Since f is positive in $[a, t]$, the function $F(t)$ monotonically increases with t and will therefore tend to a finite limit if and only if it is bounded above, *i.e.*, there exists a positive number M, independent of t, such that $F(t) < M\ \forall\ t \geq a$.

$$\Rightarrow \qquad \int_a^t f(x)\,dx < M \quad \forall \quad t \geq a.$$

Note. If no such number M exists, then the monotonic increasing function $F(t)$ is unbounded above and therefore tends to ∞ as $t \to \infty$.

$\therefore\ \int_a^t f(x)\,dx$ diverges to ∞.

11.10. COMPARISON TEST I

If f and g are two functions such that $0 < f(x) \leq g(x)\ \forall\ x \in [a, \infty)$, then

(i) $\int_a^\infty g(x)\,dx$ *is convergent* $\quad\Rightarrow\quad \int_a^\infty f(x)\,dx$ *is convergent*

(ii) $\int_a^\infty f(x)\,dx$ *is divergent* $\quad\Rightarrow\quad \int_a^\infty g(x)\,dx$ *is divergent.*

Proof. Since f and g are both positive and $f(x) \leq g(x)\ \forall\ x \in [a, t]$

$$\therefore \qquad \int_a^t f(x)\,dx \leq \int_a^t g(x)\,dx \qquad \text{...(1)}$$

(i) Let $\int_a^\infty g(x)\,dx$ be convergent.

Then there exists a positive number M such that $\quad \int_a^t g(x)\,dx < M\ \forall\ t \geq a$

$\therefore$ From (1), $\quad \int_a^t f(x)\,dx < M\ \forall\ t \geq a$

Hence $\int_a^\infty f(x)\,dx$ is convergent.

(ii) Let $\int_a^\infty f(x)dx$ be divergent. Then $\int_a^t f(x)dx$ is unbounded above.

$\therefore$ From (1), $\int_a^t g(x)dx$ is unbounded above.

Hence $\int_a^\infty g(x)dx$ is divergent.

11.11. COMPARISON TEST II

If f and g are positive functions on $[a, \infty)$ and $\lim_{x \to \infty} \frac{f(x)}{g(x)} = l$

then (i) *if l is non-zero and finite, the two integrals*

$$\int_a^\infty f(x)\,dx \text{ and } \int_a^\infty g(x)\,dx \text{ converge of diverge together.}$$

(ii) *If $l = 0$ and $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges.*

(iii) *If $l = \infty$ and $\int_a^\infty g(x)\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ diverges.*

Proof. (i) Since $\quad \frac{f(x)}{g(x)} > 0\ \forall\ x \geq a$ and $l \neq 0$

$\therefore \qquad l > 0.$

Choose a positive number ε such that $l - \varepsilon > 0$.

Since $\lim_{x \to \infty} \frac{f(x)}{g(x)} = l$, therefore, there exists a number $k\ (> a)$ such that

$$\left|\frac{f(x)}{g(x)} - l\right| < \varepsilon\ \forall\ x \geq k \ \Rightarrow\ l - \varepsilon < \frac{f(x)}{g(x)} < l + \varepsilon\ \forall\ x \geq k$$

$\Rightarrow \qquad (l - \varepsilon)g(x) < f(x) < (l + \varepsilon)g(x)\ \forall\ x \geq k \qquad$ ...(1) $\quad [\because\ g$ is positive]

From (1), $\quad (l - \varepsilon)g(x) < f(x)$ and $f(x) < (l + \varepsilon)g(x)$

$\therefore$ By comparison test I,

Convergence of $\int_a^\infty f(x)\,dx \ \Rightarrow\ $ convergence of $\int_a^\infty g(x)\,dx$

Convergence of $\int_a^\infty g(x)\,dx \ \Rightarrow\ $ convergence of $\int_a^\infty f(x)\,dx$

Divergence of $\int_a^\infty f(x)\,dx \ \Rightarrow\ $ divergence of $\int_a^\infty g(x)\,dx$

Divergence of $\int_a^\infty g(x)\,dx \ \Rightarrow\ $ divergence of $\int_a^\infty f(x)\,dx$.

(ii) $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0$

$\therefore$ Given $\varepsilon > 0$, we can find a number k such that

$$\frac{f(x)}{g(x)} < \varepsilon\ \forall\ x \geq k \ \Rightarrow\ f(x) < \varepsilon g(x)\ \forall\ x\ \varepsilon \geq k$$

$\therefore \quad$ Convergence of $\int_a^\infty g(x)\,dx \ \Rightarrow\ $ convergence of $\int_a^\infty f(x)\,dx$.

(iii) $\lim_{x \to \infty} \frac{f(x)}{g(x)} = \infty$

$\therefore$ Given M > 0, we can find a number k such that

$$\frac{f(x)}{g(x)} > M \ \forall\, x \geq k \quad \Rightarrow \quad f(x) > Mg(x) \ \forall\, x \geq k$$

$\therefore$ Divergence of $\int_a^\infty g(x)\, dx \ \Rightarrow$ divergence of $\int_a^\infty f(x)\, dx$.

11.12. A USEFUL COMPARISON INTEGRAL

Theorem. *The improper integral* $\int_a^\infty \frac{dx}{x^n}$, *(a > 0) convergent if and only if n > 1.*

Proof. $$\int_a^\infty \frac{dx}{x^n} = \lim_{t \to \infty} \int_a^t \frac{dx}{x^n}$$

Case I. When n = 1

$$\int_a^\infty \frac{dx}{x^n} = \lim_{t \to \infty} \int_a^t \frac{dx}{x} = \lim_{t \to \infty} [\log x]_a^t = \lim_{t \to \infty} [\log t - \log a] = \infty$$

$\therefore$ $\int_a^\infty \frac{dx}{x^n}$ diverges if $n = 1$.

Case II. When n ≠ 1

$$\int_a^\infty \frac{dx}{x^n} = \lim_{t \to \infty} \int_a^t x^{-n}\, dx = \lim_{t \to \infty} \left[\frac{x^{1-n}}{1-n}\right]_a^t$$

Sub-case I. ~~When $n < 1$, $1 - n > 0$~~

$\therefore$ $$\int_a^\infty \frac{dx}{x^n} = \lim_{t \to \infty} \frac{1}{1-n}\, [t^{1-n} - a^{1-n}] = \infty$$

$\therefore$ $\int_a^\infty \frac{dx}{x^n}$ diverges if $n < 1$.

Sub-case II. When $n > 1$, $n - 1 > 0$

$$\int_a^\infty \frac{dx}{x^n} = \lim_{t \to \infty} -\frac{1}{n-1}\left[\frac{1}{x^{n-1}}\right]_a^t$$

$$= \lim_{t \to \infty} -\frac{1}{n-1}\left[\frac{1}{t^{n-1}} - \frac{1}{a^{n-1}}\right] = \frac{1}{(n-1)\, a^{n-1}} \text{ which is finite.}$$

$\therefore$ $\int_a^\infty \frac{dx}{x^n}$ converges if $n > 1$.

Hence $\int_a^\infty \frac{dx}{x^n}$ converges iff $n > 1$.

ILLUSTRATIVE EXAMPLES

Example 1. *Examine the convergence of*

(i) $\int_1^\infty \frac{x^3}{(1+x)^5}\, dx$ (ii) $\int_1^\infty \frac{dx}{(2+x)\sqrt{x}}$

(iii) $\int_0^\infty \frac{x}{x^3+1}\, dx$ (iv) $\int_1^\infty \frac{x^3+1}{x^4}\, dx$.

Sol. (i) Let $$f(x) = \frac{x^3}{(1+x)^5} = \frac{x^3}{x^5\left(1+\frac{1}{x}\right)^5} = \frac{1}{x^2\left(1+\frac{1}{x}\right)^5}$$

Take $$g(x) = \frac{1}{x^2}$$

$\therefore$ $$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{1}{\left(1+\frac{1}{x}\right)^5} = 1 \text{ which is non-zero and finite.}$$

By comparison test, the two integrals $\int_1^\infty f(x)\, dx$ and $\int_1^\infty g(x)\, dx$ converge or diverge together.

But $$\int_1^\infty g(x)\, dx = \int_1^\infty \frac{dx}{x^2} \text{ is convergent.} \qquad (\because\ n = 2 > 1)$$

$\therefore$ $$\int_1^\infty f(x)\, dx = \int_1^\infty \frac{x^3}{(1+x)^5}\, dx \text{ is convergent.}$$

(ii) Let $$f(x) = \frac{1}{(2+x)\sqrt{x}} = \frac{1}{x^{3/2}\left(1+\frac{2}{x}\right)}$$

Take $$g(x) = \frac{1}{x^{3/2}}$$

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{1}{1+\frac{2}{x}} = 1 \text{ which is non-zero and finite.}$$

$\therefore$ By comparison test, $\int_1^\infty f(x)\, dx$ and $\int_1^\infty g(x)\, dx$ converge or diverge together.

But $$\int_1^\infty g(x)\, dx = \int_1^\infty \frac{dx}{x^{3/2}} \text{ is convergent.} \qquad \left(\because\ n = \frac{3}{2} > 1\right)$$

$\therefore$ $$\int_1^\infty f(x)\, dx = \int_1^\infty \frac{dx}{(2+x)\sqrt{x}} \text{ is convergent.}$$

(*iii*) $$\int_0^\infty \frac{x}{x^3+1}\,dx = \int_0^1 \frac{x}{x^3+1}\,dx + \int_1^\infty \frac{x}{x^3+1}\,dx \quad ...(1)$$

(Since the lower limit is 0 and Theorem 11.13 can be applied when $a > 0$).

The first integral of R.H.S. of (1) is a proper integral and hence convergent.

Let $$f(x) = \frac{x}{x^3+1} = \frac{x}{x^3\left(1+\frac{1}{x^3}\right)} = \frac{1}{x^2\left(1+\frac{1}{x^3}\right)}$$

Take $$g(x) = \frac{1}{x^2}$$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{1}{1+\frac{1}{x^3}} = 1 \text{ which is non-zero and finite.}$$

$\therefore$ By comparison test, $\int_1^\infty f(x)\,dx$ and $\int_1^\infty g(x)\,dx$ converge or diverge together.

But $$\int_1^\infty g(x)\,dx = \int_1^\infty \frac{dx}{x^2} \text{ is convergent.} \qquad (\because\ n = 2 > 1)$$

$\therefore$ $$\int_1^\infty f(x)\,dx = \int_1^\infty \frac{x}{x^3+1}\,dx \text{ is convergent.}$$

Since both the integrals on the R.H.S. in (1) are convergent therefore, $\int_0^\infty \frac{x}{x^3+1}\,dx$ is convergent.

(*iv*) Let $$f(x) = \frac{x^3+1}{x^4} = \frac{x^3\left(1+\frac{1}{x^3}\right)}{x^4} = \frac{1}{x}\left(1+\frac{1}{x^3}\right)$$

Take $g(x) = \frac{1}{x}$ etc. **[Ans. Divergent]**

Example 2. *Examine the convergence of*

(*i*) $\int_1^\infty \frac{dx}{x^{1/2}(1+x)^{1/3}}$ (*ii*) $\int_1^\infty \frac{dx}{x\sqrt{1+x^2}}$ (*iii*) $\int_0^\infty \frac{x^2}{\sqrt{x^5+1}}\,dx$

(*iv*) $\int_1^\infty \frac{x^2+2}{x^4}\,dx$ (*v*) $\int_1^\infty \frac{\sqrt{x}}{(1+x)^3}\,dx$ (*vi*) $\int_1^\infty \frac{x}{\sqrt{x^8+1}}\,dx.$

Sol. (*i*) Let $$f(x) = \frac{1}{x^{1/2}(1+x)^{1/3}} = \frac{1}{x^{1/2}\cdot x^{1/3}\left(1+\frac{1}{x}\right)^{1/3}} = \frac{1}{x^{5/6}\left(1+\frac{1}{x}\right)^{1/3}}$$

Take $g(x) = \frac{1}{x^{5/6}}$ etc. **[Ans. Divergent]** $\left(\because\ n = \frac{5}{6} < 1\right)$

(*ii*) *Please try yourself.* **[Ans. Convergent]**

(*iii*) $$\int_0^\infty \frac{x^2}{\sqrt{x^5+1}}\,dx = \int_0^1 \frac{x^2}{\sqrt{x^5+1}}\,dx + \int_1^\infty \frac{x^2}{\sqrt{x^5+1}}\,dx \quad ...(1)$$

The first integral on the right is a proper integral and therefore convergent.

Let $$f(x) = \frac{x^2}{\sqrt{x^5+1}} = \frac{x^2}{x^{5/2}\sqrt{1+\frac{1}{x^5}}} = \frac{1}{x^{1/2}\sqrt{1+\frac{1}{x^5}}}$$

Take $g(x) = \frac{1}{x^{1/2}}$ etc.

The second integral on the right is divergent. $\left(\because\ n = \frac{1}{2} < 1\right)$

$\therefore$ From (1), $\int_0^\infty \frac{x^2}{\sqrt{x^5+1}}\,dx$ is divergent.

(*iv*) *Please try yourself.* **[Ans. Convergent]**

(*v*) Let $$f(x) = \frac{\sqrt{x}}{(1+x)^3} = \frac{\sqrt{x}}{x^3\left(1+\frac{1}{x}\right)^3} = \frac{1}{x^{5/2}\left(1+\frac{1}{x}\right)^3}$$

Take $g(x) = \frac{1}{x^{5/2}}$ etc. **[Ans. Convergent]**

(*vi*) Let $$f(x) = \frac{x}{\sqrt{x^8+1}} = \frac{x}{x^4\sqrt{1+\frac{1}{x^8}}} = \frac{1}{x^3\sqrt{1+\frac{1}{x^8}}}$$

Take $g(x) = \frac{1}{x^3}$ etc. **[Ans. Convergent]**

Example 3. *Examine the convergence of*

(*i*) $\int_1^\infty \frac{dx}{\sqrt{x}\,(1+x)^n}$ (*ii*) $\int_2^\infty \frac{dx}{\sqrt{x^2-1}}$ (*iii*) $\int_0^\infty \frac{dx}{x^{1/3}(1+x^{1/2})}$

(*iv*) $\int_b^\infty \frac{x^{3/2}\,dx}{\sqrt{x^4-a^4}}$ *where* $b > a$ (*v*) $\int_0^\infty \frac{x^{3/2}}{b^2x^2+c^2}\,dx$ (*vi*) $\int_0^\infty \frac{x^2}{(a^2+x^2)^2}\,dx.$

Sol. (*i*) Let $$f(x) = \frac{1}{\sqrt{x}\,(1+x)^n} = \frac{1}{\sqrt{x}\cdot x^n\left(1+\frac{1}{x}\right)^n} = \frac{1}{x^{n+1/2}\left(1+\frac{1}{x}\right)^n}$$

Take $$g(x) = \frac{1}{x^{n+1/2}}$$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{1}{\left(1+\frac{1}{x}\right)^n} = 1 \text{ which is non-zero and finite.}$$

$\therefore$ By comparison test, $\int_1^\infty f(x)\,dx$ and $\int_1^\infty g(x)\,dx$ converge or diverge together.

But $\int_1^\infty g(x)\,dx = \int_1^\infty \frac{dx}{x^{n+1/2}}$ converges iff $n + \frac{1}{2} > 1$ *i.e.*, $n > \frac{1}{2}$.

$\therefore$ $\int_1^\infty f(x)\,dx = \int_1^\infty \frac{dx}{\sqrt{x}\,(1+x)^n}$ converges iff $n > \frac{1}{2}$.

(*ii*) Take $g(x) = \frac{1}{x}$ **[Ans. Divergent]**

(*iii*) Take $g(x) = \frac{1}{x^{1/3}\,.\,x^{1/2}} = \frac{1}{x^{5/6}}$ **[Ans. Divergent]**

(*iv*) Take $g(x) = \frac{x^{3/2}}{x^2} = \frac{1}{\sqrt{x}}$ **[Ans. Divergent]**

(*v*) $$\int_0^\infty \frac{x^{3/2}}{b^2x^2+c^2}\,dx = \int_0^1 \frac{x^{3/2}}{b^2x^2+c^2}\,dx + \int_1^\infty \frac{x^{3/2}}{b^2x^2+c^2}\,dx$$

First integral on right is a proper integral. For the second integral, take

$g(x) = \frac{x^{3/2}}{x^2} = \frac{1}{\sqrt{x}}$ so that $\int_1^\infty g(x)\,dx$ is divergent. **[Ans. Divergent]**

(*vi*) $$\int_0^\infty \frac{x^2}{(a^2+x^2)^2}\,dx = \int_0^1 \frac{x^2}{(a^2+x^2)^2}\,dx + \int_1^\infty \frac{x^2}{(a^2+x^2)^2}\,dx$$

The first is proper and the second converges because

$\int_1^\infty g(x)\,dx = \int_1^\infty \frac{dx}{x^2}$ is convergent. **[Ans. Convergent]**

Example 4. *Examine the convergence of*

(*i*) $\int_0^\infty \frac{x^{2m}}{1+x^{2n}}\,dx,\ m, n > 0$ (*ii*) $\int_0^\infty \frac{x^{p-1}}{1+x}\,dx.$

Sol. (*i*) $$\int_0^\infty \frac{x^{2m}}{1+x^{2n}}\,dx = \int_0^a \frac{x^{2m}}{1+x^{2n}}\,dx + \int_a^\infty \frac{x^{2m}}{1+x^{2n}}\,dx \text{ where } a > 0$$

The first integral on the right is a proper integral and, therefore, convergent. The given integral will be convergent or divergent according as $\int_a^\infty \frac{x^{2m}}{1+x^{2n}}\,dx$ is convergent or divergent.

Let $$f(x) = \frac{x^{2m}}{1+x^{2n}} = \frac{x^{2m}}{x^{2n}\left(1+\frac{1}{x^{2n}}\right)} = \frac{1}{x^{2n-2m}\left(1+\frac{1}{x^{2n}}\right)}$$

Take $$g(x) = \frac{1}{x^{2n-2m}}$$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{1}{1+\frac{1}{x^{2n}}} = 1 \qquad (\because\ n > 0)$$

which is non-zero and finite.

$\therefore$ By comparison test, $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ converge or diverge together.

But $\int_0^\infty g(x)\,dx = \int_0^\infty \frac{dx}{x^{2n-2m}}$ converges iff $2n - 2m > 1$ *i.e.*, $n - m > \frac{1}{2}$.

$\therefore$ $\int_a^\infty f(x)\,dx$ converges iff $n - m > \frac{1}{2}$. Hence the given integral converges iff $n - m > \frac{1}{2}$.

(*ii*) The given integral may be written as

$$\int_0^\infty \frac{x^{p-1}}{1+x}\,dx = \int_0^a \frac{x^{p-1}}{1+x}\,dx + \int_a^\infty \frac{x^{p-1}}{1+x}\,dx \text{ where } a > 0$$
$$= I_1 + I_2 \text{ (say).}$$

When p ≥ 1, I_1 is a proper integral and therefore, convergent. The given integral will be convergent or divergent according as I_2 is convergent or divergent.

Let $$f(x) = \frac{x^{p-1}}{1+x} = \frac{x^{p-1}}{x\left(1+\frac{1}{x}\right)} = \frac{1}{x^{2-p}\left(1+\frac{1}{x}\right)}$$

Take $$g(x) = \frac{1}{x^{2-p}}$$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} = 1 \text{ which is non-zero and finite.}$$

Since $p \geq 1, -p \leq -1$ so that $2 - p \leq 1$

$\therefore$ $\int_a^\infty g(x)\,dx = \int_a^\infty \frac{dx}{x^{2-p}}$ is divergent.

By comparison test, $\int_a^\infty f(x)\,dx = I_2$ is divergent.

Hence the given integral is divergent if $p \geq 1$.

When p < 1, 0 is a point of infinite discontinuity of $f(x)$ on $[0, a]$.

Now $$f(x) = \frac{1}{x^{1-p}\,(1+x)}$$

Take $$g(x) = \frac{1}{x^{1-p}}$$

$$\lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} \frac{1}{1+x} = 1 \text{ which is finite and non-zero.}$$

If we take $1 - p$ between 0 and 1 so that $0 < p < 1$, $\int_0^a g(x)\,dx$ is convergent.

$\therefore$ I_1 is convergent when $0 < p < 1$.

Also $$f(x) = \frac{1}{x^{2-p}\left(1+\frac{1}{x}\right)}$$

Take $$g(x) = \frac{1}{x^{2-p}}$$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} = 1 \text{ and } 2-p > 1 \text{ if } 0 < p < 1.$$

$\therefore$ $\int_0^\infty g(x)\, dx$ is convergent.

$\therefore$ I_2 is convergent when $0 < p < 1$.

Hence $\int_0^\infty \frac{x^{p-1}}{1+x}\, dx$ is convergent if $0 < p < 1$ and divergent if $p \geq 1$.

Note. When $p < 1$, we can also proceed as under :

$$\lim_{x\to\infty} \frac{x^{p-1}}{1+x} = \lim_{x\to\infty} \frac{1}{x^{1-p}(1+x)} = 0$$

$\therefore$ I_2 is bounded and hence convergent.

Example 5. *Test for convergent the integrals*

(i) $\int_0^\infty \frac{x \tan^{-1}}{(1+x^4)^{1/3}}\, dx$ (ii) $\int_0^\infty \frac{\sin^2 x}{x^2}\, dx$

(iii) $\int_e^\infty \frac{dx}{x(\log x)^{3/2}}$ (iv) $\int_e^\infty \frac{dx}{x(\log x)^{n+1}}$.

Sol. (i) $\int_0^\infty \frac{x \tan^{-1} x}{(1+x^4)^{1/3}}\, dx = \int_0^a \frac{x \tan^{-1} x}{(1+x^4)^{1/3}}\, dx + \int_a^\infty \frac{x \tan^{-1} x}{(1+x^4)^{1/3}} dx$ ...(1) where $a > 0$

The first integral on the right is a proper integral and, therefore, convergent.

Let $$f(x) = \frac{x \tan^{-1} x}{(1+x^4)^{1/3}} = \frac{x \tan^{-1} x}{x^{4/3}\left(1+\frac{1}{x^4}\right)^{1/3}} = \frac{\tan^{-1} x}{x^{1/3}\left(1+\frac{1}{x^4}\right)^{1/3}}$$

Take $$g(x) = \frac{1}{x^{1/3}}$$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\tan^{-1} x}{\left(1+\frac{1}{x^4}\right)^{1/3}} = \frac{\pi}{2} \text{ which is non-zero and finite.}$$

Since $\int_a^\infty g(x)\, dx = \int_a^\infty \frac{dx}{x^{1/3}}$ is divergent. $\left(\because\ n = \frac{1}{3} < 1\right)$

$\therefore$ $\int_a^\infty f(x)\, dx = \int_a^\infty \frac{x \tan^{-1} x}{(1+x^4)^{1/3}}\, dx$ is divergent.

Hence, from (1), $\int_0^\infty \frac{x \tan^{-1} x}{(1+x^4)^{1/3}}\, dx$ is divergent.

(ii) Since $\lim_{x\to 0+} \frac{\sin^2 x}{x^2} = \lim_{x\to 0+}\left(\frac{\sin x}{x}\right) = 1$, therefore, 0 is not a point infinite discontinuity.

Now $$\int_0^\infty \frac{\sin^2 x}{x^2}\, dx = \int_0^1 \frac{\sin^2 x}{x^2}\, dx + \int_1^\infty \frac{\sin^2 x}{x^2}\, dx \quad ...(1)$$

The first integral on right is a proper integral and, therefore, convergent.

Now we test the second integral on right for convergence at ∞.

Since $\frac{\sin^2 x}{x^2} \leq \frac{1}{x^2}$ and $\int_1^\infty \frac{dx}{x^2}$ is convergent. $(\because\ n = 2 > 1)$

$\therefore$ $\int_1^\infty \frac{\sin^2 x}{x^2}\, dx$ is also convergent.

Hence, from (1), $\int_0^\infty \frac{\sin^2 x}{x^2}\, dx$ is convergent.

(iii) Put $\log x = t$ so that $\frac{1}{x} dx = dt$ when $x = e$, $t = 1$; when $x \to \infty$, $t \to \infty$.

$\therefore$ $$\int_e^\infty \frac{dx}{x(\log x)^{3/2}} = \int_1^\infty \frac{dt}{t^{3/2}} \text{ which is convergent.} \quad \left(\because\ n = \frac{3}{2} > 1\right)$$

(iv) Put $\log x = t$, so that $\frac{1}{x} dx = dt$ when $x = e$, $t = 1$; when $x \to \infty$, $t \to \infty$.

$\therefore$ $\int_e^\infty \frac{dx}{x(\log x)^{n+1}} = \int_1^\infty \frac{dt}{t^{n+1}}$ which is convergent if $n+1 > 1$ i.e., $n > 0$ and divergent if $n+1 \leq 1$ i.e., $n \leq 0$.

Example 6. *Test for convergence the integrals*

(i) $\int_0^\infty e^{-x^2}\, dx$ (ii) $\int_1^\infty x^n e^{-x}\, dx$ (iii) $\int_1^\infty \frac{\log x}{x^2}\, dx$ (iv) $\int_0^\infty \frac{\cos x}{1+x^2}\, dx$.

Sol. (i) $$\int_0^\infty e^{-x^2}\, dx = \int_0^1 e^{-x^2}\, dx + \int_1^\infty e^{-x^2}\, dx \quad ...(1)$$

The first integral on the right is a proper integral and, therefore, convergent.

Let us consider $\int_1^\infty e^{-x^2}\, dx$.

We know that $e^{x^2} > x^2\ \forall\ x \in \mathbb{R}$

$\therefore$ $e^{-x^2} < \frac{1}{x^2}$

As $\int_1^\infty \frac{1}{x^2}\, dx$ converges at ∞, the integral $\int_1^\infty e^{-x^2}\, dx$ also converges.

$\therefore$ From (1), $\int_0^\infty e^{-x^2} dx$ is convergent.

Note. $\int_0^\infty e^{-x^2} dx$ is called the Euler-Poison integral and its value is $\frac{\sqrt{\pi}}{2}$.

(See Gamma Functions)

(ii) Let $\quad f(x) = x^n e^{-x}$

Take $\quad g(x) = \frac{1}{x^2}$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^{n+2}}{e^x} = 0 \text{ for all } n.$$

Since $\int_1^\infty g(x)\,dx = \int_1^\infty \frac{dx}{x^2}$ is convergent. $\quad (\because\ n = 2 > 1)$

$\therefore$ By comparison test, $\int_1^\infty f(x)\,dx = \int_1^\infty x^n e^{-x}\,dx$ is convergent.

(iii) Let $\quad f(x) = \frac{\log x}{x^2}$

Take $\quad g(x) = \frac{1}{x^{3/2}}$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\log x}{\sqrt{x}} = 0.$$

Since $\int_1^\infty g(x)\,dx = \int_1^\infty \frac{dx}{x^{3/2}}$ is convergent. $\quad \left(\because\ n = \frac{3}{2} > 1\right)$

$\therefore$ By comparison test, $\int_1^\infty f(x)\,dx = \int_1^\infty \frac{\log x}{x^2}\,dx$ is convergent.

(iv) $\quad \left|\frac{\cos x}{1+x^2}\right| \le \frac{1}{1+x^2}$ since $|\cos x| \le 1$

Also $\quad \int_0^\infty \frac{dx}{1+x^2} = \lim_{t\to\infty} \int_0^t \frac{dx}{1+x^2} = \lim_{t\to\infty} \left[\tan^{-1} x\right]_0^t = \lim_{t\to\infty} (\tan^{-1} t) = \frac{\pi}{2}$

$\therefore \int_0^\infty \frac{dx}{1+x^2}$ is convergent and, hence, by comparison test, $\int_0^\infty \frac{\cos x}{1+x^2}\,dx$ is convergent.

Example 7. *Show that* $\int_0^\infty \left(\frac{1}{1+x} - e^{-x}\right)\frac{dx}{x}$ *is convergent.*

Sol. Let $\quad f(x) = \left(\frac{1}{1+x} - e^{-x}\right).\frac{1}{x} = \frac{e^x - 1 - x}{x(1+x)e^x}$

$$= \frac{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right) - 1 - x}{x(1+x)e^x} = \frac{\frac{x^2}{2!} + \frac{x^3}{3!} + \ldots}{x(1+x)e^x} > 0\ \forall\ x > 0$$

Also $\quad \lim_{x\to 0+} f(x) = \lim_{x\to 0+} \frac{\frac{x}{2!} + \frac{x^2}{3!} + \ldots}{(1+x)e^x} = 0$

$\therefore$ 0 is not a point of infinite discontinuity.

$\therefore \quad \int_0^\infty f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^\infty f(x)\,dx \quad \ldots(1)$

Now $\quad f(x) = \frac{e^x - 1 - x}{e^x} \cdot \frac{x}{1+x} \cdot \frac{1}{x^2}$

and $\quad \lim_{x\to\infty} \frac{e^x - 1 - x}{e^x} = \lim_{x\to\infty}\left(1 - \frac{1+x}{e^x}\right) = 1 - \lim_{x\to\infty} \frac{1+x}{e^x} \quad \left|\frac{\infty}{\infty}\right.$

$$= 1 - \lim_{x\to\infty} \frac{1}{e^x} = 1 - 0 = 1$$

$$\lim_{x\to\infty} \frac{x}{1+x} = \lim_{x\to\infty} \frac{1}{\frac{1}{x}+1} = 1$$

$\therefore$ Take $\quad g(x) = \frac{1}{x^2}$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty}\left(\frac{e^x - 1 - x}{e^x} \cdot \frac{x}{1+x}\right) = 1 \times 1 = 1 \text{ and } \int_1^\infty g(x)\,dx \text{ is convergent.}$$

$\therefore$ By comparison test, $\int_1^\infty f(x)\,dx$ is convergent.

Also $\int_0^1 f(x)\,dx$ is a proper integral and, therefore, convergent.

Hence, from (1), $\int_0^\infty f(x)\,dx$ is convergent.

11.13. GENERAL TEST FOR CONVERGENCE AT ∞ *(Integrand may change sign)*

Cauchy's Test. *The improper integral* $\int_a^\infty f(x)\,dx$ *converges at* ∞ *if and only if to each* $\varepsilon > 0$*, there corresponds a positive real number* k *such that* $\left|\int_{t_1}^{t_2} f(x)\,dx\right| < \varepsilon\ \forall\ t_1, t_2 > k$.

Proof. We know that the improper integral $\int_a^\infty f(x)\,dx$ exists if $\lim_{t\to\infty} \int_a^t f(x)\,dx$ exists finitely.

Let $\quad F(t) = \int_a^t f(x)\,dx$

Now, the necessary and sufficient condition for $\lim_{t\to\infty} F(t)$ to exist finitely is that to every $\varepsilon > 0$ there corresponds a real number $k > 0$ such that

$$| F(t_1) - F(t_2) | < \varepsilon \quad \forall\, t_1, t_2 > k$$

$$\Rightarrow \quad \left| \int_a^{t_1} f(x)\,dx - \int_a^{t_2} f(x)\,dx \right| < \varepsilon \quad \Rightarrow \quad \left| -\int_{t_1}^{a} f(x)\,dx - \int_a^{t_2} f(x)\,dx \right| < \varepsilon$$

$$\Rightarrow \quad \left| \int_{t_1}^{t_2} f(x)\,dx \right| < \varepsilon \ \forall\, t_1, t_2 > k.$$

11.14. ABSOLUTE CONVERGENCE

Definition. The improper integral $\int_a^\infty f(x)\,dx$ is said to be absolutely convergent if $\int_a^\infty | f(x) |\,dx$ is convergent.

Theorem. *Every absolutely convergent integral is convergent*

or $\quad \int_a^\infty | f(x) |\,dx$ *exists* $\Rightarrow \int_a^\infty f(x)\,dx$ *exists.*

Proof. Since $\int_a^\infty | f(x) |\,dx$ exists, therefore, by Cauchy's test, for every $\varepsilon > 0$, there corresponds a real number $k > 0$ such that

$$\left| \int_{t_1}^{t_2} | f(x) |\,dx \right| < \varepsilon \ \forall\, t_1, t_2 < k \qquad \ldots(1)$$

Also, we know that $\quad \left| \int_{t_1}^{t_2} f(x)\,dx \right| \le \left| \int_{t_1}^{t_2} | f(x) |\,dx \right| \qquad \ldots(2)$

From (1) and (2), we have $\left| \int_{t_1}^{t_2} f(x)\,dx \right| < \varepsilon \ \forall\, t_1, t_2 > k$

$\therefore$ By Cauchy's test, $\int_a^\infty f(x)\,dx$ exists.

Note. The converse of the above theorem is not true. Every convergent integral is not absolutely convergent. A convergent integral which is not absolutely convergent is called a **conditionally convergent integral.**

11.15. TESTS FOR CONVERGENCE OF THE INTEGRAL OF A PRODUCT OF TWO FUNCTIONS

Test I. Abel's Test

If $\int_a^\infty f(x)\,dx$ is convergent at ∞ and $g(x)$ is bounded and monotonic for $x \ge a$, then $\int_a^\infty f(x)\,g(x)\,dx$ is convergent at ∞.

Or

An infinite integral which converges (not necessarily absolutely) will remain convergent after the insertion of a factor which is bounded and monotonic.

Proof. Since g is monotonic on $[a, \infty)$, it is integrable on $[a, t]$, for all $t \ge a$.

Also, since f is integrable on $[a, t]$, we have by Second Mean Value Theorem.

$$\int_{t_1}^{t_2} f(x)\,g(x)\,dx = g(t_1) \int_{t_1}^{\xi} f(x)\,dx + g(t_2) \int_{\xi}^{t_2} f(x)\,dx \qquad \ldots(1)$$

where $\qquad a < t_1 \le \xi \le t_2.$

Since g is bounded on $[a, \infty)$, there exists a positive number k such that

$$| g(x) | \le k \quad \forall\, x \ge a$$

In particular $\quad | g(t_1) | \le k,\ | g(t_2) | \le k \qquad \ldots(2)$

Let $\varepsilon > 0$ be given,

Since $\int_a^\infty f(x)\,dx$ is convergent, there exists a number t_0 such that

$$\left| \int_{t_1}^{t_2} f(x)\,dx \right| \le \frac{\varepsilon}{2k} \ \forall\, t_1, t_2 \ge t_0 \qquad \ldots(3)$$

Let the numbers t_1, t_2 in (1) be $\ge t_0$ so that the number ξ which lies between t_1 and t_2, is also $\ge t_0$. Hence from (3),

$$\left| \int_{t_1}^{\xi} f(x)\,dx \right| < \frac{\varepsilon}{2k}, \left| \int_{\xi}^{t_2} f(x)\,dx \right| < \frac{\varepsilon}{2k} \qquad \ldots(4)$$

From (1), (2) and (4), it follows that a positive number t_0 exists such that for all $t_1, t_2 \ge t_0$.

$$\left| \int_{t_1}^{t_2} f(x)\,g(x)\,dx \right| = \left| g(t_1) \int_{t_1}^{\xi} f(x)\,dx + g(t_2) \int_{\xi}^{t_2} f(x)\,dx \right|$$

$$\le | g(t_1) | \cdot \left| \int_{t_1}^{\xi} f(x)\,dx \right| + | g(t_2) | \cdot \left| \int_{\xi}^{t_2} f(x)\,dx \right| < k \cdot \frac{\varepsilon}{2k} + k \cdot \frac{\varepsilon}{2k} = \varepsilon$$

Hence, by Cauchy's test, $\int_a^\infty f(x)\,g(x)\,dx$ is convergent at ∞.

Test II. Dirichlet's Test

If $\int_a^t f(x)\,dx$ is bounded for all $t \ge a$ and $g(x)$ is a bounded and monotonic function for $x \ge a$, tending to 0 as $x \to \infty$, then $\int_\infty^a f(x)\,g(x)\,dx$ is convergent at ∞.

Or

An infinite integral which oscillates finitely becomes convergent after the insertion of a monotonic factor which tends to zero as a limit.

Proof. Since g is monotonic on $[a, \infty)$, it is integrable on $[a, t]$, for all $t \ge a$.

Also, since f is integrable on $[a, t]$, we have by Second Mean Value Theorem,

$$\int_{t_1}^{t_2} f(x)\,g(x)\,dx = g(t_1) \int_{t_2}^{\xi} f(x)\,dx + g(t_2) \int_{\xi}^{t_2} f(x)\,dx \qquad \ldots(1)$$

where $\qquad a < t_1 \le \xi \le t_2.$

Since $\int_a^t f(x)\,dx$ is bounded for all $t \ge a$, there exists a positive number k such that

$$\left| \int_a^t f(x)\,dx \right| \le k \quad \forall\, t \ge a \qquad \ldots(2)$$

Now $\quad \left| \int_{t_1}^{\xi} f(x)\,dx \right| = \left| \int_{t_1}^{a} f(x)\,dx + \int_a^{\xi} f(x)\,dx \right| = \left| \int_a^{\xi} f(x)\,dx - \int_a^{t_1} f(x)\,dx \right|$

$$\le \left| \int_a^{\xi} f(x)\,dx \right| + \left| \int_a^{t_1} f(x)\,dx \right|$$

$$\le k + k = 2k \ \forall\, t_1, \xi \ge a \qquad \ldots(3) \quad \text{[Using (2)]}$$

Similarly, $\left|\int_{\xi}^{t_2} f(x)\,dx\right| \le 2k \quad \forall\, t_2, \xi \ge a$ ...(4)

Let $\varepsilon > 0$ be given.

Since $\lim\limits_{x\to\infty} g(x) = 0$, there exists a number t_0 such that $|g(x)| < \dfrac{\varepsilon}{4k}\ \forall\, x \ge t_0$

Let the number t_1, t_2 in (1) be $\ge t_0$, then $|g(t_1)| < \dfrac{\varepsilon}{4k}$ and $|g(t_2)| < \dfrac{\varepsilon}{4k}$ ...(5)

From (1), (3), (4) and (5), it follows that a positive number t_0 exists such that for all $t_1, t_2 \ge t_0$,

$$\left|\int_{t_1}^{t_2} f(x)\,g(x)\,dx\right| = \left|g(t_1)\int_{t_1}^{\xi} f(x)\,dx + g(t_2)\int_{\xi}^{t_2} f(x)\,dx\right|$$

$$\le |g(t_1)|\left|\int_{t_1}^{\xi} f(x)\,dx\right| + |g(t_2)|\cdot\left|\int_{\xi}^{t_2} f(x)\,dx\right| < \frac{\varepsilon}{4k}\cdot 2k + \frac{\varepsilon}{4k}\cdot 2k = \varepsilon$$

Hence, by Cauchy's test, $\int_a^{\infty} f(x)\,g(x)\,dx$ is convergent at ∞.

ILLUSTRATIVE EXAMPLES

Example 1. *Examine the convergence of the integrals :*

(i) $\int_0^{\infty} \dfrac{\sin x}{x}\,dx$ (ii) $\int_0^{\infty} \dfrac{\sin x}{\sqrt{x}}\,dx$ (iii) $\int_0^{\infty} \dfrac{\sin x}{x^{3/2}}\,dx$

(iv) $\int_a^{\infty} \dfrac{\sin x}{x^m}\,dx$ *where a and m are both positive.*

(v) $\int_0^{\infty} \dfrac{\sin kx}{x}\,dx$ (vi) $\int_1^{\infty} \dfrac{\sin x^m}{x^n}\,dx.$

Sol. (i) Since $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$, therefore, 0 is not a point of infinite discontinuity.

Now $$\int_0^{\infty} \frac{\sin x}{x}\,dx = \int_0^1 \frac{\sin x}{x}\,dx + \int_1^{\infty} \frac{\sin x}{x}\,dx \quad ...(1)$$

Also $\int_0^1 \dfrac{\sin x}{x}\,dx$ is a proper integral. So, let us examine the convergence of $\int_1^{\infty} \dfrac{\sin x}{x}\,dx$ at ∞.

Let $f(x) = \sin x$ and $g(x) = \dfrac{1}{x}$.

Since $\left|\int_1^t f(x)\,dx\right| = \left|\int_1^t \sin x\,dx\right| = |\cos 1 - \cos t| \le |\cos 1| + |\cos t| \le 2$

$\therefore$ $\int_1^t f(x)\,dx$ is bounded for all $t \ge 1$.

Also $g(x)$ is a bounded and monotonically decreasing function tending to 0 as $x \to \infty$.

By Dirichlet's test, $\int_1^{\infty} f(x)\,g(x)\,dx = \int_1^{\infty} \dfrac{\sin x}{x}\,dx$ is convergent.

Hence, from (1), $\int_0^{\infty} \dfrac{\sin x}{x}\,dx$ is convergent.

(ii) Since $\lim\limits_{x\to 0} \dfrac{\sin x}{\sqrt{x}} = \lim\limits_{x\to 0} \dfrac{\sin x}{x} \times \sqrt{x} = 1 \times 0 = 0$

$\therefore$ 0 is not a point of infinite discontinuity.

Now $$\int_0^{\infty} \frac{\sin x}{\sqrt{x}}\,dx = \int_0^1 \frac{\sin x}{\sqrt{x}}\,dx + \int_1^{\infty} \frac{\sin x}{\sqrt{x}}\,dx \quad ...(1)$$

Also $\int_0^1 \dfrac{\sin x}{\sqrt{x}}\,dx$ is a proper integral. So, let us examine the convergence of $\int_1^{\infty} \dfrac{\sin x}{\sqrt{x}}\,dx$ at ∞.

Let $f(x) = \sin x$ and $g(x) = \dfrac{1}{\sqrt{x}}$

Since $\left|\int_1^t f(x)\,dx\right| \le 2$ [See part (i)]

$\therefore$ $\int_1^t f(x)\,dx$ is bounded for all $t \ge 1$.

Also $g(x)$ is a bounded and monotonically decreasing function tending to 0 as $x \to \infty$.

$\therefore$ By Dirichlet's test, $\int_1^{\infty} f(x)\,g(x)\,dx = \int_1^{\infty} \dfrac{\sin x}{\sqrt{x}}\,dx$ is convergent.

Hence, from (1), $\int_0^{\infty} \dfrac{\sin x}{\sqrt{x}}\,dx$ is convergent.

(iii) $$\int_0^{\infty} \frac{\sin x}{x^{3/2}}\,dx = \int_0^1 \frac{\sin x}{x^{3/2}}\,dx + \int_1^{\infty} \frac{\sin x}{x^{3/2}}\,dx \quad ...(1)$$

For the integral $\int_0^1 \dfrac{\sin x}{x^{3/2}}\,dx$, 0 is a point of infinite discontinuity.

Let $f(x) = \dfrac{\sin x}{x^{3/2}} = \dfrac{\sin x}{x}\cdot\dfrac{1}{\sqrt{x}}$

Take $g(x) = \dfrac{1}{\sqrt{x}}$

$$\lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} \frac{\sin x}{x} = 1 \text{ which is non-zero and finite.}$$

Since $\int_0^1 g(x)\,dx = \int_0^1 \dfrac{dx}{\sqrt{x}}$ is convergent. $\left(\because\ n = \dfrac{1}{2} < 1\right)$

$\therefore$ By comparison test, $\int_0^1 f(x)\,dx = \int_0^1 \dfrac{\sin x}{x^{3/2}}\,dx$ is convergent.

Convergence of $\int_1^{\infty} \dfrac{\sin x}{x^{3/2}}\,dx$ at ∞

Let $\quad f(x) = \sin x \quad$ and $\quad g(x) = \dfrac{1}{x^{3/2}}$

Since $\quad \left|\int_1^t f(x)\,dx\right| \le 2 \qquad$ [See part (*i*)]

$\therefore \quad \int_1^t f(x)\,dx$ is bounded for all $t \ge 1$.

Also $g(x)$ is a bounded and monotonically decreasing function tending to 0 as $x \to \infty$.

$\therefore \quad$ By Dirichlet's test, $\int_1^\infty f(x)\,g(x)\,dx = \int_1^\infty \dfrac{\sin x}{x^{3/2}}\,dx$ is convergent.

Hence, from (1), $\int_0^\infty \dfrac{\sin x}{x^{3/2}}\,dx$ is convergent.

(*iv*) Let $\quad f(x) = \sin x \quad$ and $\quad g(x) = \dfrac{1}{x^m},\ m > 0$

Since $\quad \left|\int_a^t f(x)\,dx\right| = \left|\int_a^t \sin x\,dx\right| = |\cos a - \cos t| \le |\cos a| + |\cos t| \le 2$

$\therefore \quad \int_a^t f(x)\,dx$ is bounded for all $t \ge a$.

Also $g(x)$ is a bounded and monotonically decreasing function tending to 0 as $x \to \infty$ for $m > 0$.

$\therefore \quad$ By Dirichlet's test, $\int_a^\infty f(x)\,g(x)\,dx = \int_a^\infty \dfrac{\sin x}{x^m}\,dx$, where m and a are both positive, is convergent.

(*v*) Since $\lim\limits_{x \to 0} \dfrac{\sin kx}{x} = \lim\limits_{x \to 0} k\left(\dfrac{\sin kx}{kx}\right) = k$, therefore, 0 is not a point of infinite discontinuity.

Now $\quad \int_0^\infty \dfrac{\sin kx}{x}\,dx = \int_0^1 \dfrac{\sin kx}{x}\,dx + \int_1^\infty \dfrac{\sin kx}{x}\,dx \qquad \ldots(1)$

Also $\int_0^1 \dfrac{\sin kx}{x}\,dx$ is a proper integral. So, let us examine the convergence of $\int_1^\infty \dfrac{\sin kx}{x}\,dx$ at ∞.

Let $\quad f(x) = \sin kx \quad$ and $\quad g(x) = \dfrac{1}{x}$

Since
$$\left|\int_1^t f(x)\,dx\right| = \left|\int_1^t \sin kx\,dx\right| = \left|\frac{\cos k - \cos kt}{k}\right|$$
$$\le \frac{1}{|k|}(|\cos k| + |\cos kt|) \le \frac{1}{|k|}(1+1) = \frac{2}{|k|}$$

$\therefore \quad \int_1^t f(x)\,dx$ is bounded for all $t \ge 1$.

Also $g(x)$ is a bounded and monotonically decreasing function tending to 0 as $x \to \infty$.

$\therefore \quad$ By Dirichlet's test, $\int_1^\infty f(x)\,g(x)\,dx = \int_1^\infty \dfrac{\sin kx}{x}\,dx$ is convergent.

Hence, from (1), $\int_0^\infty \dfrac{\sin kx}{x}\,dx$ is convergent.

(*vi*) **For m = 0,** the given integral reduces to $\sin 1 \int_1^\infty \dfrac{dx}{x^n}$ which converges at ∞ for $n > 1$.

For m ≠ 0, substituting $x^m = t$ so that $x = t^{1/m}$ and $dx = \dfrac{1}{m} t^{\frac{1}{m}-1}\,dt$, we get

$$\int_1^\infty \frac{\sin x^m}{x^n}\,dx = \frac{1}{m}\int_1^\infty \frac{\sin t}{t^{n/m}} \times t^{\frac{1}{m}-1}\,dt = \frac{1}{m}\int_1^\infty \frac{\sin t}{t^{\frac{n-1}{m}+1}}\,dt$$

Let $\quad f(t) = \sin t \quad$ and $\quad g(t) = \dfrac{1}{t^{\frac{n-1}{m}+1}}$

Since $\quad \left|\int_1^\lambda f(t)\,dt\right| = \left|\int_1^\lambda \sin t\,dt\right| = |\cos 1 - \cos\lambda| \le |\cos 1| + |\cos\lambda| \le 2$

$\therefore \quad \int_1^\lambda f(t)\,dt$ is bounded for all $\lambda \ge 1$.

Also $g(t)$ is a bounded and monotonically decreasing function tending to 0 as $t \to \infty$ when $\dfrac{n-1}{m} + 1 > 0$, *i.e.*, $n > 1 - m$.

$\therefore \quad$ By Dirichlet's test, $\int_1^\infty f(t)\,g(t)\,dt = \int_1^\infty \dfrac{\sin t}{t^{\frac{n-1}{m}+1}}\,dt$ is convergent at ∞ when $n > 1 - m$.

Hence $\int_1^\infty \dfrac{\sin x^m}{x^n}\,dx$ is convergent when $n > 1 - m$.

Example 2. *Examine the convergence of the integrals :*

(*i*) $\int_0^\infty \sin x^2\,dx \qquad$ (*ii*) $\int_0^\infty \dfrac{x}{1+x^2} \sin x\,dx \qquad$ (*iii*) $\int_0^\infty \cos x^3\,dx$

(*iv*) $\int_0^\infty \dfrac{\cos x}{\sqrt{x+x^2}}\,dx \qquad$ (*v*) $\int_a^\infty \dfrac{\cos \alpha x - \cos \beta x}{x}\,dx,\ a > 0 \qquad$ (*vi*) $\int_e^\infty \dfrac{\log x \sin x}{x}\,dx$.

Sol. (*i*) We have $\quad \int_0^\infty \sin x^2\,dx = \int_0^1 \sin x^2\,dx + \int_1^\infty \sin x^2\,dx \qquad \ldots(1)$

But $\int_0^1 \sin x^2\,dx$ is a proper integral and, therefore, convergent.

Convergence of $\int_1^\infty \sin x^2\,dx$ at ∞.

$$\int_1^\infty \sin x^2\,dx = \int_1^\infty (2x \sin x^2) \cdot \frac{1}{2x}\,dx$$

Let $\qquad f(x) = 2x \sin x^2$ and $g(x) = \dfrac{1}{2x}$

Since $\qquad \left|\int_1^t f(x)\,dx\right| = \left|\int_1^t 2x \sin x^2\,dx\right| = \left|\left\{-\cos x^2\right\}_1^t\right|$

$$= \left|\cos 1 - \cos t^2\right| \le |\cos 1| + |\cos t^2| \le 2$$

$\therefore \quad \int_1^t f(x)\,dx$ is bounded for all $t \ge 1$.

Also $g(x)$ is a bounded and monotonically decreasing function tending to 0 as $x \to \infty$.

$\therefore \quad$ By Dirichlet's test, $\int_1^\infty f(x)\,g(x)\,dx = \int_1^\infty \sin x^2\,dx$ is convergent.

Hence, from (1), $\int_0^\infty \sin x^2\,dx$ is convergent.

(*ii*) We have $\int_0^\infty \dfrac{x}{1+x^2} \sin x\,dx = \int_0^1 \dfrac{x}{1+x^2} \sin x\,dx + \int_1^\infty \dfrac{x}{1+x^2} \sin x\,dx \qquad \ldots(1)$

But $\int_0^1 \dfrac{x}{1+x^2} \sin x\,dx$ is proper integral and, therefore, convergent.

Convergence of $\int_1^\infty \dfrac{x}{1+x^2} \sin x\,dx$ at ∞.

Let $\qquad f(x) = \sin x$ and $g(x) = \dfrac{x}{1+x^2}$

Since $\qquad \left|\int_1^t f(x)\,dx\right| \le 2 \qquad$ [See Ex. 1 (*i*)]

$\therefore \quad \int_1^t f(x)\,dx$ is bounded for all $t \ge 1$.

Also $\qquad \lim_{x\to\infty} g(x) = \lim_{x\to\infty} \dfrac{x}{1+x^2} = 0$

$g(x)$ is a bounded and monotonically decreasing function tending to 0 as $x \to \infty$.

$\therefore \quad$ By Dirichlet's test, $\int_1^\infty f(x)\,g(x)\,dx = \int_1^\infty \dfrac{x}{1+x^2} \sin x\,dx$ is convergent.

Hence, from (1), $\int_0^\infty \dfrac{x}{1+x^2} \sin x\,dx$ is convergent.

(*iii*) *Please try yourself.* **[Ans. Convergent]**

$$\left[\int_0^\infty \cos x^3\,dx = \int_0^1 \cos x^3\,dx + \int_1^\infty \cos x^3\,dx\right.$$

~~and~~ $\qquad \left.\cos x^3 = (3x^2 \cos x^3) \cdot \dfrac{1}{3x^2} \text{ etc.}\right]$

(*iv*) Since $x = 0$ is also a point of infinite discontinuity of the integrand, we have to test the convergence of the given integral both at 0 and ∞.

Now $\qquad \int_0^\infty \dfrac{\cos x}{\sqrt{x+x^2}}\,dx = \int_0^a \dfrac{\cos x}{\sqrt{x+x^2}}\,dx + \int_a^\infty \dfrac{\cos x}{\sqrt{x+x^2}}\,dx,\ a > 0 \qquad \ldots(1)$

Convergence of $\int_0^a \dfrac{\cos x}{\sqrt{x+x^2}}\,dx$ at 0.

Let $\qquad f(x) = \dfrac{\cos x}{\sqrt{x+x^2}} = \dfrac{\cos x}{\sqrt{x}\sqrt{1+x}}$

Take $\qquad g(x) = \dfrac{1}{\sqrt{x}}$

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{\cos x}{\sqrt{1+x}} = 1 \text{ which is non-zero and finite.}$$

Also $\qquad \int_0^a g(x)\,dx = \int_0^a \dfrac{dx}{\sqrt{x}}$ is convergent $\qquad \left(\because\ n = \dfrac{1}{2} < 1\right)$

$\therefore \quad$ By comparison test, $\int_0^a f(x)\,dx = \int_0^a \dfrac{\cos x}{\sqrt{x+x^2}}\,dx$ is convergent.

Convergence of $\int_a^\infty \dfrac{\cos x}{\sqrt{x+x^2}}\,dx$ at ∞.

Let $\qquad f(x) = \cos x$ and $g(x) = \dfrac{1}{\sqrt{x+x^2}}$

Since $\qquad \left|\int_a^t f(x)\,dx\right| = \left|\int_a^t \cos x\,dx\right| = |\sin t - \sin a| \le |\sin t| + |\sin a| \le 2$

$\therefore \quad \int_a^t f(x)\,dx$ is bounded for all $t \ge a > 0$.

Also $g(x)$ is a bounded and monotonically decreasing function tending to 0 as $x \to \infty$.

$\therefore \quad$ By Dirichlet's test, $\int_a^\infty f(x)\,g(x)\,dx = \int_a^\infty \dfrac{\cos x}{\sqrt{x+x^2}}\,dx$ is convergent.

Hence, from (1), $\int_0^\infty \dfrac{\cos x}{\sqrt{x+x^2}}\,dx$ is convergent.

(*v*) $\int_a^\infty \dfrac{\cos \alpha x - \cos \beta x}{x}\,dx = \int_a^\infty \dfrac{\cos \alpha x}{x}\,dx - \int_a^\infty \dfrac{\cos \beta x}{x}\,dx$

$$= I_1 - I_2 \text{ (say)} \qquad \ldots(1)$$

Consider $\qquad I_1 = \int_a^\infty \dfrac{\cos \alpha x}{x}\,dx$

Let $\qquad f(x) = \cos \alpha x$ and $g(x) = \dfrac{1}{x}$

Since $\left|\int_a^t f(x)\,dx\right| = \left|\int_a^t \cos\alpha x\,dx\right| = \left|\dfrac{\sin\alpha t - \sin\alpha a}{\alpha}\right|$

$$\le \frac{1}{|\alpha|}(|\sin\alpha t| + |\sin\alpha a|) \le \frac{1}{|\alpha|}(1+1) = \frac{2}{|\alpha|}$$

$\therefore$ $\int_a^t f(x)\,dx$ is bounded for all $t \ge a$.

Also $g(x)$ is a bounded and monotonically decreasing function tending to 0 as $x \to \infty$.

$\therefore$ By Dirichlet's test, $\int_a^\infty f(x)\,g(x)\,dx = I_1$ is convergent.

Similarly, I_2 is convergent. Hence, from (1), the given integral is convergent.

(*vi*) Let $f(x) = \sin x$ and $g(x) = \dfrac{\log x}{x}$

Since $\left|\int_e^t f(x)\,dx\right| = \left|\int_e^t \sin x\,dx\right| = |\cos e - \cos t| \le |\cos e| + |\cos t| \le 2$

$\therefore$ $\int_e^t f(x)\,dx$ is bounded for all $t \ge e$.

Also $g'(x) = \dfrac{1-\log x}{x^2} < 0$ for $x > e$

and $\lim\limits_{x\to\infty} g(x) = \lim\limits_{x\to\infty} \dfrac{\log x}{x} \quad \left|\dfrac{\infty}{\infty}\right.$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$$

$\therefore$ $g(x)$ is a bounded $\left[\text{in fact, } 0 < g(x) \le \dfrac{1}{e}\right]$ and monotonically decreasing function tending to 0 as $x \to \infty$.

$\therefore$ By Dirichlet's test, $\int_e^\infty f(x)\,g(x)\,dx = \int_e^\infty \dfrac{\log x \sin x}{x}\,dx$ is convergent.

Example 3. *Examine the convergence of the integrals :*

(*i*) $\int_0^\infty e^{-ax}\dfrac{\sin x}{x}\,dx,\ a \ge 0$ (*ii*) $\int_0^\infty e^{-ax^2}\cos bx\,dx,\ a > 0$

(*iii*) $\int_a^\infty e^{-x}\cdot\dfrac{\sin x}{x^2}\,dx,\ a > 0$ (*iv*) $\int_a^\infty (1-e^{-x})\cdot\dfrac{\cos x}{x^2}\,dx,\ a > 0.$

Sol. (*i*) Let $f(x) = \dfrac{\sin x}{x}$ and $g(x) = e^{-ax},\ a \ge 0.$

Since $\int_0^\infty f(x)\,dx$ is convergent [See example 1 (*i*)]

and $g(x)$ is a bounded and monotonically decreasing function of x for $x > 0$.

$\therefore$ By Abel's test, $\int_0^\infty f(x)\,g(x)\,dx = \int_0^\infty e^{-ax}\cdot\dfrac{\sin x}{x}\,dx$ is convergent.

(*ii*) $\int_0^\infty e^{-ax^2}\cos bx\,dx = \int_0^1 e^{-ax^2}\cos bx\,dx + \int_1^\infty e^{-ax^2}\cos bx\,dx$ ...(1)

Now $\int_0^1 e^{-ax^2}\cos bx\,dx$ is a proper integral and, hence, convergent.

Convergence of $\int_1^\infty e^{-ax^2}\cos bx\,dx$ at ∞.

Let $f(x) = \cos bx$ and $g(x) = e^{-ax^2},\ a > 0$

Since $\left|\int_1^t f(x)\,dx\right| = \left|\int_1^t \cos bx\,dx\right| = \left|\dfrac{\sin bt - \sin b}{b}\right|$

$$\le \frac{1}{|b|}(|\sin bt| + |\sin b|) \le \frac{2}{|b|}$$

$\therefore$ $\int_1^t f(x)\,dx$ is bounded for all $t \ge 1$.

Also $g(x)$ is a bounded and monotonically decreasing function tending to 0 as $x \to \infty$.

$\therefore$ By Dirichlet's test, $\int_1^\infty f(x)\,g(x)\,dx = \int_1^\infty e^{-ax^2}\cos bx\,dx$ is convergent.

Hence, from (1), $\int_0^\infty e^{-ax^2}\cos bx\,dx$ is convergent.

(*iii*) Let $f(x) = \dfrac{\sin x}{x^2}$ and $g(x) = e^{-x}$

Since $|f(x)| = \left|\dfrac{\sin x}{x^2}\right| \le \dfrac{1}{x^2}$ and $\int_a^\infty \dfrac{dx}{x^2}$ is convergent

($\because$ $n = 2 > 1$), it follows by comparison test that

$\int_a^\infty f(x)\,dx$ is also convergent.

Again $g(x)$ is monotonic decreasing and bounded function for $x > a$.

$\therefore$ By Abel's test, $\int_a^\infty f(x)\,g(x)\,dx = \int_a^\infty e^{-x}\cdot\dfrac{\sin x}{x^2}\,dx$ is convergent.

(*iv*) *Please try yourself.* **[Ans.** Convergent]

$\left[\text{Take } f(x) = \dfrac{\cos x}{x^2} \text{ and } g(x) = 1 - e^{-x}\right]$

12

Beta and Gamma Functions

12.1. BETA FUNCTION

Definition. If $m > 0, n > 0$ then the integral $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$, which is obviously a function of m and n, is called a **Beta function** and is denoted by $B(m, n)$.

Thus $B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \;\forall\; m > 0, n > 0$. Beta function is also called the **First Eulerian Integral.**

For example,

(i) $\int_0^1 x^3(1-x)^5\,dx = B(3+1, 5+1) = B(4, 6)$

(ii) $\int_0^1 \sqrt{x}(1-x)^3\,dx = B\left(\frac{1}{2}+1, 3+1\right) = B\left(\frac{3}{2}, 4\right)$

(iii) $\int_0^1 x^{-2/3}(1-x)^{-\frac{1}{2}}\,dx = B\left(-\frac{2}{3}+1, -\frac{1}{2}+1\right) = B\left(\frac{1}{3}, \frac{1}{2}\right)$

(iv) $\int_0^1 x^{-3}(1-x)^5\,dx$ is not a Beta function since $m = -3+1 = -2 < 0$.

12.2. CONVERGENCE OF BETA FUNCTION

Theorem. *Show that $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ exists if and only if m and n are both positive.*

Proof. The integral is proper if $m \ge 1$ and $n \ge 1$. 0 is the only point of infinite discontinuity if $m < 1$ and 1 is the only point of infinite discontinuity if $n < 1$.

For m < 1 and n < 1.

Take a number, $\frac{1}{2}$ (say), between 0 and 1 and examine the convergence of the improper integrals.

$$\int_0^{\frac{1}{2}} x^{m-1}(1-x)^{n-1}\,dx, \quad \int_{\frac{1}{2}}^1 x^{m-1}(1-x)^{n-1}\,dx \text{ at 0 and 1 respectively.}$$

Convergence at 0, when m < 1

Let $\qquad f(x) = x^{m-1}(1-x)^{n-1} = \dfrac{(1-x)^{n-1}}{x^{1-m}}$

Take $\qquad g(x) = \dfrac{1}{x^{1-m}}$

Then $\qquad \underset{x\to 0+}{\text{Lt}} \dfrac{f(x)}{g(x)} = \underset{x\to 0+}{\text{Lt}} (1-x)^{n-1} = 1$ which is non-zero, finite.

Also $\qquad \int_0^{\frac{1}{2}} g(x)\,dx = \int_0^{\frac{1}{2}} \dfrac{dx}{x^{1-m}}$ is convergent if and only if $1-m < 1$ i.e., $m > 0$.

$$\left[\because \int_a^b \frac{dx}{(x-a)^n} \text{ is convergent iff } n < 1\right]$$

$\therefore$ By comparison test, $\int_0^{\frac{1}{2}} f(x)\,dx = \int_0^{\frac{1}{2}} x^{m-1}(1-x)^{n-1}\,dx$ is convergent at $x = 0$ if $m > 0$.

Convergence at 1, when n < 1

Let $\qquad f(x) = x^{m-1}(1-x)^{n-1} = \dfrac{x^{m-1}}{(1-x)^{1-n}}$

Take $\qquad g(x) = \dfrac{1}{(1-x)^{1-n}}$

Let $\qquad \underset{x\to 1-}{\text{Lt}} \dfrac{f(x)}{g(x)} = \underset{x\to 1-}{\text{Lt}} x^{m-1} = 1$ which is non-zero, finite.

Also $\qquad \int_{\frac{1}{2}}^1 g(x)\,dx = \int_{\frac{1}{2}}^1 \dfrac{dx}{(1-x)^{1-n}}$ is convergent if and only if $1-n < 1$, i.e., $n > 0$

$$\left[\because \int_a^b \frac{dx}{(b-x)^n} \text{ is convergent iff } n < 1\right]$$

$\therefore$ By comparison test, $\int_{\frac{1}{2}}^1 f(x)\,dx = \int_{\frac{1}{2}}^1 x^{m-1}(1-x)^{n-1}\,dx$ is convergent at $x = 1$ if $n > 0$.

Hence $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ converges iff $m > 0, n > 0$.

12.3. PROPERTIES OF BETA FUNCTION

Property I. *Symmetry of Beta function i.e., $B(m, n) = B(n, m)$.*

Proof. By definition, $\quad B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx, m > 0, n > 0$

Changing x to $1-x$ $\qquad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

$$B(m, n) = \int_0^1 (1-x)^{m-1}[1-(1-x)]^{n-1}\,dx$$

$$= \int_0^1 (1-x)^{m-1}x^{n-1}\,dx = \int_0^1 x^{n-1}(1-x)^{m-1}\,dx = B(n, m)$$

Hence $\qquad B(m, n) = B(n, m).$

Property II. *If m, n are positive integers, then* $B(m, n) = \dfrac{(m-1)!\,(n-1)!}{(m+n-1)!}$.

Proof. $B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

Integrating by parts
$$= \left[x^{m-1}\cdot\frac{(1-x)^n}{n(-1)}\right]_0^1 - \int_0^1 (m-1)\,x^{m-2}\cdot\frac{(1-x)^n}{n(-1)}\,dx$$
$$= \frac{m-1}{n}\int_0^1 x^{m-2}(1-x)^n\,dx = \frac{m-1}{n}\int_0^1 x^{m-2}(1-x)^{n-1}(1-x)\,dx$$
$$= \frac{m-1}{n}\int_0^1 [x^{m-2}(1-x)^{n-1} - x^{m-1}(1-x)^{n-1}]\,dx$$
$$= \frac{m-1}{n}\int_0^1 x^{m-2}(1-x)^{n-1}\,dx - \frac{m-1}{n}\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$
$$= \frac{m-1}{n}\,B(m-1, n) - \frac{m-1}{n}\,B(m, n)$$
$$\Rightarrow \left(1+\frac{m-1}{n}\right)B(m, n) = \frac{m-1}{n}\,B(m-1, n)$$
$$\Rightarrow B(m, n) = \frac{m-1}{m+n-1}\,B(m-1, n) \qquad \ldots(1)$$

Changing m to $(m-1)$, we have $B(m-1, n) = \dfrac{m-2}{m+n-2}\,B(m-2, n)$

Putting this value of $B(m-1, n)$ in (1), we have
$$B(m, n) = \frac{(m-1)(m-2)}{(m+n-1)(m+n-2)}\,B(m-2, n) \qquad \ldots(2)$$

Generalising from (1) and (2)
$$B(m, n) = \frac{(m-1)(m-2)\ldots\ldots 1}{(m+n-1)(m+n-2)\ldots\ldots(n+1)}\,B(1, n) \qquad \ldots(3)$$

But $\quad B(1, n) = \int_0^1 x^0(1-x)^{n-1}\,dx = \left[\dfrac{(1-x)^n}{n(-1)}\right]_0^1 = \dfrac{1}{n}$

$\therefore$ From (3), we get
$$B(m, n) = \frac{(m-1)(m-2)\ldots\ldots 1}{(m+n-1)(m+n-2)\ldots\ldots(n+1)n} = \frac{(m-1)!}{(m+n-1)(m+n-2)\ldots\ldots(n+1)\,n}$$

Multiplying the num. and denom. by $(n-1)!$, we have
$$B(m, n) = \frac{(m-1)!\,(n-1)!}{(m+n-1)(m+n-2)\ldots\ldots(n+1)\,n\,.\,(n-1)!} = \frac{(m-1)!\,(n-1)!}{(m+n-1)!}$$

Property III. $B(m, n) = \displaystyle\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx,\ m > 0, n > 0.$

Proof. $B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

Put $x = \dfrac{z}{1+z}$ then $dx = \dfrac{(1+z).1 - z.1}{(1+z)^2}\,dz = \dfrac{dz}{(1+z)^2}$
$$1 - x = 1 - \frac{z}{1+z} = \frac{1}{1+z}$$

Also $\quad x(1+z) = z \Rightarrow x = z(1-x)$ or $z = \dfrac{x}{1-x}$

when $x = 0, z = 0$

When $x \to 1, z \to \infty$
$$\therefore\quad B(m, n) = \int_0^\infty \left(\frac{z}{1+z}\right)^{m-1}\left(\frac{1}{1+z}\right)^{n-1}\frac{dz}{(1+z)^2}$$
$$= \int_0^\infty \frac{z^{m-1}}{(1+z)^{m+n}}\,dz = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

[Second Method]

Put $\dfrac{x}{1+x} = z$ then $x = z(1+x)$ or $x(1-z) = z$
$$\therefore\quad x = \frac{z}{1-z}$$
$$dx = \frac{(1-z).1 - z(-1)}{(1-z)^2}\,dz = \frac{dz}{(1-z)^2}$$
$$1 + x = 1 + \frac{1}{1-z} = \frac{1}{1-z}$$

When $x = 0, z = 0$; When $x \to \infty$,
$$z = \operatorname*{Lt}_{x\to\infty} \frac{x}{1+x} \qquad \left| \text{Form } \frac{\infty}{\infty}\right.$$
$$= \operatorname*{Lt}_{x\to\infty} \frac{1}{1} = 1$$
$$\therefore\quad \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \int_0^1 \left(\frac{z}{1-z}\right)^{m-1}(1-x)^{m+n}\cdot\frac{dz}{(1-z)^2}$$
$$= \int_0^1 z^{m-1}(1-z)^{n-1}\,dz = B(m, n)$$

Cor. We have proved that
$$B(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$
$$\therefore\quad B(n, m) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

But $\quad B(m, n) = B(n, m)$

$$\therefore \quad B(m, n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\, dx$$

Hence $$B(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\, dx.$$

ILLUSTRATIVE EXAMPLES

Example 1. *Express the following integrals in terms of Beta functions :*

(i) $\int_0^1 x^m(1-x^2)^n\, dx$ *if* $m > -1, n > -1$ (ii) $\int_0^1 \frac{x^2}{\sqrt{1-x^5}}\, dx$

(iii) $\int_0^2 (8-x^3)^{-1/3}\, dx.$

Sol. (i) Put $x^2 = z$ i.e., $x = z^{1/2}$ so that $dx = \frac{1}{2} z^{-1/2}\, dz$

When $x = 0, z = 0$; when $x = 1, z = 1$

$$\therefore \quad \int_0^1 x^m (1-x^2)^n\, dx = \int_0^1 z^{\frac{m}{2}} (1-z)^n \cdot \frac{1}{2} z^{-1/2}\, dz$$

$$= \frac{1}{2}\int_0^1 z^{\frac{m-1}{2}} (1-z)^n\, dz = \frac{1}{2} B\left(\frac{m-1}{2}+1, n+1\right)$$

$$= \frac{1}{2} B\left(\frac{m+1}{2}, n+1\right).$$

(ii) Put $x^5 = z$, i.e., $x = z^{1/5}$ so that $dx = \frac{1}{5} z^{-4/5}\, dz$

When $x = 0, z = 0$; when $x = 1, z = 1$.

$$\therefore \quad \int_0^1 \frac{x^2}{\sqrt{1-x^5}}\, dx = \int_0^1 x^2(1-x^5)^{-1/2}\, dx$$

$$= \int_0^1 z^{2/5} (1-z)^{-1/2} \cdot \frac{1}{5} z^{-4/5}\, dz = \frac{1}{5}\int_0^1 z^{-2/5} (1-z)^{-1/2}\, dz$$

$$= \frac{1}{5} B\left(-\frac{2}{5}+1, -\frac{1}{2}+1\right) = \frac{1}{5} B\left(\frac{3}{5}, \frac{1}{2}\right).$$

(iii) Put $x^3 = 8z$, i.e., $x = 2z^{1/3}$ so that $dx = \frac{2}{3} z^{-2/3}\, dz$

When $x = 0, z = 0$; when $x = 2, z = 1$.

$$\therefore \quad \int_0^2 (8-x^3)^{-1/3}\, dx = \int_0^1 (8-8z)^{-1/3} \cdot \frac{2}{3} z^{-2/3}\, dz = \int_0^1 \frac{2}{3} z^{-2/3} \frac{1}{2} (1-z)^{-1/3}\, dz$$

$$= \frac{1}{3}\int_0^1 z^{-2/3} (1-z)^{-1/3}\, dz = \frac{1}{3} B\left(-\frac{2}{3}+1, -\frac{1}{3}+1\right) = \frac{1}{3} B\left(\frac{1}{3}, \frac{2}{3}\right).$$

Example 2. *Express the following as Beta functions :*

(i) $\int_0^2 \sqrt{x}\, (4-x^2)^{-1/4}\, dx$ (ii) $\int_0^1 x^3 (1-x^2)^{3/2}\, dx$

(iii) $\int_0^2 x^3 (8-x^3)^{-1/3}\, dx$ (iv) $\int_0^1 x^{m-1} (1-x^2)^{n-1}\, dx.$

Sol. (i) Put $x^2 = 4z$, i.e., $x = 2z^{1/2}$ so that $dx = z^{-1/2}\, dz$

when $x = 0, z = 0$; when $x = 2, z = 1$

$$\therefore \quad \int_0^2 \sqrt{x}\, (4-x^2)^{-1/4}\, dx = \int_0^1 2^{1/2} z^{1/4} (4-4z)^{-1/4} z^{-1/2}\, dz$$

$$= \int_0^1 2^{1/2} z^{-1/4} \cdot 4^{-1/4} (1-z)^{-1/4}\, dz = \int_0^1 z^{-1/4} (1-z)^{-1/4}\, dz$$

$$= B\left(-\frac{1}{4}+1, -\frac{1}{4}+1\right) = B\left(\frac{3}{4}, \frac{3}{4}\right)$$

(ii) *Please try yourself.* $\left[\textbf{Ans. } B\left(4, \frac{5}{2}\right)\right]$

(iii) *Please try yourself.* $\left[\textbf{Ans. } \frac{8}{3} B\left(\frac{4}{3}, \frac{2}{3}\right)\right]$

(iv) *Please try yourself.* $\left[\textbf{Ans. } \frac{1}{2} B\left(\frac{1}{2}m, n\right)\right]$

Example 3. *Show that* $\int_0^p x^m (p^q - x^q)^n\, dx = \frac{p^{qn+m+1}}{q} B\left(n+1, \frac{m+1}{q}\right)$ *if* $p > 0, q > 0,$ $m > -1, n > -1.$

Sol. Put $x^q = p^q . z$, i.e., $x = pz^{\frac{1}{q}}$ so that $dx = \frac{p}{q} z^{\frac{1}{q}-1}\, dz$

When $x = 0, z = 0$; when $x = p, z = 1$

$$\therefore \quad \int_0^p x^m (p^q - x^q)^n\, dx = \int_0^1 p^m z^{\frac{m}{q}} (p^q - p^q z)^n \cdot \frac{p}{q} z^{\frac{1}{q}-1}\, dz$$

$$= \int_0^1 p^m \cdot z^{\frac{m}{q}} \cdot p^{qn} (1-z)^n \cdot \frac{p}{q} z^{\frac{1}{q}-1}\, dz$$

$$= \frac{p^{qn+m+1}}{q}\int_0^1 z^{\frac{m+1}{q}-1} (1-z)^n\, dz$$

$$= \frac{p^{qn+m+1}}{q} B\left(\frac{m+1}{q}, n+1\right) = \frac{p^{qn+m+1}}{q} B\left(n+1, \frac{m+1}{q}\right)$$

$| \because \; B(m, n) = B(n, m)$

Example 4. *Prove that* $\int_0^a (a-x)^{m-1} . x^{n-1}\, dx = a^{m+n-1} B(m, n).$

Sol. *Please try yourself.* (Put $x = az$)

Example 5. *Show that* $\int_0^n \left(1-\frac{x}{n}\right)^n . x^{t-1}\,dx = n^t B(t, n+1)$ *when* $t > 0, n > -1$.

Sol. Put $\frac{x}{n} = z$ so that $dx = n dz$

When $x = 0, z = 0$; when $x = n, z = 1$

$$\therefore \int_0^n \left(1-\frac{x}{n}\right)^n . x^{t-1}\,dx = \int_0^1 (1-z)^n . (nz)^{t-1} . n\,dz = n^t \int_0^1 z^{t-1}(1-z)^n\,dz = n^t B(t, n+1).$$

Example 6. *Show that if* $m > 0, n > 0$ *then* $\int_a^b (x-a)^{m-1}(b-x)^{n-1}\,dx = (b-a)^{m+n-1} B(m, n)$.

Sol. Put $x = a + (b-a)z$; so that $dx = (b-a)\,dz$

When $x = a, z = 0$; when $x = b, z = 1$

$$\therefore \int_a^b (x-a)^{m-1}(b-x)^{n-1}\,dx = \int_0^1 [(b-a)z]^{m-1}[b-a-(b-a)z]^{n-1}(b-a)\,dz$$

$$= \int_0^1 (b-a)^{m-1} . z^{m-1} . (b-a)^{n-1} . (1-z)^{n-1} . (b-a)\,dz$$

$$= (b-a)^{m+n-1}\int_0^1 z^{m-1}(1-z)^{n-1}\,dz = (b-a)^{m+n-1} B(m, n).$$

Example 7. *Show that* $\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+bx)^{m+n}}\,dx = \frac{1}{(a+b)^m . a^n} B(m, n)$.

Sol. Put $$\frac{x}{a+bx} = \frac{z}{a+b.1}$$

so that $$\frac{(a+bx).1 - x.b}{(a+bx)^2}\,dx = \frac{dz}{a+b} \text{ or } \frac{a}{(a+bx)^2}\,dx = \frac{dz}{a+b}$$

$$\therefore \frac{dx}{(a+bx)^2} = \frac{dz}{a(a+b)}$$

When $x = 0, z = 0$; when $x = 1, z = 1$

$$\therefore \int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+bx)^{m+n}}\,dx = \int_0^1 \left(\frac{x}{a+bx}\right)^{m-1} \cdot \left(\frac{1-x}{a+bx}\right)^{n-1} \cdot \frac{1}{(a+bx)^2}\,dx$$

$$= \int_0^1 \left(\frac{z}{a+b}\right)^{m-1} \cdot \left(\frac{1-z}{a}\right)^{n-1} \cdot \frac{dz}{a(a+b)}$$

$$\left[\because (a+b)x = az + bxz \text{ or } x = \frac{az}{a+b-bz} \therefore \frac{1-x}{a+bx} = \frac{1-z}{a}\right]$$

$$= \frac{1}{(a+b)^m . a^n}\int_0^1 z^{m-1}(1-z)^{n-1}\,dz = \frac{1}{(a+b)^m . a^n} B(m, n).$$

Example 8. *Show that* $\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+bx)^{m+n}}\,dx = \frac{B(m, n)}{a^n (1+a)^m}$.

Sol. Please try yourself (same as Ex. 7 with $b = 1$).

Example 9. *Express* $\int_0^1 x^m (1-x^n)^p\,dx$ *in terms of Beta function and hence evaluate* $\int_0^1 x^5(1-x^3)^3\,dx$.

Sol. Put $x^n = z$, *i.e.*, $x = z^{1/n}$ so that $dx = \frac{1}{n} z^{\frac{1}{n}-1}\,dz$

When $x = 0, z = 0$; when $x = 1, z = 1$

$$\therefore \int_0^1 x^m (1-x^n)^p\,dx = \int_0^1 z^{\frac{m}{n}}(1-z)^p . \frac{1}{n} z^{\frac{1}{n}-1}\,dz$$

$$= \frac{1}{n}\int_0^1 z^{\frac{m+1}{n}-1}(1-z)^p\,dz = \frac{1}{n} B\left(\frac{m+1}{n}, p+1\right) \quad \ldots(1)$$

Comparing $\int_0^1 x^5(1-x^3)^3\,dx$ with $\int_0^1 x^m(1-x^n)^p\,dx$, we have $m = 5, n = 3, p = 3$

$\therefore$ From (1), $\int_0^1 x^5(1-x^3)^3\,dx$

$$= \frac{1}{3} B\left(\frac{5+1}{3}, 3+1\right) = \frac{1}{3} B(2, 4) = \frac{1}{3}\int_0^1 x^1(1-x)^3\,dx$$

$$= \frac{1}{3}\int_0^1 (1-x)[1-(1-x)]^3\,dx = \frac{1}{3}\int_0^1 (1-x)^3 x\,dx$$

$$= \frac{1}{3}\int_0^1 (x^3 - x^4)\,dx = \frac{1}{3}\left[\frac{x^4}{4} - \frac{x^5}{5}\right]_0^1 = \frac{1}{3}\left[\frac{1}{4} - \frac{1}{5}\right] = \frac{1}{60}.$$

Example 10. *Prove that* $\int_0^{\pi/2} \frac{\cos^{2m-1}\theta \sin^{2n-1}\theta}{(a\cos^2\theta + b\sin^2\theta)^{m+n}}\,d\theta = \frac{B(m, n)}{2a^m b^n}$.

Sol.
$$I = \int_0^{\pi/2} \frac{\cos^{2m-1}\theta \sin^{2n-1}\theta}{(a\cos^2\theta + b\sin^2\theta)^{m+n}}\,d\theta$$

$$= \int_0^{\pi/2} \frac{\cos^{2m-2}\theta \sin^{2n-2}\theta . \cos\theta\sin\theta}{(a\cos^2\theta + b\sin^2\theta)^{m+n}}\,d\theta$$

$$= \int_0^{\pi/2} \frac{(\cos^2\theta)^{m-1}(\sin^2\theta)^{n-1}\cos\theta\sin\theta}{(a\cos^2\theta + b\sin^2\theta)^{m+n}}\,d\theta$$

Put $\sin^2\theta = x$ so that $2\sin\theta\cos\theta\,d\theta = dx$ and $\cos^2\theta = 1 - \sin^2\theta = 1 - x$

When $\theta = 0, x = 0$; when $\theta = \pi/2, x = 1$

$$\therefore I = \int_0^1 \frac{(1-x)^{m-1} x^{n-1} . \frac{1}{2}dx}{[a(1-x)+bx]^{m+n}} = \frac{1}{2}\int_0^1 \frac{(1-x)^{m-1} x^{n-1}}{[a+(b-a)x]^{m+n}}\,dx$$

Put $$\frac{x}{a+(b-a)x} = \frac{z}{a+(b-a).1} = \frac{z}{b}$$

$\therefore$ $$\frac{[a+(b-a)x].1 - x.(b-a)}{[a+(b-a)x]^2}\,dx = \frac{dz}{b} \Rightarrow \frac{dx}{[a+(b-a)x]^2} = \frac{dz}{ab}$$

When $x = 0, z = 0$; when $x = 1, z = 1$

Also $$\frac{x}{a+(b-a)x} = \frac{z}{b} \Rightarrow bx = az + (b-a)xz$$

$$\Rightarrow \quad [b-(b-a)z] = azx \Rightarrow x = \frac{az}{b-(b-a)z}$$

$$1 - x = 1 - \frac{az}{b-(b-z)z} = \frac{b(1-z)}{b-(b-a)z}$$

$$a+(b-a)x = \frac{bx}{z} = \frac{abz}{b-(b-a)z} \quad \text{so that} \quad \frac{1-x}{a+(b-a)x} = \frac{1-z}{az}.$$

$\therefore$ $$I = \frac{1}{2}\int_0^1 \frac{(1-x)^{m-1}x^{n-1}}{[a+(b-a)x]^{m+n}}\,dx$$

$$= \frac{1}{2}\int_0^1 \left[\frac{1-x}{a+(b-a)x}\right]^{n-1}\left[\frac{x}{a+(b-a)x}\right]^{n-1}\cdot\frac{dx}{[a+(b-a)x]^2}$$

$$= \frac{1}{2}\int_0^1 \left(\frac{1-z}{az}\right)^{m-1}\left(\frac{z}{b}\right)^{n-1}\frac{dz}{ab} = \frac{1}{2a^m b^n}\int_0^1 z^{n-1}(1-z)^{m-1}\,dz$$

$$= \frac{B(n,m)}{2a^m b^n} = \frac{B(m,n)}{2a^m b^n} \qquad [\because \ B(n,m) = B(m,n)]$$

Example 11. *Prove that if p, q are positive then*

(*i*) $\dfrac{B(p,q+1)}{q} = \dfrac{B(p+1,q)}{p} = \dfrac{B(p,q)}{p+q}$ (*ii*) $B(p,q) = B(p+1,q) + B(p,q+1)$.

Sol. (*i*) $$\frac{B(p,q+1)}{q} = \frac{1}{q}\int_0^1 x^{p-1}(1-x)^q\,dx = \frac{1}{q}\int_0^1 (1-x)^q . x^{p-1}\,dx$$

Integrating by parts $$= \frac{1}{q}\left[\left\{(1-x)^q . \frac{x^p}{p}\right\}_0^1 - \int_0^1 q(1-x)^{q-1}(-1).\frac{x^p}{p}\,dx\right]$$

$$= \frac{1}{q}.\frac{q}{p}\int_0^1 x^p(1-x)^{q-1}\,dx \qquad ...(I)$$

$$= \frac{B(p+1,q)}{p} \qquad ...(II)$$

Also from (1), $$\frac{B(p,q+1)}{q} = \frac{1}{p}\int_0^1 x^p(1-x)^{q-1}\,dx = \frac{1}{p}\int_0^1 x^{p-1}.x(1-x)^{q-1}\,dx$$

$$= \frac{1}{p}\int_0^1 x^{p-1}[1-(1-x)](1-x)^{q-1}\,dx$$

$$= \frac{1}{p}\int_0^1 x^{p-1}(1-x)^{q-1}\,dx - \frac{1}{p}\int_0^1 x^{p-1}(1-x)^q\,dx$$

$$= \frac{1}{p}B(p,q) - \frac{1}{p}B(p,q+1)$$

$$\frac{B(p,q+1)}{q} + \frac{B(p,q+1)}{p} = \frac{B(p,q)}{p} \quad \text{or} \quad \frac{p+q}{pq}B(p,q+1) = \frac{1}{p}B(p,q)$$

$\therefore$ $$\frac{B(p,q+1)}{q} = \frac{B(p,q)}{p+q} \qquad ...(III)$$

From (II) and (III),

$$\frac{B(p,q+1)}{q} = \frac{B(p+1,q)}{p} = \frac{B(p,q)}{p+q}$$

Note. For another method, see Gamma function.

(*ii*) R.H.S. $= B(p+1,q) + B(p,q+1)$

$$= \int_0^1 x^p(1-x)^{q-1}\,dx + \int_0^1 x^{p-1}(1-x)^q\,dx$$

$$= \int_0^1 [x^p(1-x)^{q-1} + x^{p-1}(1-x)^q]\,dx$$

$$= \int_0^1 x^{p-1}(1-x)^{q-1}[x + (1-x)]\,dx$$

$$= \int_0^1 x^{p-1}(1-x)^{q-1}\,dx = B(p,q) = \text{L.H.S.}$$

Example 12. *Prove that* $\dfrac{B(m+1,n)}{B(m,n)} = \dfrac{m}{m+n}$, $m > 0, n < 0$.

Sol. $$B(m+1,n) = \int_0^1 x^m(1-x)^{n-1}\,dx$$

Integrating by parts $$= \left[x^m.\frac{(1-x)^n}{-n}\right]_0^1 - \int_0^1 mx^{m-1}.\frac{(1-x)^n}{-n}\,dx$$

$$= \frac{m}{n}\int_0^1 x^{m-1}(1-x)^n\,dx = \frac{m}{n}\int_0^1 x^{m-1}(1-x)^{n-1}(1-x)\,dx$$

$$= \frac{m}{n}\left[\int_0^1 x^{m-1}(1-x)^{n-1}\,dx - \int_0^1 x^m(1-x)^{n-1}\,dx\right]$$

$$= \frac{m}{n}[B(m,n) - B(m+1,n)]$$

$$\Rightarrow \quad \left(1+\frac{m}{n}\right)B(m+1,n) = \frac{m}{n}B(m,n) \Rightarrow \frac{B(m+1,n)}{B(m,n)} = \frac{m}{m+n}.$$

Note. For another method, see Gamma function.

Example 13. *Using the property $B(m, n) = B(n, m)$, evaluate* $\int_0^1 x^3(1-x)^{4/3}\,dx$.

Sol. $\int_0^1 x^3(1-x)^{4/3}\,dx = B\left(3+1, \frac{4}{3}+1\right)$

$$= B\left(4, \frac{7}{3}\right) = B\left(\frac{7}{3}, 4\right) = \int_0^1 x^{4/3}(1-x)^3\,dx$$

$$= \int_0^1 x^{4/3}(1-3x+3x^2-x^3)\,dx = \int_0^1 (x^{4/3} - 3x^{7/3} + 3x^{10/3} - x^{13/3})\,dx$$

$$= \left[\frac{x^{7/3}}{\frac{7}{3}} - 3\cdot\frac{x^{10/3}}{\frac{10}{3}} + 3\cdot\frac{x^{13/3}}{\frac{13}{3}} - \frac{x^{16/3}}{\frac{16}{3}}\right]_0^1 = \frac{3}{7} - \frac{9}{10} + \frac{9}{13} - \frac{3}{16} = \frac{243}{7280}.$$

Example 14. *Prove that* $B(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta$.

Sol. $B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

Put $x = \sin^2\theta$ so that $dx = 2\sin\theta\cos\theta\,d\theta$

When $x = 0$, $\theta = 0$; when $x = 1$, $\theta = \frac{\pi}{2}$

$$\therefore \quad B(m, n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1}(1-\sin^2\theta)^{n-1}\,.\,2\sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-2}\theta\cos^{2n-2}\theta\,.\,\sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta.$$

Example 15. *Show that* $\int_0^{\pi/2} \sin^p\theta\cos^q\theta\,d\theta = \frac{1}{2}B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$ *where* $p > -1$, $q > -1$.

Deduce that $\int_0^2 x^4(8-x^3)^{-1/3}\,dx = \frac{16}{3}B\left(\frac{5}{3}, \frac{2}{3}\right)$.

Sol. Put $\sin^2\theta = z$ so that $2\sin\theta\cos\theta\,d\theta = dz$

When $\theta = 0$, $z = 0$; when $\theta = \frac{\pi}{2}$, $z = 1$

Also $\quad \cos^2\theta = 1 - \sin^2\theta = 1 - z$

$$\therefore \quad \int_0^{\pi/2} \sin^p\theta\cos^q\theta\,d\theta = \int_0^{\pi/2} (\sin^{p-1}\theta\cos^{q-1}\theta)\sin\theta\cos\theta\,d\theta$$

$$= \int_0^{\pi/2} (\sin^2\theta)^{\frac{p-1}{2}}(\cos^2\theta)^{\frac{q-1}{2}}\,.\,\sin\theta\cos\theta\,d\theta$$

$$= \int_0^1 z^{\frac{p-1}{2}}(1-z)^{\frac{q-1}{2}}\,.\,\frac{1}{2}\,dz = \frac{1}{2}\int_0^1 z^{\frac{p-1}{2}}(1-z)^{\frac{q-1}{2}}\,dz$$

$$= \frac{1}{2}B\left(\frac{p-1}{2}+1, \frac{q-1}{2}+1\right) = \frac{1}{2}B\left(\frac{p+1}{2}, \frac{q+1}{2}\right) \quad \ldots(1)$$

Second Part. Put $x^3 = 8z$ *i.e.*, $x = 2z^{1/3}$ so that $dx = \frac{2}{3}z^{-2/3}\,dz$

When $x = 0$, $z = 0$; when $x = 2$, $z = 1$

$$\therefore \quad \int_0^2 x^4(8-x^3)^{-1/3}\,dx = \int_0^1 16z^{4/3}(8-8z)^{-1/3}\,.\,\frac{2}{3}z^{-2/3}\,dz$$

$$= \int_0^1 \frac{32}{3}\times 8^{-1/3}z^{2/3}(1-z)^{-1/3}\,dz = \frac{32}{3\times 2}\int_0^1 z^{2/3}(1-z)^{-1/3}\,dz$$

$$= \frac{16}{3}\int_0^{\pi/2} \sin^{4/3}\theta(\cos^2\theta)^{-1/3}\times 2\sin\theta\cos\theta\,d\theta \text{ where } z = \sin^2\theta$$

$$= \frac{32}{3}\int_0^{\pi/2} \sin^{7/3}\theta\cos^{1/3}\theta\,d\theta$$

$$= \frac{32}{3}\cdot\frac{1}{2}B\left(\frac{\frac{7}{3}+1}{2}, \frac{\frac{1}{3}+1}{2}\right) \qquad \left|\text{ Here } p = \frac{7}{3}, q = \frac{1}{3} \text{ [using I]}\right.$$

$$= \frac{16}{3}B\left(\frac{5}{3}, \frac{2}{3}\right).$$

Example 16. *By putting* $\frac{x}{1-x} = \frac{at}{1-t}$, *where the constant a is suitably selected, show that*

$$\int_0^1 x^{-1/3}(1-x)^{-2/3}(1+2x)^{-1}\,dx = \frac{1}{9^{1/3}}B\left(\frac{2}{3}, \frac{1}{3}\right).$$

Sol. $\quad \frac{x}{1-x} = \frac{at}{1-t} \Rightarrow x - tx = at - atx$

$\Rightarrow \quad x[1-(1-a)t] = at \quad \Rightarrow \quad x = \frac{at}{1-(1-a)t}$

$$\therefore \quad 1 - x = 1 - \frac{at}{1-(1-a)t} = \frac{1-t}{1-(1-a)t}$$

$$1 + 2x = 1 + \frac{2at}{1-(1-a)t} = \frac{1-(1-3a)t}{1-(1-a)t}$$

Also $\quad dx = \frac{[1-(1-a)t]\,.\,a - at[-(1-a)]}{[1-(1-a)t]^2}\,dt = \frac{a\,dt}{[1-(1-a)t]^2}$

When $x = 0$, $t = 0$; when $x = 1$, $\frac{at}{1-(1-a)t} = 1$ so that $t = 1$

$$\therefore \int_0^1 x^{-1/3}(1-x)^{-2/3}(1+2x)^{-1}\,dx$$

$$= \int_0^1 \left[\frac{at}{1-(1-a)t}\right]^{-\frac{1}{3}} \cdot \left[\frac{1-t}{1-(1-a)t}\right]^{-\frac{2}{3}} \cdot \left[\frac{1-(1-3a)t}{1-(1-a)t}\right]^{-1} \cdot \frac{a\,dt}{[1-(1-a)t]^2}$$

$$= a^{2/3}\int_0^1 t^{-\frac{1}{3}}(1-t)^{-\frac{2}{3}}[1-(1-3a)t]^{-1}\,dt$$

Choosing $1-3a=0$ *i.e.,* $a=\frac{1}{3}$, we get

$$\int_0^1 x^{-1/3}(1-x)^{-2/3}(1+2x)^{-1}\,dx = \left(\frac{1}{3}\right)^{2/3}\int_0^1 t^{-1/3}(1-t)^{-2/3}\,dt$$

$$= \frac{1}{9^{1/3}}B\left(-\frac{1}{3}+1, -\frac{2}{3}+1\right) = \frac{1}{9^{1/3}}B\left(\frac{2}{3},\frac{1}{3}\right).$$

Example 17. *Show that* $\int_0^1 \frac{(1-x^4)^{3/4}}{(1+x^4)^2}\,dx = \frac{1}{4(2)^{1/4}}B\left(\frac{7}{4},\frac{1}{4}\right).$

Sol. Put $\frac{1-x^4}{1+x^4} = z$ so that $x^4 = \frac{1-z}{1+z}$ *i.e.,* $x = \left(\frac{1-z}{1+z}\right)^{1/4}$

$$dx = \frac{1}{4}\left(\frac{1-z}{1+z}\right)^{-3/4} \times \frac{(1+z)(-1)-(1-z).1}{(1+z)^2}\,dz$$

$$= \frac{1}{4}\left(\frac{1+z}{1-z}\right)^{3/4} \cdot \frac{-z}{(1+z)^2}\,dz = \frac{-dz}{2(1-z)^{3/4}(1+z)^{5/4}}$$

Also

$$1-x^4 = 1-\frac{1-z}{1+z} = \frac{2z}{1+z}$$

$$1+x^4 = 1+\frac{1-z}{1+z} = \frac{2}{1+z}$$

When $x=0, z=1$; When $x=1, z=0$

$$\therefore \int_0^1 \frac{(1-x^4)^{3/4}}{(1+x^4)^2}\,dx = \int_0^1 \frac{\left(\frac{2z}{1+z}\right)^{3/4}}{\left(\frac{2}{1+z}\right)^2} \times \frac{-dz}{2(1-z)^{3/4}(1+z)^{5/4}}$$

$$= \int_0^1 \frac{1}{4(2)^{1/4}} z^{3/4}(1-z)^{-3/4}\,dz = \frac{1}{4(2)^{1/4}}B\left(\frac{7}{4},\frac{1}{4}\right).$$

Example 18. *Show that* $\int_0^\pi \frac{\sin^{n-1}x}{(a+b\cos x)^n}\,dx = \frac{2^{n-1}}{(a^2-b^2)^{n/2}}B\left(\frac{1}{2}n,\frac{1}{2}n\right)$, *if* $a^2 > b^2$.

Sol. Let

$$I = \int_0^\pi \frac{\sin^{n-1}x}{(a+b\cos x)^n}\,dx = \int_0^\pi \frac{\left(2\sin\frac{x}{2}\cos\frac{x}{2}\right)^{n-1}}{\left[a+b\left(1-2\sin^2\frac{x}{2}\right)\right]^n}\,dx$$

$$= 2^{n-1}\int_0^\pi \frac{\sin^{n-1}\frac{x}{2}\cos^{n-1}\frac{x}{2}}{\left(a+b-2b\sin^2\frac{x}{2}\right)^n}\,dx$$

Put $\frac{x}{2} = \theta$ then $dx = 2d\theta$

When $x=0, \theta=0$; when $x=\pi, \theta=\frac{\pi}{2}$

$$\therefore \quad I = 2^{n-1}\int_0^{\pi/2} \frac{\sin^{n-1}\theta\cos^{n-1}\theta}{(a+b-2b\sin^2\theta)^n}\,.\,2d\theta$$

$$= 2^{n-1}\int_0^{\pi/2} \frac{\sin^{n-2}\theta\cos^{n-2}\theta\,.\,2\sin\theta\cos\theta}{(a+b-2b\sin^2\theta)^n}\,d\theta$$

Put $\sin^2\theta = t$ so that $2\sin\theta\cos\theta\,d\theta = dt$

When $\theta=0, t=0$, when $\theta=\frac{\pi}{2}, t=1$

$$\therefore \quad I = 2^{n-1}\int_0^{\pi/2} \frac{(\sin^2\theta)^{\frac{n-2}{2}}(1-\sin^2\theta)^{\frac{n-2}{2}}\,.\,2\sin\theta\cos\theta}{(a+b-2b\sin^2\theta)^n}\,d\theta$$

$$= 2^{n-1}\int_0^1 \frac{t^{\frac{n-2}{2}}(1-t)^{\frac{n-2}{2}}}{(a+b-2bt)^n}\,dt$$

Put $\frac{1-t}{a+b-2bt} = \frac{z}{a+b}$, *i.e.,* $t = \frac{(a+b)(1-z)}{a+b-2bz}$

$$dt = \frac{a+b}{(a+b-2bz)^2}[(a+b-2bz)(-1)-(1-z)(-2b)]\,dz$$

$$= \frac{(a+b)(-a+b)}{(a+b-2bz)^2}\,dz = -\frac{a^2-b^2}{(a+b-2bz)^2}\,dz$$

Also

$$1-t = 1-\frac{(a+b)(1-z)}{a+b-2bz} = \frac{(a-b)z}{a+b-2bz}$$

and

$$a+b-2bt = a+b-\frac{2b(a+b)(1-z)}{a+b-2bz} = \frac{a^2-b^2}{a+b-2bz}$$

When $t = 0, z = 1$, when $t = 1, z = 0$

$$\therefore \quad I = -2^{n-1} \int_1^0 \frac{\left[\frac{(a+b)(1-z)}{a+b-2bz}\right]^{\frac{n-2}{2}} \left[\frac{(a-b)z}{a+b-2bz}\right]^{\frac{n-2}{2}}}{\left[\frac{a^2-b^2}{a+b-2bz}\right]^n} \times \frac{a^2-b^2}{(a+b-2bz)^2}\, dz$$

$$= 2^{n-1} \int_0^1 \frac{(a^2-b^2)^{\frac{n-2}{2}} (1-z)^{\frac{n-2}{2}} . z^{\frac{n-2}{2}}}{(a^2-b^2)^{n-1}}\, dz$$

$$= 2^{n-1} \int_0^1 \frac{z^{n/2-1}(1-z)^{n/2-1}}{(a^2-b^2)^{n/2}}\, dz = \frac{2^{n-1}}{(a^2-b^2)^{n/2}} B\left(\frac{n}{2}, \frac{n}{2}\right).$$

Example 19. *Prove that* $\int_0^\infty \frac{t^3}{(1+t)^7}\, dt = \frac{1}{60}$.

Sol. $\int_0^\infty \frac{t^3}{(1+t)^7}\, dt = \int_0^\infty \frac{t^{4-1}}{(1+t)^{4+3}}\, dt$ $\qquad \left[\text{Form} \int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}}\, dt = B(m,n)\right]$

$$= B(4, 3) = \int_0^1 t^3(1-t)^2\, dt = \int_0^1 t^3(1-2t+t^2)\, dt$$

$$= \int_0^1 (t^3 - 2t^4 + t^5)\, dt = \left[\frac{t^4}{4} - \frac{2t^5}{5} + \frac{t^6}{6}\right]_0^1 = \frac{1}{4} - \frac{2}{5} + \frac{1}{6} = \frac{1}{60}.$$

Example 20. *Express* $\int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}}\, dx$ *in terms of Beta function, where* $m > 0, n > 0$; $a > 0, b > 0$.

Sol. Put $bx = az$ or $x = \frac{az}{b}$ so that $dx = \frac{a}{b}\, dz$

When $x = 0, z = 0$ and when $x \to \infty, z \to \infty$

$$\therefore \quad \int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}}\, dx = \int_0^\infty \left(\frac{az}{b}\right)^{m-1} \cdot \frac{1}{(a+az)^{m+n}} \cdot \frac{a}{b}\, dz$$

$$= \int_0^\infty \frac{a^{m-1} . z^{m-1} . a}{b^{m-1} . a^{m+n} (1+z)^{m+n} . b}\, dz$$

$$= \frac{1}{a^n b^m} \int_0^\infty \frac{z^{m-1}}{(1+z)^{m+n}}\, dz = \frac{1}{a^n b^m} B(m, n).$$

Example 21. *Show that* $\int_0^\infty \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}\, dx = 2B(m, n)$, *where* $m > 0, n > 0$.

Sol. $\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx = B(m, n)$

Also $\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\, dx = B(m, n)$

Adding $\int_0^\infty \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}\, dx = 2B(m, n)$.

Example 22. *Prove that* $\int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}\, dx = B(m, n)$, *where m, n are both positive.*

Sol. $$B(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\, dx + \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx \qquad ...(i)$$

In the second integral on R.H.S. of (*i*), put $x = \frac{1}{t}$, so that $dx = -\frac{1}{t^2}\, dt$

When $x = 1, t = 1$; when $x \to \infty, t = 0$

$$\therefore \quad \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx = \int_1^0 \frac{\left(\frac{1}{t}\right)^{m-1}}{\left(1+\frac{1}{t}\right)^{m+n}} \left(-\frac{1}{t^2}\right) dt = \int_0^1 \frac{1}{t^{m-1}} \cdot \frac{t^{m+n}}{(1+t)^{m+n}} \cdot \frac{1}{t^2}\, dt$$

$$= \int_0^1 \frac{t^{n-1}}{(1+t)^{m+n}}\, dt = \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\, dx$$

$$\left[\because \int_a^b f(x)\, dx = \int_a^b f(z)\, dz\right]$$

$$\therefore \text{ From } (i), \quad B(m, n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\, dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\, dx = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}\, dx.$$

Example 23. *For* $m > 0, n > 0$, *show that* $\int_0^\infty \frac{x^{m-1} - x^{n-1}}{(1+x)^{m+n}}\, dx = 0$.

Sol. $$\int_0^\infty \frac{x^{m-1} - x^{n-1}}{(1+x)^{m+n}}\, dx = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx - \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\, dx$$

$$= B(m, n) - B(n, m) = 0.$$

12.4. GAMMA FUNCTION

Definition. If $n > 0$, then the integral $\int_0^\infty x^{n-1} e^{-x}\,dx$, which is obviously a function of n, is called a **Gamma function** and is denoted by $\Gamma(n)$.

Thus $$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\,dx, \forall\, n > 0$$

Gamma function is also called the **Second Eulerian Integral.**

For example,

(i) $\int_0^\infty x^3 e^{-x}\,dx = \Gamma(3+1) = \Gamma(4)$ (ii) $\int_0^\infty x^{2/3} e^{-x}\,dx = \Gamma\left(\frac{2}{3}+1\right) = \Gamma\left(\frac{5}{3}\right)$.

12.5. CONVERGENCE OF GAMMA FUNCTION

Theorem. *Show that* $\int_0^\infty x^{n-1} e^{-x}\,dx$ *converge iff* $n > 0$.

Proof. If $n \geq 1$, then integrand $x^{n-1} e^{-x}$ is continuous at $x = 0$.

If $n < 1$, the integrand $\frac{e^{-x}}{x^{1-n}}$ has infinite discontinuity at $x = 0$.

Thus we have to examine the convergence at 0 and ∞ both. Consider any positive number, say 1, and examine the convergence of $\int_0^1 x^{n-1} e^{-x}\,dx$ and $\int_1^\infty x^{n-1} e^{-x}\,dx$ at 0 and ∞ respectively.

Convergence at 0, when n < 1

Let $$f(x) = \frac{e^{-x}}{x^{1-n}}$$

Take $$g(x) = \frac{1}{x^{1-n}}$$

Then $$\operatorname*{Lt}_{x\to 0+} \frac{f(x)}{g(x)} = \operatorname*{Lt}_{x\to 0+} e^{-x} = 1 \text{ which is non-zero, finite.}$$

Also $$\int_0^1 g(x)\,dx = \int_0^1 \frac{dx}{x^{1-n}} \text{ is convergent iff } 1-n<1 \text{ i.e., } n>0$$

$\therefore$ By comparison test, $\int_0^1 f(x)\,dx = \int_0^1 \frac{e^{-x}}{x^{1-n}}\,dx = \int_0^1 x^{n-1} e^{-x}\,dx$ is convergent at $x = 0$ if $n = 0$.

Convergence at ∞

We know that $e^x > x^{n+1}$ whatever value n may have

$$\therefore \quad e^{-x} < x^{-n-1} \text{ and } x^{n-1} e^{-x} < x^{n-1} \cdot x^{-n-1} = \frac{1}{x^2}$$

Since $\int_1^\infty \frac{1}{x^2}\,dx$ is convergent at ∞.

$\therefore$ $\int_1^\infty x^{n-1} e^{-x}\,dx$ is convergent at ∞ for every value of n.

Now $$\int_0^\infty x^{n-1} e^{-x}\,dx = \int_0^1 x^{n-1} e^{-x}\,dx + \int_1^\infty x^{n-1} e^{-x}\,dx$$

$\therefore$ $\int_0^\infty x^{n-1} e^{-x}\,dx$ converges iff $n > 0$.

12.6. RECURRENCE FORMULA FOR GAMMA FUNCTION

Prove that $\Gamma(n) = (n-1)\,\Gamma(n-1)$, *when* $n > 1$

Proof. By def. $$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\,dx$$

Integrating by parts
$$= \left[x^{n-1} \cdot \frac{e^{-x}}{-1}\right]_0^\infty - \int_0^\infty (n-1) x^{n-2} \cdot \left(\frac{e^{-x}}{-1}\right) dx$$
$$= -\left[\operatorname*{Lt}_{x\to\infty} \frac{x^{n-1}}{e^x} - 0\right] + (n-1)\int_0^\infty e^{-x} \cdot x^{n-2}\,dx$$
$$= (n-1)\int_0^\infty e^{-x} \cdot x^{n-2}\,dx \qquad \left[\because \operatorname*{Lt}_{x\to\infty} \frac{x^{n-1}}{e^x} = 0 \text{ for } n>0\right]$$
$$= (n-1)\,\Gamma(n-1)$$

Hence $$\Gamma(n) = (n-1)\,\Gamma(n-1).$$

Cor. If n is a positive integer, then $\Gamma(n) = (n-1)\,!$

When n is a +ve integer, then by repeated application of above formula, we get
$$\Gamma(n) = (n-1)\,\Gamma(n-1)$$
$$= (n-1)\,.\,(n-2)\,\Gamma(n-2)$$
$$= (n-1)(n-2)\,.\,(n-3)\,\Gamma(n-3)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$= (n-1)(n-2)\ldots\ldots 1\,\Gamma(1)$$
$$= (n-1)\,!\,\Gamma(1)$$

But $$\Gamma(1) = \int_0^\infty x^0 e^{-x}\,dx \qquad \text{(By def.)}$$
$$= \int_0^\infty e^{-x}\,dx = \left[\frac{e^{-x}}{-1}\right]_0^\infty = -\left[\operatorname*{Lt}_{x\to\infty} \frac{1}{e^x} - e^0\right] = -[0-1] = 1$$

Hence $\Gamma(n) = (n-1)\,!$ when n is a +ve integer.

Note. (i) **If n is a +ve fraction**, then $\Gamma(n) = (n \times 1) \times$ go on decreasing by 1 ..., the series of factors being continued so long as the factors remain positive, the last factor being Γ(last factor).

For example, $\Gamma\left(\frac{7}{2}\right) = \frac{5}{2}\,\Gamma\left(\frac{5}{2}\right) = \frac{5}{2}\cdot\frac{3}{2}\,\Gamma\left(\frac{3}{2}\right) = \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)$

(ii) **If n is a +ve integer,** $\Gamma(n) = (n-1)\,!$

12.7. RELATION BETWEEN BETA AND GAMMA FUNCTIONS

To show that $B(m, n) = \dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$ *where* $m > 0, n > 0$.

Proof. We know that for $n > 0$, $\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\,dx$

Putting $x = az$ so that $dx = a\,dz$, we have

$$\Gamma(n) = \int_0^\infty (az)^{n-1} e^{-az} \,.\, a\,dz = \int_0^\infty a^n z^{n-1} e^{-az}\,dz$$

Replacing z by x, $\quad = \int_0^\infty a^n x^{n-1} e^{-ax}\,dx$

Replacing a by z, we have $\quad \Gamma(n) = \int_0^\infty z^n x^{n-1} e^{-zx}\,dx$

Multiplying both sides by $e^{-z} z^{m-1}$, we have

$$\Gamma(n) \,.\, e^{-z} z^{m-1} = \int_0^\infty x^{n-1} z^{m+n-1} e^{-z(1+x)}\,dx$$

Integrating both sides w.r.t. z between the limits 0 to ∞, we have

$$\Gamma(n) \int_0^\infty e^{-z} z^{m-1}\,dx = \int_0^\infty \int_0^\infty x^{n-1} z^{m+n-1} e^{-z(1+x)}\,dx\,dz$$

$$= \int_0^\infty \int_0^\infty x^{n-1} z^{m+n-1} e^{-z(1+x)}\,dz\,dx$$

$$\Rightarrow \quad \Gamma(n)\,\Gamma(m) = \int_0^\infty x^{n-1} \left[\int_0^\infty z^{m+n-1} e^{-z(1+x)}\,dz\right] dx$$

Putting $z(1+x) = y$ so that $dz = \dfrac{dy}{1+x}$

$$\Gamma(n)\,\Gamma(m) = \int_0^\infty x^{n-1} \left[\int_0^\infty \left(\frac{y}{1+x}\right)^{m+n-1} e^{-y}\,\frac{dy}{1+x}\right] dx$$

$$= \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} \left[\int_0^\infty y^{m+n-1} e^{-y}\,dy\right] dx$$

$$= \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} [\Gamma(m+n)]\,dx = \Gamma(m+n) \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

$$= \Gamma(m+n)\,B(m, n) \qquad \left[\because \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx = B(m, n)\right]$$

Hence $\quad B(m, n) = \dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$.

(Second Method)

We know that $\quad \Gamma(m) = \int_0^\infty e^{-t} \,.\, t^{m-1}\,dt$

Putting $t = x^2$ so that $dt = 2x\,dx$

$$\Gamma(m) = 2\int_0^\infty e^{-x^2} \,.\, x^{2m-1}\,dx \qquad \text{...(1)}$$

Similarly $\quad \Gamma(n) = 2\int_0^\infty e^{-y^2} \,.\, y^{2n-1}\,dy \qquad \text{...(2)}$

Now we use the following result from double integrals :

If $f(x)$ and $g(y)$ are functions of x and y only, and the limits of integration are constants, then the double integral can be represented as a product of two integrals. Thus

$$\int_c^d \int_a^b f(x)\,g(y)\,dx\,dy = \int_a^b f(x)\,dx \times \int_c^d g(y)\,dy$$

$\therefore$ From (1) and (2), we have

$$\Gamma(m)\,\Gamma(n) = 4\int_0^\infty e^{-x^2} \,.\, x^{2m-1}\,dx \times \int_0^\infty e^{-y^2} \,.\, y^{2n-1}\,dy$$

$$= 4\int_0^\infty e^{-(x^2+y^2)} \,.\, x^{2m-1} \,.\, y^{2n-1}\,dx\,dy$$

Changing to polar coordinates with $x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$; the region of integration being the complete positive quadrant, r must vary from 0 to ∞ and θ from 0 to $\dfrac{\pi}{2}$.

$$\Gamma(m)\,\Gamma(n) = 4\int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2} (r^{2m-1}\cos^{2m-1}\theta)(r^{2n-1}\sin^{2n-1}\theta)\,r\,dr\,d\theta$$

$$= 4\int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2} \,.\, r^{2(m+n)-1} \,.\, \cos^{2m-1}\theta \sin^{2n-1}\theta\,dr\,d\theta$$

$$= 2\int_0^\infty e^{-r^2} \,.\, r^{2(m+n)-1}\,dr \times 2\int_0^{\frac{\pi}{2}} \cos^{2m-1}\theta \sin^{2n-1}\theta\,d\theta \qquad \text{...(3)}$$

Now $\quad 2\int_0^\infty e^{-r^2} \,.\, r^{2(m+n)-1}\,dr = \Gamma(m+n)$ from (1)

and putting $\sin^2\theta = z$ so that $2\sin\theta\cos\theta\,d\theta = dz$

$$2\int_0^{\frac{\pi}{2}} \cos^{2m-1}\theta \sin^{2n-1}\theta\,d\theta = 2\int_0^{\frac{\pi}{2}} (\cos^{2m-2}\theta \sin^{2n-2}\theta)\cos\theta\sin\theta\,d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} (\sin^2\theta)^{n-1} (\cos^2\theta)^{m-1} \cos\theta\sin\theta\,d\theta$$

$$= \int_0^1 z^{n-1} (1-z)^{m-1}\,dz$$

$= B(n, m) = B(m, n)$ by symmetry of Beta function.

$\therefore$ From (3), we have

$$\Gamma(m)\,\Gamma(n) = \Gamma(m+n)\,B(m, n)$$

Hence $\quad B(m, n) = \dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$.

12.8. PROVE THAT $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

Proof. We know that $B(m, n) = \dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$

Taking $m = n = \frac{1}{2}$, $\quad B(\frac{1}{2}, \frac{1}{2}) = \dfrac{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+\frac{1}{2})} = \dfrac{[\Gamma(\frac{1}{2})]^2}{\Gamma(1)}$

or $\quad B(\frac{1}{2}, \frac{1}{2}) = [\Gamma(\frac{1}{2})]^2 \qquad [\because \Gamma(1) = 1] \quad ...(i)$

Now $\quad B(\frac{1}{2}, \frac{1}{2}) = \int_0^1 x^{1/2-1}(1-x)^{1/2-1}\,dx = \int_0^1 x^{-1/2}(1-x)^{1/2}\,dx$

Putting $x = \sin^2\theta$ so that $dx = 2\sin\theta\cos\theta\,d\theta$

When $x = 0, \theta = 0$; when $x = 1, \theta = \dfrac{\pi}{2}$

$$\therefore \quad B(\tfrac{1}{2}, \tfrac{1}{2}) = \int_0^{\pi/2} \frac{1}{\sin\theta} \cdot \frac{1}{\cos\theta} \cdot 2\sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\pi/2} d\theta = 2[\theta]_0^{\pi/2} = 2\left(\frac{\pi}{2} - 0\right) = \pi$$

$\therefore$ From (i), $\left[\Gamma\left(\dfrac{1}{2}\right)\right]^2 = \pi$ or $\Gamma\left(\dfrac{1}{2}\right) = \sqrt{\pi}$.

12.9. PROVE THAT $\int_0^\infty e^{-x^2}\,dx = \frac{1}{2}\sqrt{\pi}$

Proof. Put $x^2 = z$ so that $2x\,dx = dz$ or $dx = \dfrac{dz}{2\sqrt{z}}$

When $x = 0, z = 0$; when $x \to \infty, z \to \infty$

$$\therefore \quad \int_0^\infty e^{-x^2}\,dx = \int_0^\infty e^{-z}\frac{dz}{2\sqrt{z}} = \frac{1}{2}\int_0^\infty e^{-z} z^{-\frac{1}{2}}\,dz = \frac{1}{2}\int_0^\infty e^{-z} z^{\frac{1}{2}-1}\,dz$$

$$= \frac{1}{2}\Gamma\left(\frac{1}{2}\right) \qquad \left[\because \Gamma(n) = \int_0^\infty e^{-x}x^{n-1}\,dx, \text{ here } n = \frac{1}{2}\right]$$

$$= \frac{1}{2}\sqrt{\pi} \qquad \left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

Cor. 1 Prove that $\int_{-\infty}^0 e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$.

Put $x = -z$ so that $dx = -dz$

When $x \to -\infty, z \to \infty$; when $x = 0, z = 0$

$$\therefore \quad \int_{-\infty}^0 e^{-x^2}\,dx = \int_\infty^0 e^{-z^2}(-dz) = -\int_\infty^0 e^{-z^2}\,dz$$

$$= \int_0^\infty e^{-z^2}\,dz = \int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2} \qquad \text{[By Art. 12.9]}$$

Cor. 2. Prove that $\int_{-\infty}^\infty e^{-x^2}\,dx = \sqrt{\pi}$.

$$\int_{-\infty}^\infty e^{-x^2}\,dx = 2\int_0^\infty e^{-x^2}\,dx$$

$\left[\because e^{-x^2}\text{ is an even function of } x \text{ and } \int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx \text{ if } f(x) \text{ is an even function of } x\right]$

$$= 2 \cdot \frac{\sqrt{\pi}}{2} \qquad \text{[By Cor. 1]}$$

$$= \sqrt{\pi}.$$

12.10. TO EVALUATE $\int_0^{\pi/2} \sin^p x \cos^q x\,dx$ where $p > -1, q > -1$

Put $\sin^2 x = z$ so that $2\sin x\cos x\,dx = dz$

When $x = 0, z = 0$; when $x = \dfrac{\pi}{2}, z = 1$

Also $\quad \cos^2 x = 1 - \sin^2 x = 1 - z$

$$\therefore \int_0^{\pi/2} \sin^p x \cos^q x\,dx = \int_0^{\pi/2} (\sin^{p-1} x \cos^{q-1} x)\sin x\cos x\,dx$$

$$= \int_0^{\pi/2} (\sin^2 x)^{\frac{p-1}{2}} \cdot (\cos^2 x)^{\frac{q-1}{2}} \cdot \sin x\cos x\,dx$$

$$= \int_0^1 z^{\frac{p-1}{2}}(1-z)^{\frac{q-1}{2}} \cdot \frac{1}{2}\,dz = \frac{1}{2}\int_0^1 z^{\frac{p-1}{2}}(1-z)^{\frac{q-1}{2}}\,dz$$

$$= \frac{1}{2}B\left(\frac{p-1}{2}+1, \frac{q-1}{2}+1\right) = \frac{1}{2} \cdot B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$= \frac{1}{2} \cdot \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+1}{2}+\frac{q+1}{2}\right)} \qquad \left[\because B(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}\right]$$

Hence $\quad \displaystyle\int_0^{\pi/2} \sin^p x \cos^q x\,dx = \frac{1}{2} \cdot \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+1}{2}+\frac{q+1}{2}\right)}.$

Example 1. *Express the following in terms of Gamma functions :*

(i) $\int_0^1 x^p (1-x^q)^n\,dx$ where $p > 0, q > 0, n > 0$

(ii) $\int_0^1 x^{p-1} (1-x^2)^{q-1}\,dx$ where $p > 0, q > 0$

(iii) $\int_0^a x^{p-1} (a-x)^{q-1}\,dx$ where $p > 0, q > 0.$

Sol. (i) Put $x^q = z$ or $x = z^{1/q}$ so that $dx = \frac{1}{q} z^{\frac{1}{q}-1}\,dz = \frac{1}{q} z^{\frac{1-q}{q}}\,dz$

When $x = 0, z = 0$ and when $x = 1, z = 1$

$$\therefore \quad \int_0^1 x^p (1-x^q)^n\,dx = \int_0^1 z^{\frac{p}{q}} (1-z)^n \cdot \frac{1}{q} z^{\frac{1-q}{q}}\,dz$$

$$= \frac{1}{q}\int_0^1 z^{\frac{p+1-q}{q}} (1-z)^n\,dz = \frac{1}{q}\,B\left(\frac{p+1-q}{q}+1, n+1\right)$$

$$= \frac{1}{q}\,B\left(\frac{p+1}{q}, n+1\right) = \frac{1}{q}\cdot\frac{\Gamma\left(\frac{p+1}{q}\right)\Gamma(n+1)}{\Gamma\left(\frac{p+1}{q}+n+1\right)}.$$

(ii) *Please try yourself.* (*Put* $x^2 = z$) $\left[\textbf{Ans. } \frac{\Gamma(p/2)\,\Gamma(q)}{2\,\Gamma(p/2+q)}\right]$

(iii) *Please try yourself.* (*Put* $x = az$) $\left[\textbf{Ans. } a^{p+q+1}\cdot\frac{\Gamma(p)\,\Gamma(q)}{2\,\Gamma(p+q)}\right]$

Example 2. *Show that* $\int_0^1 \frac{dx}{\sqrt{1-x^n}} = \frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{n}\right)}{n\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}.$

Sol. Put $x^n = z$ i.e., $x = z^{\frac{1}{n}}$ so that $dx = \frac{1}{n} z^{\frac{1}{n}-1}\,dz = \frac{1}{n} z^{\frac{1-n}{n}}\,dz$

When $x = 0, z = 0$; when $x = 1, z = 1$

$$\therefore \quad \int_0^1 \frac{dx}{\sqrt{1-x^n}} = \int_0^1 \frac{\frac{1}{n} z^{\frac{1-n}{n}}}{\sqrt{1-z}}\,dz = \frac{1}{n}\int_0^1 z^{\frac{1}{n}-1} (1-z)^{-1/2}\,dz$$

$$= \frac{1}{n}\,B\left(\frac{1}{n},\frac{1}{2}\right) = \frac{1}{n}\,\frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)} = \frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{n}\right)}{n\,\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}. \qquad \left[\because\ \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

Example 3. *Show that* $\int_0^1 \frac{x^{m-1} (1-x)^{n-1}}{(a+x)^{m+n}}\,dx = \frac{\Gamma(m)\,\Gamma(n)}{a^n (1+a)^m\,\Gamma(m+n)}.$

Sol. $\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+x)^{m+n}}\,dx = \frac{B(m,n)}{a^n (1+a)^m}$ [See Beta functions]

$$= \frac{\Gamma(m)\,\Gamma(n)}{a^n (1+a)^m\,\Gamma(m+n)}.$$

Example 4. *Evaluate*

(i) $\int_0^\infty e^{-x^3}\,dx$ (ii) $\int_0^\infty x^3 e^{-x^3}\,dx$

(iii) $\int_0^\infty \sqrt{x}\,.\,e^{-x^3}\,dx$ (iv) $\int_0^\infty e^{-a^2x^2}\,dx,\ a > 0.$

Sol. (i) Put $x^3 = z$ or $x = z^{1/3}$ so that $dx = \frac{1}{3} z^{-\frac{2}{3}}\,dz$

When $x = 0, z = 0$; when $x \to \infty, z \to \infty$

$$\therefore \quad \int_0^\infty e^{-x^3}\,dx = \int_0^\infty e^{-z}\,.\,\tfrac{1}{3} z^{-\frac{2}{3}}\,dz = \frac{1}{3}\int_0^\infty e^{-z}\,.\,z^{\frac{1}{3}-1}\,dz = \tfrac{1}{3}\,\Gamma\left(\tfrac{1}{3}\right).$$

Note. $\int_0^\infty e^{-x^3}\,dx = \frac{1}{3}\,\Gamma\left(\frac{1}{3}\right) = \Gamma\left(\frac{4}{3}\right)$ $[\because\ (n-1)\,\Gamma(n-1) = \Gamma(n)]$

(ii) Proceeding as in part (i)

$$\int_0^\infty x^3 e^{-x^3}\,dx = \int_0^\infty z e^{-z}\,.\,\tfrac{1}{3} z^{-\frac{2}{3}}\,dz = \frac{1}{3}\int_0^\infty e^{-z}\,.\,z^{1/3}\,dz$$

$$= \frac{1}{3}\int_0^\infty e^{-z}\,.\,z^{\frac{4}{3}-1}\,dz = \frac{1}{3}\,\Gamma\left(\frac{4}{3}\right)$$

$$= \frac{1}{3}\cdot\frac{1}{3}\,\Gamma\left(\frac{1}{3}\right) \qquad [\because\ \Gamma(n) = (n-1)\,\Gamma(n-1)]$$

$$= \frac{1}{9}\,\Gamma\left(\frac{1}{3}\right).$$

(iii) Proceeding as in part (i)

$$\int_0^\infty \sqrt{x}\,.\,e^{-x^3}\,dx = \int_0^\infty z^{1/6} e^{-z}\,.\,\tfrac{1}{3} z^{-\frac{2}{3}}\,dz = \frac{1}{3}\int_0^\infty e^{-z}\,.\,z^{-\frac{1}{2}}\,dz$$

$$= \frac{1}{3}\int_0^\infty e^{-z}\,.\,z^{\frac{1}{2}-1}\,dz = \frac{1}{3}\,\Gamma\left(\frac{1}{2}\right) = \frac{1}{3}\sqrt{\pi}.$$

(iv) Put $a^2x^2 = z$ i.e., $x = \frac{\sqrt{z}}{a}$ so that $dz = \frac{z^{-\frac{1}{2}}}{2a}\,dz.$

When $x = 0, z = 0$; when $x \to \infty, z \to \infty$

$$\therefore \quad \int_0^\infty e^{-a^2x^2}\,dx = \int_0^\infty e^{-z}\cdot\frac{z^{-\frac{1}{2}}}{2a}\,dz = \frac{1}{2a}\int_0^\infty e^{-z}\cdot z^{\frac{1}{2}-1}\,dz = \frac{1}{2a}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2a}.$$

$$\left[\because \ \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}\right]$$

Example 5. *Show that* $\int_0^{\pi/2}\sqrt{\sin\theta}\,d\theta\times\int_0^{\pi/2}\frac{d\theta}{\sqrt{\sin\theta}} = \pi.$

Sol. $\int_0^{\pi/2}\sqrt{\sin\theta}\,d\theta\times\int_0^{\pi/2}\frac{d\theta}{\sqrt{\sin\theta}}$

$$= \int_0^{\pi/2}\sin^{1/2}\theta\cos^0\theta\,d\theta\times\int_0^{\pi/2}\sin^{-1/2}\theta\cos^0\theta\,d\theta$$

$$= \frac{\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{\frac{1}{2}+1}{2}+\frac{0+1}{2}\right)}\times\frac{\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{-\frac{1}{2}+1}{2}+\frac{0+1}{2}\right)}$$

$$= \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma(5/4)}\times\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{3}{4}\right)} = \frac{1}{4}\cdot\frac{[\Gamma(\frac{1}{2})]^2\,\Gamma(\frac{1}{4})}{\Gamma(\frac{5}{4})} = \frac{1}{4}\cdot\frac{(\sqrt{\pi})^2\cdot\Gamma(\frac{1}{4})}{\frac{1}{4}\Gamma(\frac{1}{4})} = \pi.$$

Example 6. *Prove that if* $n > -1$, $\int_0^\infty x^n e^{-a^2x^2}\,dx = \frac{1}{2a^{n+1}}\Gamma\left(\frac{n+1}{2}\right).$

Hence or otherwise show that $\int_{-\infty}^\infty e^{-a^2x^2}\,dx = \frac{\sqrt{\pi}}{a}.$

Sol. Put $a^2x^2 = z$, *i.e.*, $x = \frac{\sqrt{z}}{a}$ so that $dx = \frac{z^{-1/2}}{2a}$

When $x = 0, z = 0$; When $x \to \infty, z \to \infty$.

$$\therefore \quad \int_0^\infty x^n e^{-a^2x^2}\,dx = \int_0^\infty \frac{z^{n/2}}{a^n}\cdot e^{-z}\cdot\frac{z^{-1/2}}{2a}\,dz = \frac{1}{2a^{n+1}}\int_0^\infty e^{-z}\cdot z^{n-1/2}\,dz \quad ...(1)$$

$$= \frac{1}{2a^{n+1}}\Gamma\left(\frac{n-1}{2}+1\right) = \frac{1}{2a^{n+1}}\Gamma\left(\frac{n+1}{2}\right)$$

Putting $n = 0$, $\int_0^\infty e^{-a^2x^2}\,dx = \frac{1}{2a}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2a}$

$$\therefore \quad \int_{-\infty}^\infty e^{-a^2x^2}\,dx = 2\int_0^\infty e^{-a^2x^2}\,dx = 2\cdot\frac{\sqrt{\pi}}{2a} = \frac{\sqrt{\pi}}{a}$$

$$\left[\because \ e^{-a^2x^2} \text{ is an even function of } x \text{ and } \int_0^a f(x)\,dx = 2\int_0^a f(x)\,dx \text{ if } f(x) \text{ is an even function}\right]$$

Example 7. *Show that if* $a > 1$, $\int_0^\infty \frac{x^a}{a^x}\,dx = \frac{\Gamma(a+1)}{(\log a)^{a+1}}.$

Sol. $\because \quad a = e^{\log a} \quad \therefore \quad a^x = e^{x\log a}$

$$= \int_0^\infty \frac{x^a}{a^x}\,dx = \int_0^\infty \frac{x^a}{e^{x\log a}}\,dx = \int_0^\infty e^{-x\log a}\cdot x^a\,dx$$

Put $x\log a = z$, *i.e.*, $x = \frac{z}{\log a}$ so that $dx = \frac{dz}{\log a}$

When $x = 0, z = 0$; when $x \to \infty, z \to \infty$

$$\therefore \quad \int_0^\infty \frac{x^a}{a^x}\,dx = \int_0^\infty e^{-z}\cdot\frac{z^a}{(\log a)^a}\cdot\frac{dz}{\log a} = \frac{1}{(\log a)^{a+1}}\int_0^\infty e^{-z}\cdot z^{(a+1)-1}\,dz = \frac{\Gamma(a+1)}{(\log a)^{a+1}}.$$

Example 8. *Prove that* $\int_0^\infty e^{-ax}x^{n-1}\,dx = \frac{\Gamma(n)}{a^n}$ *where a, n are positive. Hence show that*

(i) $\int_0^\infty e^{-ax}x^{n-1}\cos bx\,dx = \frac{\Gamma(n)}{r^n}\cos n\theta$ *(ii)* $\int_0^\infty e^{-ax}x^{n-1}\sin bx\,dx = \frac{\Gamma(n)}{r^n}\sin n\theta$

where $r^2 = a^2 + b^2$ *and* $\theta = \tan^{-1}\frac{b}{a}.$

Sol. Put $ax = z$ so that $dx = \frac{dz}{a}$

When $x = 0, z = 0$, when $x \to \infty, z \to \infty$

$$\therefore \quad \int_0^\infty e^{-ax}x^{n-1}\,dx = \int_0^\infty e^{-z}\cdot\left(\frac{z}{a}\right)^{n-1}\cdot\frac{dz}{a} = \frac{1}{a^n}\int_0^\infty e^{-z}z^{n-1}\,dz = \frac{\Gamma(n)}{a^n}$$

Replacing a by $a + ib$, we have $\int_0^\infty e^{-(a+ib)x}x^{n-1}\,dx = \frac{\Gamma(n)}{(a+ib)^n}$ ...(1)

Now $\quad e^{-(a+ib)x} = e^{-ax-ibx} = e^{-ax}(\cos bx - i\sin bx)$ $\quad [\because \ e^{p-iq} = e^p(\cos q - i\sin q)]$

Also putting $\quad a = r\cos\theta$ and $b = r\sin\theta$

$$a^2 + b^2 = r^2 \quad \text{and} \quad \frac{b}{a} = \tan\theta \quad i.e., \quad \theta = \tan^{-1}\frac{b}{a}$$

and $\quad (a+ib)^n = (r\cos\theta + i\,.\,r\sin\theta)^n = r^n(\cos\theta + i\sin\theta)^n$

$$= r^n(\cos n\theta + i\sin n\theta) \quad \text{[De Moivre's Theorem]}$$

From (1), $\int_0^\infty e^{-ax}(\cos bx - i\sin bx)\,x^{n-1}\,dx = \frac{\Gamma(n)}{r^n(\cos n\theta + i\sin n\theta)}$

$$= \frac{\Gamma(n)}{r^n}(\cos n\theta + i\sin n\theta)^{-1} = \frac{\Gamma(n)}{r^n}(\cos n\theta - i\sin n\theta)$$

Equating real parts $\int_0^\infty e^{-ax}x^{n-1}\sin bx\,dx = \frac{\Gamma(n)}{r^n}\cos n\theta$

Equating imaginary parts $\int_0^\infty e^{-ax}x^{n-1}\sin bx\,dx = \frac{\Gamma(n)}{r^n}\sin n\theta.$

Example 9. *Evaluate*

(*i*) $\int_0^{\pi/2} \sin^3 x \cos^{5/2} x\, dx$ (*ii*) $\int_0^{\pi/2} \sin^7 x\, dx$ (*iii*) $\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta$.

Sol. (*i*) $\int_0^{\pi/2} \sin^3 x \cos^{5/2} x\, dx = \frac{1}{2} \cdot \frac{\Gamma\left(\frac{3+1}{2}\right)\Gamma\left(\frac{5/2+1}{2}\right)}{\Gamma\left(\frac{3+1}{2}+\frac{5/2+1}{2}\right)}$

$$\left[\because \int_0^{\pi/2} \sin^p x \cos^q x\, dx = \frac{1}{2} \cdot \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+1}{2}+\frac{q+1}{2}\right)}\right]$$

$$= \frac{1}{2} \cdot \frac{\Gamma(2)\,\Gamma\left(\frac{7}{4}\right)}{\Gamma\left(\frac{15}{4}\right)} = \frac{1}{2} \cdot \frac{1!\,\Gamma\left(\frac{7}{4}\right)}{\frac{11}{4}\cdot\frac{7}{4}\Gamma\left(\frac{7}{4}\right)}$$

$$= \frac{8}{77} \qquad [\because \Gamma(n) = (n-1)! \text{ if } n \text{ is a +ve integer}]$$

and $\Gamma(n) = (n-1)\,\Gamma(n-1)$

$$\therefore \quad \Gamma\left(\frac{15}{4}\right) = \frac{11}{4}\Gamma\left(\frac{11}{4}\right) = \frac{11}{4}\cdot\frac{7}{4}\Gamma\left(\frac{7}{4}\right)\Bigg]$$

(*ii*) $\int_0^{\pi/2} \sin^7 x\, dx = \int_0^{\pi/2} \sin^7 x \cos^0 x\, dx$

$$= \frac{1}{2} \cdot \frac{\Gamma\left(\frac{7+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{\Gamma\left(\frac{7+1}{2}+\frac{0+1}{2}\right)} = \frac{1}{2} \cdot \frac{\Gamma(4)\,\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{9}{2}\right)}$$

$$= \frac{1}{2} \cdot \frac{(4-1)!\,\Gamma\left(\frac{1}{2}\right)}{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} = \frac{6}{\frac{105}{8}} = \frac{16}{35}.$$

(*iii*) $\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta = \int_0^{\pi/2} \sqrt{\frac{\sin\theta}{\cos\theta}}\, d\theta = \int_0^{\pi/2} \sin^{1/2}\theta \cos^{-1/2}\theta\, d\theta$

$$= \frac{1}{2} \cdot \frac{\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)}{\Gamma\left(\frac{\frac{1}{2}+1}{2}+\frac{-\frac{1}{2}+1}{2}\right)} = \frac{1}{2} \cdot \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma(1)}$$

$$= \frac{1}{2} \cdot \frac{\sqrt{2}\pi}{1} \qquad \left[\because \Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right) = \sqrt{2}\pi\right.$$

[See Cor. with Duplication Formula]

$$= \frac{\pi}{\sqrt{2}}.$$

Example 10. *Show that* $\int_0^{\pi/2} \sin^n\theta\, d\theta = \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}$ *where* $n > -1$.

Sol. $\int_0^{\pi/2} \sin^n\theta\, d\theta = \int_0^{\pi/2} \sin^n\theta \cos^0\theta\, d\theta$

$$= \frac{1}{2} \cdot \frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{\Gamma\left(\frac{n+1}{2}+\frac{0+1}{2}\right)} = \frac{1}{2} \cdot \frac{\Gamma\left(\frac{n+1}{2}\right)\sqrt{\pi}}{\Gamma\left(\frac{n+2}{2}\right)} \qquad [\because \Gamma(\tfrac{1}{2}) = \sqrt{\pi}]$$

$$= \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}.$$

Example 11. *Prove that* $\int_0^1 \frac{1}{\sqrt{1-x^4}}\, dx = \frac{1}{8}\sqrt{\frac{2}{\pi}}\left[\Gamma\left(\frac{1}{4}\right)\right]^2$.

Sol. Put $x^4 = z$ *i.e.*, $x = z^{1/4}$ so that $dx = \frac{1}{4} z^{-3/4}\, dz$

When $x = 0, z = 0$; when $x = 1, z = 1$

$$\therefore \quad \int_0^1 \frac{1}{\sqrt{1-x^4}}\, dx = \int_0^1 \frac{1}{\sqrt{1-z}} \cdot \frac{1}{4} z^{-3/4}\, dz$$

$$= \frac{1}{4}\int_0^1 z^{-3/4}(1-z)^{-1/2}\, dz = \frac{1}{4} B\left(\frac{1}{4}, \frac{1}{2}\right)$$

$$= \frac{1}{4} \cdot \frac{\Gamma(\frac{1}{4})\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{4}+\frac{1}{2})} = \frac{1}{4} \cdot \frac{\Gamma(\frac{1}{4}).\sqrt{\pi}}{\Gamma(\frac{3}{4})}$$

$$= \frac{\sqrt{\pi}}{4} \cdot \frac{\Gamma(\frac{1}{4})}{\sqrt{2}\pi/\Gamma(\frac{1}{4})} \qquad \left[\because \Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right) = \sqrt{2}\pi\right]$$

$$= \frac{1}{8}\sqrt{\frac{2}{\pi}}\left[\Gamma\left(\frac{1}{4}\right)\right]^2.$$

Example 12. *Prove that* $\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\, dx = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$.

Sol. $\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\, dx = B(m, n)$ [See examples with Beta Function]

$$= \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}.$$

Example 13. *Evaluate* $\int_0^1 x^3 (1-x)^{4/3}\, dx.$

Sol. Put $x = \sin^2\theta$; so that $dx = 2\sin\theta\cos\theta\, d\theta$

When $x = 0, \theta = 0$; when $x = 1, \theta = \pi/2$

$$\therefore \quad \int_0^1 x^3 (1-x)^{4/3}\, dx = \int_0^{\pi/2} \sin^6\theta \cos^{8/3}\theta \,.\, 2\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^7\theta \cos^{11/3}\theta\, d\theta = 2 \,.\, \frac{1}{2} \frac{\Gamma\left(\frac{7+1}{2}\right)\Gamma\left(\frac{11/3+1}{2}\right)}{\Gamma\left(\frac{7+1}{2}+\frac{11/3+1}{2}\right)}$$

$$= \frac{\Gamma(4)\,\Gamma\left(\frac{7}{3}\right)}{\Gamma\left(\frac{19}{3}\right)} = \frac{(4-1)!\,.\,\Gamma\left(\frac{7}{3}\right)}{\frac{16}{3}\cdot\frac{13}{3}\cdot\frac{10}{3}\cdot\frac{7}{3}\,\Gamma\left(\frac{7}{3}\right)} = \frac{6\times 81}{16\times 13\times 10\times 7} = \frac{243}{7280}.$$

Example 14. *Express* $\int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}}\, dx$ *in terms of Beta and Gamma functions ; where* $m > 0, n > 0, a > 0, b > 0.$

Sol. Put $bx = az$ so that $dx = \frac{a}{b}\, dz.$

When $x = 0, z = 0$; when $x \to \infty, z \to \infty$

$$\therefore \quad \int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}}\, dx = \int_0^\infty \frac{\left(\frac{az}{b}\right)^{m-1}}{(a+az)^{m+n}} \cdot \frac{a}{b}\, dz = \int_0^\infty \frac{a^{m-1}\,.\,a}{b^{m-1}\,.\,b\,.\,a^{m+n}} \cdot \frac{z^{m-1}}{(1+z)^{m+n}}\, dz$$

$$= \frac{1}{a^n b^m}\int_0^\infty \frac{z^{m-1}}{(1+z)^{m+n}}\, dz = \frac{1}{a^n b^m}\, \mathrm{B}(m, n) \qquad | \text{ By def.}$$

$$= \frac{1}{a^n b^m} \cdot \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}.$$

Example 15. *Prove that* $\Gamma(n) = \int_0^1 \left(\log \frac{1}{y}\right)^{n-1} dy.$

Sol. Put $\log \frac{1}{y} = z$, *i.e.*, $\frac{1}{y} = e^z$ or $y = e^{-z}$ so that $dy = -e^{-z}\, dz$

When $y = 0, z \to \infty$; when $y = 1, z = 0$

$$\therefore \quad \int_0^1 \left(\log\frac{1}{y}\right)^{n-1} dy = \int_\infty^0 z^{n-1}\,(-e^{-z})\, dz = \int_0^\infty z^{n-1} e^{-z}\, dz = \Gamma(n).$$

Example 16. *Show that*

(*i*) $\int_0^\infty \sqrt{x}\, e^{-x^2}\, dx \times \int_0^\infty \frac{e^{-x^2}}{\sqrt{x}}\, dx = \frac{\pi}{2\sqrt{2}}$ (*ii*) $\int_0^\infty \frac{e^{-y^2}}{\sqrt{y}}\, dy \times \int_0^\infty y^2 e^{-y^4}\, dy = \frac{\pi}{4\sqrt{2}}.$

Sol. (*i*) Put $x^2 = z$ *i.e.*, $x = z^{1/2}$ so that $dx = \frac{1}{2} z^{-\frac{1}{2}}\, dz$

When $x = 0, z = 0$; when $x \to \infty, z \to \infty$

$$\therefore \quad \int_0^\infty \sqrt{x}\, e^{-x^2}\, dx = \int_0^\infty z^{1/4} e^{-z} \,.\, \tfrac{1}{2} e^{-\frac{1}{2}}\, dz = \frac{1}{2}\int_0^\infty z^{-1/4} e^{-z}\, dz = \frac{1}{2}\,\Gamma\left(\frac{3}{4}\right)$$

and $$\int_0^\infty \frac{e^{-x^2}}{\sqrt{x}}\, dx = \int_0^\infty \frac{e^{-z}}{z^{1/4}} \cdot \tfrac{1}{2} z^{-\frac{1}{4}}\, dz = \frac{1}{2}\int_0^\infty z^{-\frac{1}{2}} e^{-z}\, dz = \tfrac{1}{2}\,\Gamma\left(\tfrac{1}{2}\right)$$

$$\therefore \quad \int_0^\infty \sqrt{x}\, e^{-x^2}\, dx \times \int_0^\infty \frac{e^{-x^2}}{\sqrt{x}}\, dx = \tfrac{1}{2}\,\Gamma\left(\frac{3}{4}\right) \cdot \tfrac{1}{2}\,\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{1}{4} \,.\, \sqrt{2}\,\pi \qquad \left[\because\ \Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right) = \sqrt{2\pi}\right]$$

$$= \frac{\pi}{2\sqrt{2}}.$$

(*ii*) Put $y^2 = z$, $\int_0^\infty \frac{e^{-y^2}}{\sqrt{y}}\, dy = \frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)$ [As in part (*i*)]

Put $y^4 = z$ *i.e.*, $y = z^{1/4}$ so that $dy = \frac{1}{2} z^{-\frac{3}{4}}\, dz$

When $y = 0, z = 0$; when $y \to \infty, z \to \infty$

$$\therefore \quad \int_0^\infty y^2 e^{-y^4}\, dy = \int_0^\infty z^{1/2} e^{-z} \,.\, \tfrac{1}{4} z^{-\frac{3}{4}}\, dz = \tfrac{1}{4}\int_0^\infty z^{-\frac{1}{4}} e^{-z}\, dz$$

$$\therefore \quad \int_0^\infty \frac{e^{-y^2}}{\sqrt{y}}\, dy \times \int_0^\infty y^2 e^{-y^4}\, dy = \frac{1}{2}\,\Gamma\left(\frac{1}{2}\right) \cdot \frac{1}{4}\,\Gamma\left(\frac{3}{4}\right)$$

$$= \frac{1}{8}\,\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{4}\right) = \frac{1}{8} \times \sqrt{2}\,\pi = \frac{\pi}{4\sqrt{2}}.$$

Example 17. *Prove that* $\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta = \frac{\Gamma(m)\,\Gamma(n)}{2\Gamma(m+n)}.$

Sol. *Please try yourself.* [See Art. 12.10]

Example 18. *Show that* $2^n\,\Gamma\left(n+\frac{1}{2}\right) = 1.3.5 \ldots\ldots (2n-1)\sqrt{\pi}$, *where n is a positive integer.*

Sol. $$\Gamma\left(n+\frac{1}{2}\right) = \left(n-\frac{1}{2}\right)\Gamma\left(n-\frac{1}{2}\right) = \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\Gamma\left(n-\frac{3}{2}\right)$$

$$= \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\left(n-\frac{5}{2}\right)\ldots\ldots\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$= \left(\frac{2n-1}{2}\right)\left(\frac{2n-3}{2}\right)\left(\frac{2n-5}{2}\right)\ldots\ldots\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}$$

$$= \frac{1}{2^n}(2n-1)(2n-3)\ldots 3.1\sqrt{\pi}$$

$$\Rightarrow \qquad 2^n\Gamma\left(n+\frac{1}{2}\right) = 1.3.5\ldots\ldots(2n-1)\sqrt{\pi}$$

(writing the factors in reverse order).

Example 19. *Show that*

(i) $\displaystyle\int_0^1 \frac{x^2\,dx}{\sqrt{1-x^4}} \times \int_0^1 \frac{dx}{\sqrt{1+x^4}} = \frac{\pi}{4\sqrt{2}}$ (ii) $\displaystyle\int_0^\infty xe^{-x^8}\,dx \times \int_0^\infty x^2e^{-x^4}\,dx = \frac{\pi}{16\sqrt{2}}$

(iii) $\displaystyle\int_0^{\pi/2} \sin^p x\,dx \times \int_0^{\pi/2} \sin^{p+1} x\,dx = \frac{\pi}{2(p+1)}$.

Sol. (i) Put $x^2 = \sin\theta$ i.e., $x = \sqrt{\sin\theta}$ so that $dx = \dfrac{\cos\theta}{2\sqrt{\sin\theta}}\,d\theta$

When $x = 0, \theta = 0$; when $x = 1, \theta = \dfrac{\pi}{2}$

$$\therefore \quad \int_0^1 \frac{x^2}{\sqrt{1-x^4}}\,dx = \int_0^{\pi/2} \frac{\sin\theta}{\cos\theta}\cdot\frac{\cos\theta}{2\sqrt{\sin\theta}}\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin^{1/2}\theta\cos^0\theta\,d\theta = \frac{1}{2}\cdot\frac{\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{\frac{1}{2}+1}{2}+\frac{0+1}{2}\right)}$$

$$= \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{4\Gamma\left(\frac{5}{4}\right)} = \frac{\Gamma\left(\frac{3}{4}\right)\sqrt{\pi}}{4\times\frac{1}{4}\Gamma\left(\frac{1}{4}\right)} = \sqrt{\pi}\,\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}$$

Now put $x^2 = \tan\phi$ i.e., $x = \sqrt{\tan\phi}$ so that $dx = \dfrac{\sec^2\phi}{2\sqrt{\tan\phi}}\,d\phi$

When $x = 0, \phi = 0$; when $x = 1, \phi = \dfrac{\pi}{4}$

$$\therefore \quad \int_0^1 \frac{dx}{\sqrt{1+x^4}} = \int_0^{\pi/4} \frac{1}{\sec\phi}\cdot\frac{\sec^2\phi}{2\sqrt{\tan\phi}}\,d\phi$$

$$= \frac{1}{2}\int_0^{\pi/4} \frac{d\phi}{\sqrt{\sin\phi\cos\phi}} = \frac{\sqrt{2}}{2}\int_0^{\pi/4} \frac{d\phi}{\sqrt{2\sin\phi\cos\phi}}$$

$$= \frac{1}{\sqrt{2}}\int_0^{\pi/4} \frac{d\phi}{\sqrt{\sin 2\phi}} = \frac{1}{2\sqrt{2}}\int_0^{\pi/2} \frac{dt}{\sqrt{\sin t}} \text{ where } t = 2\phi$$

$$= \frac{1}{2\sqrt{2}}\int_0^{\pi/2} \sin^{-1/2} t\cos^0 t\,dt = \frac{1}{2\sqrt{2}}\cdot\frac{\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{-\frac{1}{2}+1}{2}+\frac{0+1}{2}\right)}$$

$$= \frac{1}{4\sqrt{2}}\cdot\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)} = \frac{\sqrt{\pi}}{4\sqrt{2}}\cdot\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}$$

$$\therefore \quad \int_0^1 \frac{x^2}{\sqrt{1+x^4}}\,dx \times \int_0^1 \frac{dx}{\sqrt{1+x^4}} = \sqrt{\pi}\,\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}\cdot\frac{\sqrt{\pi}}{4\sqrt{2}}\cdot\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} = \frac{\pi}{4\sqrt{2}}.$$

(ii) Put $x^8 = z$ i.e., $x = z^{1/8}$ so that $dx = \dfrac{1}{8}z^{-7/8}\,dz$

When $x = 0, z = 0$; when $x \to \infty, z \to \infty$.

$$\therefore \quad \int_0^\infty xe^{-x^8}\,dx = \int_0^\infty z^{1/8}e^{-z}\cdot\frac{1}{8}z^{-7/8}\,dz = \frac{1}{8}\int_0^\infty z^{-3/4}e^{-z}\,dz = \frac{1}{8}\Gamma\left(\frac{1}{4}\right)$$

Now put $x^4 = t$ i.e., $x = t^{1/4}$ so that $dx = \dfrac{1}{4}t^{-3/4}\,dt$

When $x = 0, t = 0$; when $x \to \infty, t \to \infty$

$$\therefore \quad \int_0^\infty x^2e^{-x^4}\,dx = \int_0^\infty t^{1/2}e^{-t}\cdot\frac{1}{4}e^{-3/4}\,dt = \frac{1}{4}\int_0^\infty t^{-1/4}e^{-t}\,dt = \frac{1}{4}\Gamma\left(\frac{3}{4}\right)$$

$$\therefore \quad \int_0^\infty xe^{-x^8}\,dx \times \int_0^\infty x^2e^{-x^4}\,dx = \frac{1}{8}\Gamma\left(\frac{1}{4}\right)\times\frac{1}{4}\Gamma\left(\frac{3}{4}\right) = \frac{1}{32}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{1}{32}\times\sqrt{2}\pi$$

[See cor. with Art. 12.11]

$$= \frac{\pi}{16\sqrt{2}}.$$

(iii) $$\int_0^{\pi/2} \sin^p x\,dx \times \int_0^{\pi/2} \sin^{p+1} x\,dx = \int_0^{\pi/2} \sin^p x\cos^0 x\,dx \times \int_0^{\pi/2} \sin^{p+1} x\cos^0 x\,dx$$

$$= \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{\Gamma\left(\frac{p+1}{2}+\frac{0+1}{2}\right)} \times \frac{\Gamma\left(\frac{p+1+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{p+1+1}{2}+\frac{0+1}{2}\right)}$$

$$= \frac{1}{4}\cdot\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{p+2}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{p+2}{2}\right)\Gamma\left(\frac{p+3}{2}\right)}$$

$$= \frac{1}{4}\cdot\frac{\Gamma\left(\frac{p+1}{2}\right)\left[\Gamma\left(\frac{1}{2}\right)\right]^2}{\Gamma\left(\frac{p+1}{2}+1\right)} = \frac{1}{4}\cdot\frac{\Gamma\left(\frac{p+1}{2}\right)(\sqrt{\pi})^2}{\frac{p+1}{2}\Gamma\left(\frac{p+1}{2}\right)} = \frac{\pi}{2(p+1)}.$$

Example 20. *Show that* $\int_0^1 \sqrt{1-x^4}\, dx = \frac{1}{12}\sqrt{\frac{2}{\pi}}\left[\Gamma\left(\frac{1}{4}\right)\right]^2$.

Sol. Put $x^4 = z$ *i.e.*, $x = z^{1/4}$ so that $dx\ \frac{1}{4}\, z^{-3/4}\, dz$

When $x = 0, z = 0$, when $x = 1, z = 1$

$$\therefore \quad \int_0^1 \sqrt{1-x^4}\, dx = \int_0^1 (1-z)^{1/2}\cdot\tfrac{1}{4} z^{-3/4}\, dz = \frac{1}{4}\int_0^1 z^{3/4}(1-z)^{1/2}\, dz$$

$$= \frac{1}{4}\,B\left(\frac{1}{4},\frac{3}{2}\right) = \frac{1}{4}\cdot\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{1}{4}+\frac{3}{2}\right)} = \frac{1}{4}\cdot\frac{\Gamma\left(\frac{1}{4}\right)\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{7}{4}\right)}$$

$$= \frac{1}{8}\cdot\frac{\Gamma\left(\frac{1}{4}\right)\sqrt{\pi}}{\frac{3}{4}\Gamma\left(\frac{3}{4}\right)} = \frac{\sqrt{\pi}}{6}\cdot\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} = \frac{\sqrt{\pi}}{6}\cdot\frac{\Gamma\left(\frac{1}{4}\right)}{\frac{\sqrt{2}\pi}{\Gamma\left(\frac{1}{4}\right)}} \qquad \left[\because\ \Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \sqrt{2}\pi\right]$$

$$= \frac{1}{6\sqrt{2\pi}}\left[\Gamma\left(\frac{1}{4}\right)\right]^2 = \frac{1}{12}\sqrt{\frac{2}{\pi}}\left[\Gamma\left(\frac{1}{4}\right)\right]^2.$$

Example 21. *Prove that*

(i) $B(p, q) = B(p+1, q) + B(p, q+1)$

(ii) $B(p, q)\, B(p+q, r) = B(q, r)\, B(q+r, p) = B(r, p)\, B(r+p, q)$

(iii) $B(p, q)\, B(p+q, r)\, B(p+q+r, s) = \dfrac{\Gamma(p)\,\Gamma(q)\,\Gamma(r)\,\Gamma(s)}{\Gamma(p+q+r+s)}$.

Sol. *(i)* R.H.S. $= B(p+1, q) + B(p, q+1)$

$$= \frac{\Gamma(p+1)\Gamma(q)}{\Gamma(p+q+1)} + \frac{\Gamma(p)\Gamma(q+1)}{\Gamma(p+q+1)} = \frac{p\,\Gamma(p)\Gamma(q) + \Gamma(p)\,.\,q\,\Gamma(q)}{(p+q)\,\Gamma(p+q)}$$

$$= \frac{(p+q)\,\Gamma(p)\,\Gamma(q)}{(p+q)\,\Gamma(p+q)} = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)} = B(p, q) = \text{L.H.S.}$$

(ii) $\quad B(p, q)\, B(p+q, r) = \dfrac{B(p)\,B(q)}{B(p+q)}\cdot\dfrac{B(p+q)\,B(r)}{B(p+q+r)} = \dfrac{B(p)\,B(q)\,B(r)}{B(p+q+r)}$

Similarly for others.

(iii) Please try yourself.

Example 22. *Prove that*

(i) $\dfrac{B(p, q+1)}{q} = \dfrac{B(p+1, q)}{p} = \dfrac{B(p, q)}{p+q}$ $\qquad$ *(ii)* $\dfrac{B(m+1, n)}{B(m, n)} = \dfrac{m}{m+n}$

(iii) $\dfrac{B(m+2, n-2)}{B(m, n)} = \dfrac{m(m+1)}{(n-1)(n-2)}$.

Sol. *(i)* $\quad \dfrac{B(p, q+1)}{q} = \dfrac{\Gamma(p)\,\Gamma(q+1)}{q\,\Gamma(p+q+1)} = \dfrac{\Gamma(p)\,q\,\Gamma(q)}{q(p+q)\,\Gamma(p+q)} = \dfrac{\Gamma(p)\,\Gamma(p)}{(p+q)\,\Gamma(p+q)} = \dfrac{B(p, q)}{p+q}$

Similarly $\quad \dfrac{B(p+1, q)}{p} = \dfrac{B(p, q)}{p+q}$

Hence the result.

(ii) $\quad B(m+1, n) = \dfrac{\Gamma(m+1)\,\Gamma(n)}{\Gamma(m+n+1)} = \dfrac{m\,\Gamma(m)\,\Gamma(n)}{(m+n)\,\Gamma(m+n)} = \dfrac{m}{(m+n)}\,B(m, n)$

$$\Rightarrow \quad \frac{B(m+1, n)}{B(m, n)} = \frac{m}{m+n}.$$

(iii) $\quad B(m+2, n-2) = \dfrac{\Gamma(m+2)\,\Gamma(n-2)}{\Gamma(m+n)} = \dfrac{(m+1)\,m\,\Gamma(m)\,\Gamma(n-2)}{\Gamma(m+n)}$

$$B(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

$$\therefore \quad \frac{B(m+2, n-2)}{B(m, n)} = \frac{m(m+1)\,\Gamma(m)\,\Gamma(n-2)}{\Gamma(m+n)}\cdot\frac{\Gamma(m+n)}{\Gamma(m)\,\Gamma(n)}$$

$$= \frac{m(m+1)\,\Gamma(n-2)}{\Gamma(n)} = \frac{m(m+1)\,\Gamma(n-2)}{(n-1)(n-2)\,\Gamma(n-2)} = \frac{m(m+1)}{(n-1)(n-2)}.$$

Example 23. *Evaluate the following integrals :*

(i) $\int_0^\infty x^6 e^{-2x}\, dx$ $\qquad$ *(ii)* $\int_0^\infty e^{-4x} x^{3/2}\, dx$ $\qquad$ *(iii)* $\int_0^2 \dfrac{x^2}{\sqrt{2-x}}\, dx$

(iv) $\int_0^3 \dfrac{dx}{\sqrt{3x-x^2}}$ $\qquad$ *(v)* $\int_0^\infty \dfrac{x^8(1-x^6)}{(1+x)^{24}}\, dx$ $\qquad$ *(vi)* $\int_0^\infty \dfrac{x^4(1+x^5)}{(1+x)^{15}}\, dx$.

Sol. *(i)* Put $2x = z$ *i.e.*, $x = \frac{1}{2}z$ then $dx = \frac{1}{2}\,dz$

when $x = 0, z = 0$; when $x \to \infty, z \to \infty$

$$\therefore \quad \int_0^\infty x^6 e^{-2x}\, dx = \int_0^\infty \left(\tfrac{1}{2}z\right)^6 e^{-z}\cdot\tfrac{1}{2}\, dz = \frac{1}{128}\int_0^\infty z^6 e^{-z}\, dz = \frac{1}{128}\,\Gamma(7)$$

$$= \frac{1}{128}\,(6\,!) \qquad |\because\ \Gamma(n) = (n-1)\,!\ \text{where } n \in \mathbb{N}$$

$$= \frac{6\times5\times4\times3\times2}{128} = \frac{45}{8}.$$

(*ii*) *Please try yourself.* $\left[\textbf{Ans. } \frac{3\sqrt{\pi}}{128}\right]$

(*iii*) Put $x = 2z$ then $dx = 2dz$

When $x = 0, z = 0$; when $x = 2, z = 1$

$$\therefore \quad \int_0^2 \frac{x^2}{\sqrt{2-x}}\,dx = \int_0^1 \frac{4z^2}{\sqrt{2(1-z)}} \cdot 2dz = 4\sqrt{2}\int_0^1 z^2(1-z)^{-1/2}\,dz$$

$$= 4\sqrt{2}\,B\left(3, \frac{1}{2}\right) = 4\sqrt{2} \cdot \frac{\Gamma(3)\,\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(3+\frac{1}{2}\right)} = 4\sqrt{2} \cdot \frac{(2!)\,\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{7}{2}\right)}$$

$$= 8\sqrt{2} \cdot \frac{\Gamma\left(\frac{1}{2}\right)}{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} = \frac{64\sqrt{2}}{15}.$$

(*iv*)
$$I = \int_0^3 \frac{dx}{\sqrt{3x-x^2}} = \int_0^3 \frac{dx}{\sqrt{x}\,\sqrt{3-x}}$$

Put $x = 3z$ then $dx = 3dz$

When $x = 0, z = 0$; when $x = 3, z = 1$

$$\therefore \quad I = \int_0^1 \frac{3dz}{\sqrt{3z}\,\sqrt{3(1-z)}} = \int_0^1 z^{-1/2}(1-z)^{-1/2}\,dz$$

$$= B\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)} = \left[\Gamma\left(\frac{1}{2}\right)\right]^2 \qquad [\because \ \Gamma(1) = 1]$$

$$= (\sqrt{\pi})^2 = \pi.$$

(*v*)
$$I = \int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}\,dx = \int_0^\infty \frac{x^8}{(1+x)^{24}}\,dx - \int_0^\infty \frac{x^{14}}{(1+x)^{24}}\,dx$$

$$= \int_0^\infty \frac{x^{9-1}}{(1+x)^{9+15}}\,dx - \int_0^\infty \frac{x^{15-1}}{(1+x)^{15+9}}\,dx$$

$$= B(9, 15) - B(15, 9) \qquad \left[\because \ \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = B(m, n)\right]$$

$$= 0. \qquad [\because \ B(m, n) = B(n, m)]$$

(*vi*)
$$I = \int_0^\infty \frac{x^4(1+x^5)}{(1+x)^{15}}\,dx = \int_0^\infty \frac{x^4}{(1+x)^{15}}\,dx + \int_0^\infty \frac{x^9}{(1+x)^{15}}\,dx$$

$$= \int_0^\infty \frac{x^{5-1}}{(1+x)^{5+10}}\,dx + \int_0^\infty \frac{x^{10-1}}{(1+x)^{10+5}}\,dx$$

$$= B(5, 10) + B(10, 5) = 2B(5, 10) = \frac{2\,\Gamma(5)\,\Gamma(10)}{\Gamma(5+10)}$$

$$= \frac{2 \times 4! \times 9!}{14!} = \frac{2 \times 4 \times 3 \times 2 \times 9!}{14 \times 13 \times 12 \times 11 \times 10 \times 9!} = \frac{1}{5005}.$$

12.11. DUPLICATION FORMULA

To prove that $\Gamma(m)\,\Gamma(m+\frac{1}{2}) = \frac{\sqrt{\pi}}{2^{2m-1}}\,\Gamma(2m)$.

Proof. We know that $B(m, n) = \dfrac{\Gamma(m)\,\Gamma(n)}{2\Gamma(m+n)}$

$$\therefore \quad B(m, m) = \frac{[\Gamma(m)]^2}{\Gamma(2m)} \qquad \ldots(i)$$

Also by def. $\quad B(m, m) = \int_0^1 x^{m-1}(1-x)^{m-1}\,dx$

Put $x = \sin^2\theta$, so that $dx = 2\sin\theta\cos\theta\,d\theta$.

When $x = 0, \theta = 0$; when $x = 1, \theta = \pi/2$

$$\therefore \quad B(m, m) = \int_0^{\pi/2} \sin^{2m-2}\theta \cdot \cos^{2m-2}\theta \cdot 2\sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2m-1}\theta\,d\theta = 2\int_0^{\pi/2}\left(\frac{2\sin\theta\cos\theta}{2}\right)^{2m-1} d\theta$$

$$= 2\int_0^{\pi/2} \frac{1}{2^{2m-1}}\sin^{2m-1}2\theta\,d\theta = \frac{1}{2^{2m-2}}\int_0^{\pi/2}\sin^{2m-1}2\theta\,d\theta$$

Put $2\theta = \phi$, so that $d\theta = \frac{1}{2}\,d\phi$

When $\theta = 0, \phi = 0$; when $x = \frac{\pi}{2}, \phi = \pi$

$$\therefore \quad B(m, n) = \frac{1}{2^{2m-2}}\int_0^\pi \sin^{2m-1}\phi \cdot \tfrac{1}{2}\,d\phi = \frac{1}{2^{2m-1}}\int_0^\pi \sin^{2m-1}\phi\,d\phi$$

$$= \frac{1}{2^{2m-1}} \cdot 2\int_0^{\pi/2} \sin^{2m-1}\phi\,d\phi$$

$$\left[\because \ \int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx, \text{ if } f(2a-x) = f(x)\right]$$

$$= \frac{1}{2^{2m-2}}\int_0^{\pi/2} \sin^{2m-1}\phi\cos^0\phi\,d\phi$$

$$= \frac{1}{2^{2m-2}} \cdot \frac{\Gamma\left(\frac{2m-1+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\,\Gamma\left(\frac{2m-1+1}{2}+\frac{0+1}{2}\right)} = \frac{1}{2^{2m-1}} \cdot \frac{\Gamma(m)\,.\,\Gamma(\frac{1}{2})}{\Gamma(m+\frac{1}{2})}$$

$$= \frac{1}{2^{2m-1}} \cdot \frac{\Gamma(m) \cdot \sqrt{\pi}}{\Gamma\left(m+\frac{1}{2}\right)} \qquad \left[\because \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}\right] \quad \ldots(ii)$$

From (*i*) and (*ii*), $\frac{[\Gamma(m)]^2}{\Gamma(2m)} = \frac{1}{2^{2m-1}} \frac{\Gamma(m)\sqrt{\pi}}{\Gamma\left(m+\frac{1}{2}\right)}$ or $\Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\,\Gamma(2m)$.

Cor. *Prove that* $\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \sqrt{2}\,.\,\pi$.

Sol. By Duplication Formula $\Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\,\Gamma(2m)$

Putting $m = \frac{1}{4}$, $\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{\sqrt{\pi}}{2^{-1/2}}\,\Gamma\left(\frac{1}{2}\right) = \sqrt{2}\,.\sqrt{\pi}\,.\sqrt{\pi} = \sqrt{2}\pi$.

Example 1. *Prove that* $B(m,m) = 2^{1-2m} B\left(m, \frac{1}{2}\right)$.

Sol. As proved in the above article

$$B(m, m) = \frac{1}{2^{2m-1}} \cdot \frac{\Gamma(m)\,\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(m+\frac{1}{2}\right)} = 2^{1-2m} B\left(m, \tfrac{1}{2}\right)$$

Example 2. *Prove that* $B(m, m) \,.\, B\left(m+\frac{1}{2}, m+\frac{1}{2}\right) = \frac{\pi m^{-1}}{2^{4m-1}}$.

Sol. $B(m, m) \,.\, B\left(m+\frac{1}{2}, m+\frac{1}{2}\right)$

$$= \frac{\Gamma(m)\,\Gamma(m)}{\Gamma(m+m)} \cdot \frac{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(m+\frac{1}{2}\right)}{\Gamma\left(m+\frac{1}{2}+m+\frac{1}{2}\right)}$$

$$= \frac{[\Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right)]^2}{\Gamma(2m)\,\Gamma(2m+1)} = \frac{[\Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right)]^2}{\Gamma(2m)\,.\,2m\,\Gamma(2m)}$$

$$= \frac{1}{2m} \cdot \left[\frac{\Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right)}{\Gamma(2m)}\right]^2 \qquad \ldots(i)$$

By Duplication Formula $\quad \Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}} \,.\, \Gamma(2m)$

$\therefore$ From (*i*), $\quad B(m, m) \,.\, B\left(m+\frac{1}{2}, m+\frac{1}{2}\right) = \frac{1}{2m} \cdot \frac{\pi}{2^{4m-2}} = \frac{\pi m^{-1}}{2^{4m-1}}$.

Example 3. *Prove that* $\Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}\,\Gamma(2n+1)}{2^{2n}\,\Gamma(n+1)}$.

Sol. By definition, $B\left(n+\frac{1}{2}, n+\frac{1}{2}\right) = \int_0^1 x^{n-\frac{1}{2}} (1-x)^{n-\frac{1}{2}}\, dx$

Put $x = \sin^2\theta$ so that $dx = 2\sin\theta\cos\theta\, d\theta$

When $x = 0$, $\theta = 0$ and when $x = 1$, $\theta = \frac{\pi}{2}$

$$\therefore\ B\left(n+\tfrac{1}{2}, n+\tfrac{1}{2}\right) = \int_0^{\frac{\pi}{2}} (\sin^2\theta)^{n-\frac{1}{2}} (\cos^2\theta)^{n-\frac{1}{2}} \,.\, 2\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \sin^{2n-1}\theta\, \cos^{2n-1}\theta\, \sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} (\sin\theta\cos\theta)^{2n}\, d\theta = 2\int_0^{\frac{\pi}{2}} \left(\frac{\sin 2\theta}{2}\right)^{2n} d\theta$$

$$= \frac{2}{2^{2n}} \int_0^{\frac{\pi}{2}} (\sin 2\theta)^{2n}\, d\theta \qquad [\text{Put } 2\theta = t \text{ so that } 2d\theta = dt]$$

$$= \frac{1}{2^{2n}} \int_0^{\pi} (\sin t)^{2n}\, dt$$

$$= \frac{2}{2^{2n}} \int_0^{\frac{\pi}{2}} (\sin t)^{2n}\, dt \qquad \left[\because \int_0^{\pi} f(t)\, dt = 2\int_0^{\frac{\pi}{2}} f(t)\, dt \text{ if } f(\pi - t) = f(t)\right]$$

$$= \frac{2}{2^{2n}} \int_0^{\frac{\pi}{2}} (\sin t)^{2n} (\cos t)^0\, dt$$

$$= \frac{2}{2^{2n}} \cdot \frac{1}{2} \frac{\Gamma\left(\frac{2n+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{2n+2}{2}\right)}$$

$$\left[\because \int_0^{\frac{\pi}{2}} \sin^m t \cos^n t\, dt = \frac{1}{2} \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{m+1+n+1}{2}\right)}\right]$$

$$= \frac{1}{2^{2n}} \cdot \frac{\Gamma\left(n+\frac{1}{2}\right)\sqrt{\pi}}{\Gamma(n+1)}$$

Putting $B\left(n+\frac{1}{2}, n+\frac{1}{2}\right) = \frac{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(n+\frac{1}{2}+n+\frac{1}{2}\right)} = \frac{[\Gamma\left(n+\frac{1}{2}\right)]^2}{\Gamma(2n+1)}$, we get

$$\frac{[\Gamma\left(n+\frac{1}{2}\right)]^2}{\Gamma(2n+1)} = \frac{1}{2^{2n}} \cdot \frac{\Gamma\left(n+\frac{1}{2}\right)\sqrt{\pi}}{\Gamma(n+1)}$$

$$\Rightarrow \qquad \Gamma\left(n+\tfrac{1}{2}\right) = \frac{\sqrt{\pi}\,\Gamma(2n+1)}{2^{2n}\,\Gamma(n+1)}.$$

13

Differentiation Under the Integral Sign

13.1. INTRODUCTION

If the integrand is a function of one or more parameters in addition to the variable of integration, then the integral between the limits which may be constants or functions of the parameters is a function of these parameters.

For example

(i) $\displaystyle\int_0^{\pi/2} \sin \alpha x \, dx = -\left[\frac{\cos \alpha x}{\alpha}\right]_0^{\pi/2} = -\frac{1}{\alpha}\left(\cos\frac{\pi}{2}\alpha - 1\right) = \frac{1}{\alpha}\left(1 - \cos\frac{\pi}{2}\alpha\right) = F(\alpha)$

(ii) $\displaystyle\int_1^2 (x+\alpha)^2 \, dx = \left[\frac{(x+\alpha)^3}{3}\right]_1^2 = \frac{1}{3}[(2+\alpha)^3 - (1+\alpha)^3] = \frac{1}{3}(3\alpha^2 + 9\alpha + 7) = F(\alpha)$

Thus, in general $\displaystyle\int_1^b f(x, \alpha)\, dx = F(\alpha)$ ...(1)

$$\int_a^b f(x, \alpha, \beta)\, dx = F(\alpha, \beta)$$

where a, b may be constants or functions of parameters.

Sometimes $f(x, \alpha)$ is such that the evaluation of the integral is very complicated or impossible. However, the integral with integrand $\dfrac{\partial f}{\partial \alpha}$ may be easily evaluated. Hence we discuss here how to differentiate the integral (1) w.r.t. the parameter α.

13.2. LEIBNITZ'S RULE FOR DIFFERENTIATION UNDER INTEGRAL SIGN

Theorem. *If $f(x, \alpha)$ and $\dfrac{\partial}{\partial\alpha} f(x, \alpha)$ are continuous functions of x and α for $a \le x \le b$, $c \le \alpha \le d$, a, b being independent of α, then*

$$\frac{d}{d\alpha}\int_a^b f(x, \alpha)\, dx = \int_a^b \frac{\partial}{\partial\alpha} f(x, \alpha)\, dx.$$

Proof. Let $\quad F(\alpha) = \displaystyle\int_a^b f(x, \alpha)\, dx$ ...(1)

Let α change to $\alpha + \delta\alpha$, (α and $\alpha + \delta\alpha$ both ~~in $[c, d]$~~) then a, b, x being independent of α, remain unaltered and $F(\alpha)$ changes to $F(\alpha + \delta\alpha)$.

$\therefore \quad F(\alpha + \delta\alpha) = \displaystyle\int_a^b f(x, \alpha + \delta\alpha)\, dx$ ...(2)

Subtracting (1) from (2), we get

$F(\alpha + \delta\alpha) - F(\alpha) = \displaystyle\int_a^b [f(x, \alpha + \delta\alpha) - f(x, \alpha)]\, dx$ ...(3)

Now $f(x, \alpha)$ is a continuous function of α in $[\alpha, \alpha + \delta\alpha]$ and $\dfrac{\partial}{\partial\alpha} f(x, \alpha)$ exists.

$\therefore$ By Lagrange's mean value theorem, we have

$$f(x, \alpha + \delta\alpha) - f(x, \alpha) = \delta\alpha \cdot \frac{\partial}{\partial\alpha} f(x, \alpha + \theta\delta\alpha), \quad \text{where } 0 < \theta < 1$$

$\therefore$ From (3), we get

$$F(\alpha + \delta\alpha) - F(\alpha) = \int_a^b \delta\alpha \cdot \frac{\partial}{\partial\alpha} f(x, \alpha + \theta\delta\alpha)\, dx = \delta\alpha \int_a^b \frac{\partial}{\partial\alpha} f(x, \alpha + \theta\,\delta\alpha)\, dx$$

$$\Rightarrow \quad \frac{F(\alpha + \delta\alpha) - F(\alpha)}{\delta\alpha} = \int_a^b \frac{\partial}{\partial\alpha} f(x, \alpha + \theta\delta\alpha)\, dx$$

Taking limits as $\delta\alpha \to 0$, we have

$$\operatorname*{Lt}_{\delta\alpha \to 0} \frac{F(\alpha + \delta\alpha) - F(\alpha)}{\delta\alpha} = \operatorname*{Lt}_{\delta\alpha \to 0} \int_a^b \frac{\partial}{\partial\alpha} f(x, \alpha + \theta\delta\alpha)\, dx = \int_a^b \operatorname*{Lt}_{\delta\alpha \to 0} \frac{\partial}{\partial\alpha} f(x, \alpha + \theta\delta\alpha)\, dx$$

(assuming the limit of the integral is equal to the integral of the limit)

$$\Rightarrow \quad \frac{d}{d\alpha}(F(\alpha)) = \int_a^b \frac{\partial}{\partial\alpha} f(x, \alpha)\, dx \qquad \left[\because \frac{\partial}{\partial\alpha} f(x, \alpha) \text{ is continuous}\right]$$

Hence $\quad \displaystyle\frac{d}{d\alpha}\int_a^b f(x, \alpha)\, dx = \int_a^b \frac{\partial}{\partial\alpha} f(x, \alpha)\, dx$

☞ **Note 1.** Thus, Leibnitz's rule states $\quad F(\alpha) = \displaystyle\int_a^b f(x, \alpha)\, dx \;\Leftrightarrow\; F'(\alpha) = \int_a^b \frac{\partial}{\partial\alpha} f(x, \alpha)\, dx$

Similarly, $\quad F(\alpha, \beta) = \displaystyle\int_a^b f(x, \alpha, \beta)\, dx \;\Rightarrow\; \frac{\partial F}{\partial\alpha} = \int_a^b \frac{\partial}{\partial\alpha} d(x, \alpha, \beta)\, dx$

and $\quad \displaystyle\frac{\partial F}{\partial\beta} = \int_a^b \frac{\partial}{\partial\beta} f(x, \alpha, \beta)\, dx$

Note 2. From definite integrals, if a function f is

(i) Continuous on $[a, b]$ then $\exists$ a number c in (a, b) such that $\displaystyle\int_a^b f(x)\, dx = (b - a) f(c)$

(ii) Continuous on $[a, a + h]$, then $\exists$ a real number θ in $(0, 1)$ such that $\displaystyle\int_0^{a+h} f(x)\, dx = hf(a + \theta h)$.

13.3. THEOREM

If $f(x, \alpha)$ and $\dfrac{\partial}{\partial\alpha} f(x, \alpha)$ are continuous functions of x and α for $a \le x \le b$, $c \le \alpha \le d$ and a, b are themselves functions of α, possessing continuous first order derivatives, then

$$\frac{d}{d\alpha}\int_a^b f(x, \alpha)\, dx = \int_a^b \frac{\partial}{\partial\alpha} f(x, \alpha)\, dx + \frac{db}{d\alpha} f(b, \alpha) - \frac{da}{d\alpha} f(a, \alpha).$$

Proof. Let $\quad F(\alpha) = \int_a^b f(x, \alpha)\, dx \quad \ldots(1)$

Since a, b are functions of α, when α changes to $\alpha + \delta\alpha$, let a change to $a + \delta a$, b change to $b + \delta b$ and $F(\alpha)$ changes to $F(\alpha + \delta\alpha)$.

$$\therefore \quad F(\alpha + \delta\alpha) = \int_{a+\delta a}^{b+\delta b} f(x, \alpha + \delta\alpha)\, dx \quad \ldots(2)$$

Subtracting (1) from (2), we get

$$F(\alpha + \delta\alpha) - F(\alpha) = \int_{a+\delta a}^{b+\delta b} f(x, \alpha + \delta\alpha)\, dx - \int_a^b f(x, \alpha)\, dx$$

$$= \int_{a+\delta a}^{b+\delta b} [f(x, \alpha + \delta\alpha) - f(x, \alpha)]\, dx + \int_{a+\delta a}^{b+\delta b} f(x, \alpha)\, dx - \int_a^b f(x, \alpha)\, dx$$

$$= \int_{a+\delta a}^{b+\delta b} [f(x, \alpha + \delta\alpha) - f(x, \alpha)]\, dx + \int_{a+\delta a}^{b} f(x, \alpha)\, dx + \int_b^{b+\delta b} f(x, \alpha)\, dx - \int_a^b f(x, \alpha)\, dx$$

$$= \int_{a+\delta a}^{b+\delta b} [f(x, \alpha + \delta\alpha) - f(x, \alpha)]\, dx - \int_b^{a+\delta a} f(x, \alpha)\, dx + \int_b^{b+\delta b} f(x, \alpha)\, dx - \int_a^b f(x, \alpha)\, dx$$

$$= \int_{a+\delta a}^{b+\delta b} [f(x, \alpha + \delta\alpha) - f(x, \alpha)]\, dx + \int_b^{b+\delta b} f(x, \alpha)\, dx - \left[\int_a^b f(x, \alpha)\, dx + \int_b^{a+\delta a} f(x, \alpha)\, dx\right]$$

$$= \int_{a+\delta a}^{b+\delta b} \left[f(x, \alpha + \delta\alpha) - f(x, \alpha)\right] dx + \int_b^{b+\delta b} f(x, \alpha)\, dx - \int_a^{a+\delta a} f(x, \alpha)\, dx \quad \ldots(3)$$

Now by Lagrange's Mean Value Theorem on $[\alpha, \alpha + \delta\alpha]$, $\exists$ a real number θ in $(0, 1)$ such that

$$f(x, \alpha + \delta\alpha) - f(x, \alpha) = \delta\alpha \cdot \frac{\partial}{\partial\alpha} f(x, \alpha + \theta\delta\alpha)$$

Since $f(x, \alpha)$ is a continuous function of x in $[a, a + \delta a]$

$\therefore$ $\exists$ a real number θ_1 in $(0, 1)$ such that

$$\int_a^{a+\delta a} f(x, \alpha)\, dx = \delta a\, f(a + \theta_1 \delta a, \alpha)$$

Also $f(x, \alpha)$ is a continuous function of x in $[b, b + \delta b]$

$\therefore$ $\exists$ a real number θ_2 in $(0, 1)$ such that $\int_b^{b+\delta b} f(x, \alpha)\, dx = \delta b\, f(b + \theta_2\, \delta b, \alpha)$

$\therefore$ From (3), we get

$$F(\alpha + \delta\alpha) - F(\alpha) = \int_{a+\delta a}^{b+\delta b} \delta\alpha \cdot \frac{\partial}{\partial\alpha} f(x, \alpha + \theta\delta\alpha)\, dx + \delta b f(b + \theta_2\, \delta b, \alpha) - \delta a f(a + \theta_1\, \delta a, \alpha)$$

$$\Rightarrow \quad \frac{F(\alpha + \delta\alpha) - F(\alpha)}{\delta\alpha} = \int_{a+\delta a}^{b+\delta b} \frac{\partial}{\partial\alpha} f(x, \alpha + \theta\delta\alpha)\, dx + \frac{\delta b}{\delta\alpha} f(b + \theta_2\, \delta b, \alpha) - \frac{\delta a}{\delta\alpha} f(a + \theta_1, \delta a, \alpha)$$

Taking limits as $\delta\alpha \to 0$ and hence $\delta a \to 0$, $\delta b \to 0$, we have

$$\underset{\delta\alpha \to 0}{\text{Lt}} \frac{F(\alpha + \delta\alpha) - F(\alpha)}{\delta\alpha} = \underset{\delta\alpha \to 0}{\text{Lt}} \int_{a+\delta a}^{b+\delta b} \frac{\partial}{\partial\alpha} f(x, \alpha + \theta\delta\alpha)\, dx$$

$$+ \underset{\delta\alpha \to 0}{\text{Lt}} \frac{\delta b}{\delta\alpha} \times \underset{\delta b \to 0}{\text{Lt}} f(b + \theta_2\, \delta b, \alpha) - \underset{\delta\alpha \to 0}{\text{Lt}} \frac{\delta a}{\delta\alpha} \times \underset{\delta a \to 0}{\text{Lt}} f(a + \theta_1, \delta a, \alpha)$$

$$\Rightarrow \quad \frac{d}{d\alpha}(F(\alpha)) = \int_a^b \frac{\partial}{\partial\alpha} f(x, \alpha)\, dx + \frac{db}{d\alpha} f(b, \alpha) - \frac{da}{d\alpha} f(a, \alpha)$$

$$\left[\because \; f(x, \alpha) \text{ and } \frac{\partial}{\partial\alpha} f(x, \alpha) \text{ are continuous}\right]$$

Hence $\quad \dfrac{d}{d\alpha} \displaystyle\int_a^b f(x, \alpha)\, dx = \int_a^b \frac{\partial}{\partial\alpha} f(x, \alpha)\, dx + \frac{db}{d\alpha} f(b, \alpha) - \frac{da}{d\alpha} f(a, \alpha).$

13.4. LEIBNITZ'S RULE IS APPLIED IN THE FOLLOWING TWO WAYS

1. *Two evaluate a given integral* $\int_a^b f(x, \alpha)\, dx$

   (*i*) Let $F(\alpha) = \int_a^b f(x, \alpha)\, dx$

   (*ii*) Differentiate both sides w.r.t. α using Leibnitz's rule. Thus

   $$F'(\alpha) = \int_a^b \frac{\partial}{\partial\alpha} f(x, \alpha)\, dx$$

   (*iii*) Evaluate the integral on R.H.S.

   (*iv*) Integrate both sides w.r.t. α, adding the constant of integration on R.H.S.

   (*v*) Evaluate the constant of integration by giving suitable value to the parameter α.

2. New integrals can be deduced by differentiating certain standard known integrals.

ILLUSTRATIVE EXAMPLES

Example 1. *If* $|\, a \,| < 1$, *show that* $\displaystyle\int_0^\pi \frac{\log(1 + a\cos x)}{\cos x}\, dx = \pi \sin^{-1} a.$

Sol. Let $\quad F(a) = \displaystyle\int_0^\pi \frac{\log(1 + a\cos x)}{\cos x}\, dx \quad \ldots(i)$

Differentiating both sides w.r.t. a (the parameter)

$$F'(a) = \int_0^\pi \frac{\partial}{\partial a}\left[\frac{\log(1 + a\cos x)}{\cos x}\right] dx = \int_0^\pi \frac{1}{\cos x} \cdot \frac{1}{1 + a\cos x} \cdot \cos x\, dx \quad |\text{ Note}$$

$$= \int_0^\pi \frac{dx}{1 + a\cos x} = \frac{1}{\sqrt{1 - a^2}}\left[\cos^{-1} \frac{a + 1\cos x}{1 + a\cos x}\right]_0^\pi$$

$$\left[\because \text{ if } a^2 > b^2, \text{ then } \int \frac{dx}{a + b\cos x} = \frac{1}{\sqrt{a^2 - b^2}} \cos^{-1} \frac{b + a\cos x}{a + b\cos x}\right.$$

$$\left.\text{Here } |\, a \,| < 1 \text{ so that } a^2 < 1 \text{ i.e., } 1 > a^2\right]$$

$$= \frac{1}{\sqrt{1-a^2}}\left[\cos^{-1}\frac{a-1}{1-a} - \cos^{-1}\frac{a+1}{1+a}\right]$$

$$= \frac{1}{\sqrt{1-a^2}}[\cos^{-1}(-1) - \cos^{-1} 1] = \frac{1}{\sqrt{1-a^2}}[\pi - 0] = \frac{\pi}{\sqrt{1-a^2}}$$

Integrating both sides w.r.t. a

$$F(a) = \pi \sin^{-1} a + c \quad \text{...(ii)}$$

Put $a = 0$ in (i) and (ii)

From (i), $F(0) = 0$

From (ii), $F(0) = \pi \sin^{-1} 0 + c \Rightarrow c = 0$

$\therefore$ $F(a) = \pi \sin^{-1} a$

Hence $\displaystyle\int_0^{\pi} \frac{\log(1 + a\cos x)}{\cos x}dx = \pi \sin^{-1} a.$

Example 2. *Prove that* $\displaystyle\int_0^{\pi/2} \frac{\log(1+\cos\alpha\cos x)}{\cos x}dx = \frac{1}{2}\left(\frac{\pi^2}{4} - \alpha^2\right).$

Sol. Let $\displaystyle F(\alpha) = \int_0^{\pi/2} \frac{\log(1+\cos\alpha\cos x)}{\cos x}dx \quad \text{...(i)}$

Differentiating both sides w.r.t α, we get

$$F'(\alpha) = \int_0^{\pi/2} \frac{\partial}{\partial\alpha}\left[\frac{\log(1+\cos\alpha\cos x)}{\cos x}\right]dx$$

$$= \int_0^{\pi/2} \frac{1}{\cos x}\frac{1}{1+\cos\alpha\cos x}.\cos x(-\sin\alpha)\,dx = -\int_0^{\pi/2}\frac{\sin\alpha}{1+\cos\alpha\cos x}dx$$

$$= -\sin\alpha.\frac{1}{\sqrt{1-\cos^2\alpha}}\left[\cos^{-1}\frac{\cos\alpha + 1.\cos x}{1+\cos\alpha\cos x}\right]_0^{\pi/2}$$

$$= -\left[\cos^{-1}\frac{\cos\alpha+0}{1+0} - \cos^{-1}\frac{\cos\alpha+1}{1+\cos\alpha}\right]$$

$$= -[\cos^{-1}(\cos\alpha) - \cos^{-1} 1] = -(\alpha - 0) = -\alpha.$$

Integrating both sides w.r.t. α

$$F(\alpha) = -\frac{\alpha^2}{2} + c \quad \text{...(ii)}$$

when $\alpha = \frac{\pi}{2}$ from (i), $F\left(\frac{\pi}{2}\right) = 0$

$\therefore$ From (ii), $\quad 0 = -\frac{1}{2}.\frac{\pi^2}{4} + c \Rightarrow c = \frac{\pi^2}{8}$

$$\therefore \quad F(\alpha) = -\frac{\alpha^2}{2} + \frac{\pi^2}{8} = \frac{1}{2}\left(\frac{\pi^2}{4} - \alpha^2\right).$$

Hence $\displaystyle\int_0^{\pi/2} \frac{\log(1+\cos\alpha\cos x)}{\cos x}dx = \frac{1}{2}\left(\frac{\pi^2}{4} - \alpha^2\right).$

Example 3. *Prove that* $\displaystyle\int_0^{\pi} \frac{\log(1+\sin\alpha\cos x)}{\cos x}dx = \alpha\pi$

Sol. *Please try yourself.*

Example 4. *Prove that* $\displaystyle\int_0^{\pi/2} \frac{\log(1+y\sin^2 x)}{\sin^2 x}dx = \pi[\sqrt{1+y} - 1], y > -1.$

Sol. Let $\displaystyle F(y) = \int_0^{\pi/2}\frac{\log(1+y\sin^2 x)}{\sin^2 x}dx \quad \text{...(i)}$

Differentiating both sides w.r.t. y, we get

$$F'(y) = \int_0^{\pi/2}\frac{\partial}{\partial y}\left[\frac{\log(1+y\sin^2 x)}{\sin^2 x}\right]dx = \int_0^{\pi/2}\frac{1}{\sin^2 x}.\frac{1}{1+y\sin^2 x}.\sin^2 x\,dx$$

$$= \int_0^{\pi/2}\frac{1}{1+y\sin^2 x}dx = \int_0^{\pi/2}\frac{dx}{\cos^2 x + \sin^2 x + y\sin^2 x}$$

$$= \int_0^{\pi/2}\frac{dx}{\cos^2 x + (1+y)\sin^2 x}$$

Dividing the numerator and denominator by $\cos^2 x$

$$= \int_0^{\pi/2}\frac{\sec^2 x\,dx}{1+(1+y)\tan^2 x}$$

Put $\tan x = t$, $\sec^2 x\,dx = dt$

When $x = 0, t = 0$; when $x = \frac{\pi}{2}, t \to \infty$

$$\therefore \quad F'(y) = \int_0^{\infty}\frac{dt}{1+(1+y)t^2} = \frac{1}{1+y}\int_0^{\infty}\frac{dt}{t^2 + \frac{1}{1+y}}$$

$$= \frac{1}{1+y}.\frac{1}{\frac{1}{\sqrt{1+y}}}\left[\tan^{-1}\frac{t}{\frac{1}{\sqrt{1+y}}}\right]_0^{\infty}$$

$$= \frac{1}{\sqrt{1+y}}[\tan^{-1}\infty - \tan^{-1} 0] = \frac{\pi}{2\sqrt{1+y}} = \frac{\pi}{2}(1+y)^{-1/2}$$

Integrating both sides w.r.t. y

$$F(y) = \frac{\pi}{2}.\frac{(1+y)^{1/2}}{\frac{1}{2}} + c = \pi\sqrt{1+y} + c \quad \text{...(ii)}$$

From (*i*) when $y = 0$, $F(0) = 0$

$\therefore$ from (*ii*), $\quad 0 = \pi + c \Rightarrow c = -\pi$

$$\therefore \quad F(y) = \pi\sqrt{1+y} - \pi = \pi[\sqrt{1+y} - 1]$$

Hence $\displaystyle\int_0^{\pi/2} \frac{\log(1 + y\sin^2 x)}{\sin^2 x}\, dx = \pi(\sqrt{1+y} - 1).$

Example 5. *Evaluate the integral* $\displaystyle\int_0^1 \frac{x^\alpha - 1}{\log x}\, dx$ *(α > – 1) by applying differentiating under the integral sign.*

Sol. Let $\quad F(\alpha) = \displaystyle\int_0^1 \frac{x^\alpha - 1}{\log x} dx \qquad ...(i)$

Differentiating both sides w.r.t. α, we get

$$F'(\alpha) = \int_0^1 \frac{\partial}{\partial \alpha}\left[\frac{x^\alpha - 1}{\log x}\right] dx = \int_0^1 \frac{1}{\log x} x^\alpha \log x\, dx$$

$$= \int_0^1 x^\alpha\, dx = \left[\frac{x^{\alpha+1}}{\alpha+1}\right]_0^1 \text{ for } \alpha > -1$$

$$= \frac{1}{1+\alpha}$$

Integrating both sides w.r.t. α

$$F(\alpha) = \log(1+\alpha) + c \qquad ...(ii)$$

From (*i*), when $\alpha = 0$, $F(0) = \displaystyle\int_0^1 \frac{1-1}{\log x} dx = 0$

$\therefore$ from (*ii*), $0 = \log 1 + c = 0 + c \Rightarrow c = 0$

$\therefore \quad F(\alpha) = \log(1+\alpha)$ where $\alpha > -1$

Hence $\quad \displaystyle\int_0^1 \frac{x^\alpha - 1}{\log x} dx = \log(1+\alpha).$

Example 6. *Evaluate* $\displaystyle\int_0^{\pi/2} \frac{dx}{(a^2\cos^2 x + b^2 \sin^2 x)^2}.$

Sol. Let $\quad I = \displaystyle\int_0^{\pi/2} \frac{dx}{(a^2\cos^2 x + b^2\sin^2 x)^2} \qquad ...(i)$

Let us first evaluate $\displaystyle\int_0^{\pi/2} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x}$

Dividing the numerator and denominator by $\cos^2 x$

$$\int_0^{\pi/2} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x} = \int_0^{\pi/2} \frac{\sec^2 x\, dx}{a^2 + b^2\tan^2 x}$$

[Put $\tan x = t$, $\sec^2 x\, dx = dt$

When $x = 0, t = 0$ when $x = \frac{\pi}{2}, t \to \infty$]

$$= \int_0^\infty \frac{dt}{a^2 + b^2 t^2} = \frac{1}{b^2}\int_0^\infty \frac{dt}{t^2 + \frac{a^2}{b^2}} = \frac{1}{b^2}\cdot\frac{1}{\frac{a}{b}}\left[\tan^{-1}\frac{t}{\frac{a}{b}}\right]_0^\infty$$

$$= \frac{1}{ab}[\tan^{-1}\infty - \tan^{-1} 0] = \frac{\pi}{2ab}$$

$$\therefore \quad \int_0^{\pi/2} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x} = \frac{\pi}{2ab} \qquad ...(ii)$$

Differentiating both sides partially w.r.t. *a*, we get

$$\int_0^{\pi/2} \frac{\partial}{\partial a}[(a^2\cos^2 x + b^2\sin^2 x)^{-1}]\, dx = \frac{\partial}{\partial a}\left[\frac{\pi}{2ab}\right]$$

$$\Rightarrow \quad \int_0^{\pi/2} -(a^2\cos^2 x + b^2\sin^2 x)^{-2} \cdot 2a\cos^2 x\, dx = \frac{\pi}{2b}\cdot(a^{-2})$$

i.e., $\quad \displaystyle\int_0^{\pi/2} \frac{-2a\cos^2 x}{(a^2\cos^2 x + b^2\sin^2 x)^2} dx = -\frac{\pi}{2a^2 b}$

Dividing both sides by $-2a$

$$\int_0^{\pi/2} \frac{\cos^2 x}{(a^2\cos^2 x + b^2\sin^2 x)^2} dx = \frac{\pi}{4a^3 b} \qquad ...(iii)$$

Similarly differentiating (*ii*), partially w.r.t. *b*, we get

$$\int_0^{\pi/2} \frac{\sin^2 x}{(a^2\cos^2 x + b^2\sin^2 x)^2} dx = \frac{\pi}{4ab^3} \qquad ...(iv)$$

Adding (*iii*) and (*iv*), $\displaystyle\int_0^{\pi/2} \frac{\cos^2 x + \sin^2 x}{(a^2\cos^2 x + b^2\sin^2 x)^2} dx = \frac{\pi}{4a^3 b} + \frac{\pi}{4ab^3}$

$$\Rightarrow \quad \int_0^{\pi/2} \frac{dx}{(a^2\cos^2 x + b^2\sin^2 x)^2} = \frac{\pi}{4a^3 b^3}(a^2 + b^2)$$

Hence $\quad I = \dfrac{\pi(a^2 + b^2)}{4a^3 b^3}.$

Example 7. *Prove that* $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{\log(1 + b\sin x)}{\sin x} dx = \pi\sin^{-1} b$ *where* $|b| < 1$.

Sol. Let $\quad F(b) = \displaystyle\int_{-\pi/2}^{\pi/2} \frac{\log(1 + b\sin x)}{\sin x} dx \qquad ...(i)$

Differentiating both sides w.r.t. *b*, we get

$$F'(b) = \int_{-\pi/2}^{\pi/2} \frac{\partial}{\partial b}\left[\frac{\log(1 + b\sin x)}{\sin x}\right] dx$$

$$= \int_{-\pi/2}^{\pi/2} \frac{1}{\sin x} \cdot \frac{1}{1+b\sin x} \sin x \, dx = \int_{-\pi/2}^{\pi/2} \frac{dx}{1+b\sin x}$$

Put $x = \frac{\pi}{2} - t$ then $dx = -dt$

When $x = -\frac{\pi}{2}, t = \pi$; when $x = \frac{\pi}{2}, t = 0$

$\therefore$ $$F'(b) = \int_{\pi}^{0} \frac{-dt}{1+b\sin\left(\frac{\pi}{2}-t\right)} = \int_{\pi}^{0} \frac{-dt}{1+b\cos t}$$

$$= \int_0^{\pi} \frac{dt}{1+b\cos t} \qquad \left| \because \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx \right.$$

$$= \frac{1}{\sqrt{1-b^2}} \left[\cos^{-1} \frac{b+1\cos t}{1+b\cos t}\right]_0^{\pi} \qquad | \because \ |b| < 1 \Rightarrow b^2 < 1$$

$$= \frac{1}{\sqrt{1-b^2}} \left[\cos^{-1} \frac{b+\cos\pi}{1+b\cos\pi} - \cos^{-1} \frac{b+\cos 0}{1+b\cos 0}\right]$$

$$= \frac{1}{\sqrt{1-b^2}} \left[\cos^{-1} \frac{b-1}{1-b} - \cos^{-1} \frac{b+1}{1+b}\right]$$

$$= \frac{1}{\sqrt{1-b^2}} [\cos^{-1}(-1) - \cos^{-1} 1] = \frac{1}{\sqrt{1-b^2}} [\pi - 0] = \frac{\pi}{\sqrt{1-b^2}}$$

Integrating w.r.t. b, we get

$$F(b) = \pi \sin^{-1} b + c \qquad \ldots(ii)$$

From (i) when $b = 0$, $F(0) = 0$

$\therefore$ from (ii), $0 = \pi \sin^{-1} 0 + c = 0 + c \Rightarrow c = 0$

$\therefore$ $F(b) = \pi \sin^{-1} b$

Hence $$\int_{-\pi/2}^{\pi/2} \frac{\log(1+b\sin x)}{\sin x} dx = \pi \sin^{-1} b.$$

Example 8. *Evaluate* $\int_0^{\infty} \frac{\tan^{-1} ax}{x(1+x^2)} dx$, $a \geq 0$ *by applying differentiation under the integral sign.*

Sol. Let $$F(a) = \int_0^{\infty} \frac{\tan^{-1} ax}{x(1+x^2)} dx \qquad \ldots(i)$$

Differentiating both sides w.r.t. a, we get

$$F'(a) = \int_0^{\infty} \frac{\partial}{\partial a}\left[\frac{\tan^{-1} ax}{x(1+x^2)}\right] dx = \int_0^{\infty} \frac{1}{x(1+x^2)} \cdot \frac{1}{1+a^2x^2} \cdot x\,dx \qquad | \textbf{ Note}$$

$$= \int_0^{\infty} \frac{dx}{(1+x^2)(1+a^2x^2)}$$

$$= \frac{1}{1-a^2} \int_0^{\infty} \left[\frac{1}{1+x^2} - \frac{a^2}{1+a^2x^2}\right] dx \qquad | \text{ Partial fractions by putting } x^2 =$$

$$= \frac{1}{1-a^2} \left[\tan^{-1} x\right]_0^{\infty} - \frac{a^2}{1-a^2} \int_0^{\infty} \frac{dx}{1+a^2x^2}$$

$$= \frac{-1}{1-a^2} [\tan^{-1} \infty - \tan^{-1} 0] - \frac{a^2}{1-a^2} \cdot \frac{1}{a^2} \int_0^{\infty} \frac{dx}{x^2 + \frac{1}{a^2}}$$

$$= \frac{1}{1-a^2} \cdot \frac{\pi}{2} - \frac{1}{1-a^2} \frac{1}{\frac{1}{a}} \left[\tan^{-1} \frac{x}{\frac{1}{a}}\right]_0^{\infty} = \frac{1}{1-a^2} \cdot \frac{\pi}{2} - \frac{a}{1-a^2} [\tan^{-1} \infty - \tan^{-1} 0]$$

$$= \frac{1}{1-a^2} \left[\frac{\pi}{2} - a \cdot \frac{\pi}{2}\right] = \frac{1}{1-a^2} \cdot \frac{\pi}{2} [1-a] = \frac{\pi}{2(1+a)}$$

Integrating both sides w.r.t. a

$$F(a) = \frac{\pi}{2} \log(1+a) + c \qquad \ldots(ii)$$

from (i), when $a = 0$, $F(0) = 0$

$\therefore$ from (ii), $0 = \frac{\pi}{2} \log 1 + c = 0 + c \Rightarrow c = 0$

$\therefore$ $F(a) = \frac{\pi}{2} \log(1+a)$

Hence $$\int_0^{\infty} \frac{\tan^{-1} ax}{x(1+x^2)} dx = \frac{\pi}{2} \log(1+a).$$

Example 9. *Prove that if* $a > b$ $\int_0^{\pi/2} \log\left(\frac{a+b\sin\theta}{a-b\sin\theta}\right) \cdot \frac{d\theta}{\sin\theta} = \pi \sin^{-1} \frac{b}{a}$.

Sol. Let $$F(a, b) = \int_0^{\pi/2} \log\left(\frac{a+b\sin\theta}{a-b\sin\theta}\right) \cdot \frac{d\theta}{\sin\theta} \qquad \ldots(i)$$

Differentiating both sides partially w.r.t. b, we get

$$\frac{\partial F}{\partial b} = \int_0^{\pi/2} \frac{\partial}{\partial b} [\log(a+b\sin\theta) - \log(a-b\sin\theta)] \cdot \frac{d\theta}{\sin\theta}$$

$$= \int_0^{\pi/2} \left[\frac{\sin\theta}{a+b\sin\theta} - \frac{-\sin\theta}{a-b\sin\theta}\right] \cdot \frac{d\theta}{\sin\theta}$$

$$= \int_0^{\pi/2} \left(\frac{1}{a+b\sin\theta} + \frac{1}{a-b\sin\theta}\right) d\theta$$

$$= \int_0^{\pi/2} \frac{2a}{a^2 - b^2\sin^2\theta} d\theta = \int_0^{\pi/2} \frac{2a}{a^2(\cos^2\theta + \sin^2\theta) - b^2\sin^2\theta} d\theta$$

$$= 2a \int_0^{\pi/2} \frac{d\theta}{a^2 \cos^2\theta + (a^2 - b^2)\sin^2\theta}$$

Dividing the numerator and denominator by

$$\cos^2\theta = 2a \int_0^{\pi/2} \frac{\sec^2\theta \, d\theta}{a^2 + (a^2 - b^2)\tan^2\theta}$$

Put $\tan\theta = t$ then $\sec^2\theta \, d\theta = dt$

When $\theta = 0, t = 0$ when $\theta = \frac{\pi}{2}, t \to \infty$

$$\therefore \quad \frac{\partial F}{\partial b} = 2a \int_0^\infty \frac{dt}{a^2 + (a^2 - b^2)t^2} = \frac{2a}{a^2 - b^2} \int_0^\infty \frac{dt}{t^2 + \frac{a^2}{a^2 - b^2}}$$

$$= \frac{2a}{a^2 - b^2} \cdot \frac{1}{\frac{a}{\sqrt{a^2 - b^2}}} \left[\tan^{-1} \frac{t}{\frac{a}{\sqrt{a^2 - b^2}}} \right]_0^\infty$$

$$= \frac{2}{\sqrt{a^2 - b^2}} [\tan^{-1}\infty - \tan^{-1} 0] = \frac{2}{\sqrt{a^2 - b^2}} \cdot \frac{\pi}{2} = \frac{\pi}{\sqrt{a^2 - b^2}}$$

Integrating both sides w.r.t. b, we get $\quad F(a, b) = \pi \sin^{-1} \frac{b}{a} + c \qquad ...(ii)$

From (i) when $b = 0$, $F(a, 0) = 0$

$\therefore$ from (ii), $\quad 0 = \pi \sin^{-1} 0 + c = 0 + c \quad \Rightarrow \quad c = 0$

$$\therefore \quad F(a, b) = \pi \sin^{-1} \frac{b}{a}$$

Hence $\displaystyle\int_0^{\pi/2} \log\left(\frac{a + b\sin\theta}{a - b\sin\theta}\right) \frac{d\theta}{\sin\theta} = \pi \sin^{-1} \frac{b}{a}.$

Example 10. *Show that if x^2 is $<$ or $= 1$,*

$$\int_0^{\pi/2} \log(1 - x^2 \cos^2\theta)\, d\theta = \pi \log\,[1 + \sqrt{1 - x^2}\,] - \pi \log 2.$$

Sol. Let $\quad F(x) = \displaystyle\int_0^{\pi/2} \log(1 - x^2\cos^2\theta)\, d\theta \qquad ...(i)$

Differentiating w.r.t. x, we get

$$F'(x) = \int_0^{\pi/2} \frac{\partial}{\partial x} [\log(1 - x^2\cos^2\theta)]\, d\theta = \int_0^{\pi/2} \frac{1}{1 - x^2\cos^2\theta} (-2x\cos^2\theta)\, d\theta$$

$$= -2x \int_0^{\pi/2} \frac{\cos^2\theta \, d\theta}{(\cos^2\theta + \sin^2\theta) - x^2\cos^2\theta}$$

$$= -2x \int_0^{\pi/2} \frac{\cos^2\theta \, d\theta}{(1 - x^2)\cos^2\theta + \sin^2\theta}$$

Dividing the numerator and denominator by $\cos^2\theta = -2x \displaystyle\int_0^{\pi/2} \frac{d\theta}{(1 - x^2) + \tan^2\theta}$

Put $\tan\theta = t$, then $\quad \sec^2\theta \, d\theta = dt \quad$ or $\quad d\theta = \dfrac{1}{\sec^2\theta} dt = \dfrac{1}{1 + \tan^2\theta} dt = \dfrac{1}{1 + t^2} dt$

when $\quad \theta = 0, t = 0$, when $\theta = \frac{\pi}{2}, t \to \infty$

$$\therefore \quad F'(x) = -2x \int_0^\infty \frac{dt}{(1 + t^2)[(1 - x^2) + t^2]}$$

$$= -2x \int_0^\infty \frac{1}{x^2} \left[\frac{1}{1 + t^2} - \frac{1}{(1 - x^2) + t^2} \right] dt \qquad \text{| Partial Fractions}$$

$$= \frac{2}{x} \left[\tan^{-1} t - \frac{1}{\sqrt{1 - x^2}} \tan^{-1} \left(\frac{t}{\sqrt{1 - x^2}} \right) \right]_0^\infty$$

$$= \frac{2}{x} \left[(\tan^{-1}\infty - \tan^{-1} 0) - \frac{1}{\sqrt{1 - x^2}} (\tan^{-1}\infty - \tan^{-1} 0) \right]$$

$$= \frac{2}{x} \left[\frac{\pi}{2} - \frac{\pi}{2\sqrt{1 - x^2}} \right] = \frac{\pi}{x} \left[1 - \frac{1}{\sqrt{1 - x^2}} \right]$$

Integrating both sides w.r.t. x, we get

$$F(x) = \pi \int \frac{1}{x} dx - \pi \int \frac{1}{x\sqrt{1 - x^2}} dx + c. \quad \text{Put } x = \frac{1}{y}$$

$$= \pi \log x - \pi \int \frac{-\frac{1}{y^2} dy}{\frac{1}{y}\sqrt{1 - \frac{1}{y^2}}} + c = \pi \log x + \pi \int \frac{dy}{\sqrt{y^2 - 1}} + c$$

$$= \pi \log x + \pi \log (y + \sqrt{y^2 - 1}) + c$$

$$= \pi \log x + \pi \log \left(\frac{1}{x} + \sqrt{\frac{1}{x^2} - 1} \right) + c$$

$$= \pi \left[\log x + \log \frac{1 + \sqrt{1 - x^2}}{x} \right] + c$$

$$= \pi \log \left[x \cdot \frac{1 + \sqrt{1 - x^2}}{x} \right] + c$$

$$= \pi \log (1 + \sqrt{1 - x^2}) + c \qquad ...(ii)$$

when $x = 0$, from (*i*), $\quad F(0) = 0$

$\therefore$ from (*ii*), $\quad 0 = \pi \log 2 + c \;\Rightarrow\; c = -\pi \log 2$

$$\therefore \quad F(x) = \pi \log (1 + \sqrt{1-x^2}) - \pi \log 2$$

Hence $\displaystyle\int_0^{\pi/2} \log (1 - x^2 \cos^2 \theta)\, d\theta = \pi \log (1 + \sqrt{1-x^2}) - \pi \log 2.$

Example 11. *If $e^2 < 1$, prove that* $\displaystyle\int_0^{\pi/2} \log (1 - e^2 \sin^2 \theta)\, d\theta = \pi \log \frac{1}{2}(1 + \sqrt{1-e^2})$.

Sol. *Please try yourself.*

Example 12. *If $|a| < 1$, prove that* $\displaystyle\int_0^{\pi} \log (1 - a \cos x)\, dx = \pi \log \left[\frac{1}{2} + \frac{1}{2}\sqrt{1-a^2}\right]$.

Sol. Let $\quad F(a) = \displaystyle\int_0^{\pi} \log (1 + a \cos x)\, dx \qquad \ldots(i)$

Differentiating w.r.t. a, we get

$$F'(a) = \int_0^{\pi} \frac{\partial}{\partial a} [\log (1 + a \cos x)]\, dx$$

$$= \int_0^{\pi} \frac{1}{1 + a \cos x} . \cos x\, dx$$

$$a \cos x + 1 \,\big)\, \cos x \,\big(\, \frac{1}{a} \qquad \cos x + \frac{1}{a} \qquad -\frac{1}{a}$$

$$= \int_0^{\pi} \left[\frac{1}{a} - \frac{\frac{1}{a}}{1 + a \cos x}\right] dx$$

$$= \left[\frac{x}{a}\right]_0^{\pi} - \frac{1}{a} . \frac{1}{\sqrt{1-a^2}} \left[\cos^{-1} \frac{a + 1 \cos x}{1 + a \cos x}\right]_0^{\pi} \qquad \because \; |a| < 1$$

$$= \frac{\pi}{a} - \frac{1}{a\sqrt{1-a^2}} \left[\cos^{-1} \frac{a-1}{1-a} - \cos^{-1} \frac{a+1}{1+a}\right]$$

$$= \frac{\pi}{a} - \frac{1}{a\sqrt{1-a^2}} [\cos^{-1}(-1) - \cos^{-1} 1]$$

$$= \frac{\pi}{a} - \frac{1}{a\sqrt{1-a^2}} [\pi - 0] = \frac{\pi}{a} - \frac{\pi}{a\sqrt{1-a^2}}$$

Integrating w.r.t. a, we get

$$F(a) = \int \frac{\pi}{a}\, da - \pi \int \frac{da}{a\sqrt{1-a^2}} + c$$

$$= \pi \log (1 + \sqrt{1+a^2}) + c \qquad \ldots(ii) \text{ [As in Example 10]}$$

From (*i*), when $a = 0$, $F(0) = 0$

$\therefore$ From (*ii*), $\quad 0 = \pi \log 2 + c \;\Rightarrow\; c = -\pi \log 2$

$$\therefore \quad F(a) = \pi \log (1 + \sqrt{1-a^2}) - \pi \log 2$$

$$= \pi \log \frac{1 + \sqrt{1-a^2}}{2} \qquad \because \; \log m - \log n = \log \frac{m}{n}$$

Hence $\displaystyle\int_0^{\pi} \log (1 + a \cos x)\, dx = \pi \log \left(\frac{1}{2} + \frac{1}{2}\sqrt{1-a^2}\right)$.

Example 13. *If $a > 0$, prove that* $\displaystyle\int_0^{\infty} \frac{e^{-ax} \sin mx}{x}\, dx = \tan^{-1} \frac{m}{a}$ *and deduce that*

$$\int_0^{\infty} \frac{\sin bx}{x}\, dx = \frac{\pi}{2} \quad or \quad -\frac{\pi}{2} \text{ according as } b \text{ is positive or negative.}$$

Sol. Let $\quad F(a, m) = \displaystyle\int_0^{\infty} \frac{e^{-ax} \sin mx}{x}\, dx \qquad \ldots(i)$

Differentiating partially w.r.t. m, we get

$$\frac{\partial F}{\partial m} = \int_0^{\infty} \frac{\partial}{\partial m} \left[\frac{e^{-ax} \sin mx}{x}\right] dx$$

$$= \int_0^{\infty} \frac{e^{-ax}}{x}\, x \cos mx\, dx = \int_0^{\infty} e^{-ax} \cos mx\, dx$$

$$= \left[\frac{e^{-ax}}{(-a)^2 + m^2} (-a \cos mx + m \sin mx)\right]_0^{\infty}$$

$$\left[\because \int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx)\right]$$

$$= 0 - \frac{1}{a^2 + m^2} (-a) = \frac{a}{a^2 + m^2}$$

Integrating both sides w.r.t. m, we get

$$F(a, m) = \int \frac{a}{a^2 + m^2}\, dm + c = a . \frac{1}{a} \tan^{-1} \frac{m}{a} + c = \tan^{-1} \frac{m}{a} + c \qquad \ldots(ii)$$

when $m = 0$ from (*i*), $F(a, 0) = 0$

$\therefore$ from (*ii*), $\quad 0 = \tan^{-1} 0 + c \;\Rightarrow\; c = 0$

$\therefore \quad F(a, m) = \tan^{-1} \dfrac{m}{a}$. Hence $\displaystyle\int_0^{\infty} \frac{e^{-ax} \sin mx}{x}\, dx = \tan^{-1} \frac{m}{a}$

Putting $a = 0$ and $m = b$

$$\int_0^{\infty} \frac{\sin bx}{x}\, dx = \tan^{-1} \frac{b}{0} = \tan^{-1} \infty \text{ or } \tan^{-1}(-\infty) \text{ according as } b > 0 \text{ or } b < 0$$

$$= \frac{\pi}{2} \text{ or } -\frac{\pi}{2} \text{ according as } b > 0 \text{ or } b < 0.$$

Example 14. *If $\alpha > 0$, $\beta > 0$, prove that*

$$\int_0^{\pi/2} \log (\alpha \cos^2 \theta + \beta \sin^2 \theta) . d\theta = \pi \log \frac{\sqrt{\alpha} + \sqrt{\beta}}{2}.$$

Sol. Let $\quad F(\alpha, \beta) = \int_0^{\pi/2} \log(\alpha\cos^2\theta + \beta\sin^2\theta)\, d\theta \quad ...(i)$

Differentiating partially w.r.t. α, we get

$$\frac{\partial F}{\partial \alpha} = \int_0^{\pi/2} \frac{\partial}{\partial \alpha}[\log(\alpha\cos^2\theta + \beta\sin^2\theta)]\, d\theta$$

$$= \int_0^{\pi/2} \frac{1}{\alpha\cos^2\theta + \beta\sin^2\theta} \cdot \cos^2\theta\, d\theta$$

Dividing the numerator and denominator by

$$\cos^2\theta = \int_0^{\pi/2} \frac{d\theta}{\alpha + \beta\tan^2\theta}$$

Put $\tan\theta = t$ so that $\sec^2\theta\, d\theta = dt$ or $(1+\tan^2\theta)\, d\theta = dt \therefore d\theta = \dfrac{dt}{1+t^2}$

when $\theta = 0$ $t = 0$ when $\theta = \dfrac{\pi}{2}, t \to \infty$

$$\therefore \quad \frac{\partial F}{\partial \alpha} = \int_0^\infty \frac{dt}{(\alpha + \beta t^2)(1+t^2)}$$

Resolving the integral into partial fractions

$$= \frac{1}{\alpha-\beta}\int_0^\infty \left(\frac{1}{1+t^2} + \frac{\beta}{\alpha+\beta t^2}\right) dt = \frac{1}{\alpha-\beta}\int_0^\infty \left(\frac{1}{1+t^2} - \frac{1}{\frac{\alpha}{\beta}+t^2}\right) dt$$

$$= \frac{1}{\alpha-\beta}\left[\tan^{-1} t - \frac{1}{\sqrt{\frac{\alpha}{\beta}}}\tan^{-1}\frac{t}{\sqrt{\frac{\alpha}{\beta}}}\right]_0^\infty$$

$$= \frac{1}{\alpha-\beta}\left[\tan^{-1} t - \sqrt{\frac{\beta}{\alpha}}\tan^{-1}\left(\sqrt{\frac{\beta}{\alpha}}\, t\right)\right]_0^\infty = \frac{1}{\alpha-\beta}\left[\frac{\pi}{2} - \sqrt{\frac{\beta}{\alpha}}\cdot\frac{\pi}{2}\right]$$

$$= \frac{\pi}{2(\alpha-\beta)}\cdot\frac{\sqrt{\alpha}-\sqrt{\beta}}{\sqrt{\alpha}} = \frac{\pi(\sqrt{\alpha}-\sqrt{\beta})}{2\sqrt{\alpha}(\sqrt{\alpha}+\sqrt{\beta})(\sqrt{\alpha}-\sqrt{\beta})} = \frac{\pi}{2\sqrt{\alpha}(\sqrt{\alpha}+\sqrt{\beta})}.$$

$$\Rightarrow \quad \frac{\partial F}{\partial \alpha} = \pi \cdot \frac{\frac{1}{2\sqrt{\alpha}}}{\sqrt{\alpha}+\sqrt{\beta}}$$

Integrating both sides w.r.t. parameter α, we get

$$F(\alpha, \beta) = \pi\int \frac{\frac{1}{2\sqrt{\alpha}}}{\sqrt{\alpha}+\sqrt{\beta}}\, d\alpha + c = \pi\log(\sqrt{\alpha}+\sqrt{\beta}) + c \quad ...(ii)$$

$$\left[\because \frac{\partial}{\partial \alpha}(\sqrt{\alpha}+\sqrt{\beta}) = \frac{1}{2\sqrt{\alpha}} \text{ and } \int \frac{f'(x)}{f(x)}\, dx = \log f(x)\right]$$

When $\alpha = \beta = 1$, from (i) $F(1, 1) = \int_0^{\pi/2} \log 1\, d\theta = 0$

$\therefore$ From (ii), $\quad F(1, 1) = \pi\log 2 + c$ or $0 = \pi\log 2 + c \Rightarrow c = -\pi\log 2$

$$\therefore \quad F(\alpha, \beta) = \pi\log(\sqrt{\alpha}+\sqrt{\beta}) - \pi\log 2 = \pi\log\frac{\sqrt{\alpha}+\sqrt{\beta}}{2}$$

Hence $\displaystyle\int_0^{\pi/2} \log(\alpha\cos^2\theta + \beta\sin^2\theta)\, d\theta = \pi\log\frac{\sqrt{\alpha}+\sqrt{\beta}}{2}.$

Example 15. *Prove that* $\displaystyle\int_0^{\pi/2} \log(\alpha^2\cos^2 + \beta^2\sin^2\theta), d\theta = \pi\log\frac{\alpha+\beta}{2}$, $\alpha > 0, \beta > 0.$

Sol. *Please try yourself.*

[See Ex. 14. Here α, β are replaced by α^2, β^2.]

Example 16. *Prove that* $\displaystyle\int_0^\infty \frac{\log(1+a^2x^2)}{1+b^2x^2}\, dx = \frac{\pi}{b}\log\left(1+\frac{a}{b}\right).$

Sol. Let $\quad F(a, b) = \displaystyle\int_0^\infty \frac{\log(1+a^2x^2)}{1+b^2x^2}\, dx \quad ...(1)$

Differentiating partially w.r.t. a, we get

$$\frac{\partial F}{\partial a} = \int_0^\infty \frac{\partial}{\partial a}\left[\frac{\log(1+a^2x^2)}{1+b^2x^2}\right] dx$$

$$= \int_0^\infty \frac{1}{1+b^2x^2}\cdot\frac{1}{1+a^2x^2}\cdot 2ax^2\, dx = 2a\int_0^\infty \frac{x^3\, dx}{(1+a^2x^2)(1+b^2x^2)}.$$

Now $\quad \dfrac{x^2}{(1+a^2x^2)(1+b^2x^2)} = \dfrac{y}{(1+a^2y)(1+b^2y)}$ where $y = x^2$

$$= \frac{-\frac{1}{a^2}}{(1+a^2y)\left(1-\frac{b^2}{a^2}\right)} + \frac{-\frac{1}{b^2}}{\left(1-\frac{a^2}{b^2}\right)(1+b^2y)} \qquad \text{[Partial Fractions]}$$

$$= \frac{-1}{(a^2-b^2)(1+a^2y)} + \frac{1}{(a^2-b^2)(1+b^2y)}$$

$$= \frac{1}{a^2-b^2}\left[\frac{1}{1+b^2x^2} + \frac{1}{1+a^2x^2}\right]$$

$$\therefore \quad \frac{\partial F}{\partial \alpha} = \frac{2a}{a^2-b^2}\int_0^\infty \left(\frac{1}{1+b^2x^2} - \frac{1}{1+a^2x^2}\right) dx$$

$$= \frac{2a}{a^2-b^2}\int_0^\infty \left[\frac{1}{b^2\left(x^2+\frac{1}{b^2}\right)} - \frac{1}{a^2\left(x^2+\frac{1}{a^2}\right)}\right] dx$$

$$= \frac{2a}{a^2 - b^2}\left[\frac{1}{b^2}\cdot\frac{1}{\frac{1}{b}}\tan^{-1}\frac{x}{\frac{1}{b}} - \frac{1}{a^2}\cdot\frac{1}{\frac{1}{a}}\tan^{-1}\frac{x}{\frac{1}{a}}\right]_0^\infty$$

$$= \frac{2a}{a^2 - b^2}\left[\frac{1}{b}\tan^{-1}(bx) - \frac{1}{a}\tan^{-1}(ax)\right]_0^\infty$$

$$= \frac{2a}{a^2 - b^2}\left[\frac{1}{b}\tan^{-1}\infty - \frac{1}{a}\tan^{-1}\infty\right] = \frac{2a}{a^2 - b^2}\left[\frac{1}{b}\cdot\frac{\pi}{2} - \frac{1}{a}\cdot\frac{\pi}{2}\right]$$

$$= \frac{2a}{a^2 - b^2}\cdot\frac{\pi}{2}\left[\frac{1}{b} - \frac{1}{a}\right] = \frac{a\pi}{a^2 - b^2}\cdot\frac{a-b}{ab} = \frac{\pi}{b(a+b)}$$

Integrating w.r.t. a, we get

$$F(a, b) = \frac{\pi}{b}\int \frac{da}{a+b} + c = \frac{\pi}{b}\log(a+b) + c \quad \ldots(1)$$

when $a = 0$ from (1), $F(0, b) = 0$

$\therefore$ from (2), $\quad 0 = \frac{\pi}{b}\log b + c \Rightarrow c = -\frac{\pi}{b}\log b$

$$\therefore \quad F(a, b) = \frac{\pi}{b}\log(a+b) - \frac{\pi}{b}\log b = \frac{\pi}{b}\log\frac{a+b}{b} = \frac{\pi}{b}\log\left(1 + \frac{a}{b}\right)$$

Hence $\quad \displaystyle\int_0^\infty \frac{\log(1 + a^2x^2)}{1 + b^2x^2}\,dx = \frac{\pi}{b}\log\left(1 + \frac{a}{b}\right).$

Example 17. *Prove that for $y > 0$, $\displaystyle\int_0^\infty e^{-xy}\frac{\sin x}{x}\,dx = \cot^{-1} y$.*

Sol. Let $\quad F(y) = \displaystyle\int_0^\infty e^{-xy}\frac{\sin x}{x}\,dx \quad \ldots(1)$

Differentiating w.r.t. y, we get

$$F'(y) = \int_0^\infty \frac{\partial}{\partial y}\left(e^{-xy}\frac{\sin x}{x}\right)dx = \int_0^\infty \frac{\sin x}{x}\cdot e^{-xy}\cdot(-x)\,dx = -\int_0^\infty e^{(-y)x}\sin x\,dx$$

$$= -\left[\frac{e^{-yx}}{(-y)^2 + 1^2}(-y\sin x - 1.\cos x)\right]_0^\infty$$

$$\left[\because \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2 + b^2}(a\sin bx - b\cos bx)\right]$$

$$= \frac{1}{y^2 + 1}\left[e^{-yx}(y\sin x + \cos x)\right]_0^\infty = \frac{1}{y^2 + 1}[0 - (0 + 1)] = \frac{-1}{y^2 + 1} \quad \ldots(2)$$

Integrating w.r.t. y, we get $\quad F(y) = \cot^{-1} y + c$

when $y \to \infty$, from (1), $F(\infty) = 0$

$\therefore$ from (2), $\quad 0 = \cot^{-1}\infty + c = 0 + c \Rightarrow c = 0$

$\therefore \quad F(y) = \cos^{-1} y$

Hence $\quad \displaystyle\int_0^\infty e^{-xy}\frac{\sin x}{x}\,dx = \cot^{-1} y$

Note. Since $\quad \tan^{-1} y + \cot^{-1} y = \frac{\pi}{2}$

$$\therefore \quad \int_0^\infty e^{-xy}\frac{\sin x}{x}\,dx = \cot^{-1} y = \frac{\pi}{2} - \tan^{-1} y.$$

Example 18. *Differentiating $\displaystyle\int_0^x \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$ under the integral sign, show that*

$$\int_0^x \frac{dx}{(x^2 + a^2)^2} = \frac{1}{2a^3}\tan^{-1}\frac{x}{a} + \frac{x}{2a^2(x^2 + a^2)}.$$

Sol. $\quad \displaystyle\int_0^x \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$

Differentiating both sides w.r.t. a, we get

$$\int_0^x -\frac{1}{(x^2 + a^2)^2}\cdot 2a\,dx = \frac{1}{a}\cdot\frac{1}{1 + (x/a)^2}\cdot\left(-\frac{x}{a^2}\right) - \frac{1}{a^2}\tan^{-1}\frac{x}{a}$$

$$\int_0^x \frac{-2a}{(x^2 + a^2)^2}\,dx = \frac{-x}{a(x^2 + a^2)} - \frac{1}{a^2}\tan^{-1}\frac{x}{a}$$

Dividing both sides by $-2a$

$$\int_0^x \frac{dx}{(x^2 + a^2)^2} = \frac{1}{2a^3}\tan^{-1}\frac{x}{a} + \frac{x}{2a^2(x^2 + a^2)}.$$

Example 19. *Evaluate $\displaystyle\int_0^\pi \frac{dx}{a + b\cos x}$ $a > 0$, $|b| < a$ and deduce that*

(i) $\displaystyle\int_0^\pi \frac{dx}{(a + b\cos x)^2} = \frac{\pi a}{(a^2 - b^2)^{3/2}}$ $\qquad (ii)$ $\displaystyle\int_0^\pi \frac{\cos x\,dx}{(a + b\cos x)^2} = -\frac{\pi b}{(a^2 - b^2)^{3/2}}.$

Sol. $\quad \displaystyle\int_0^\pi \frac{dx}{a + b\cos x} = \frac{1}{\sqrt{a^2 - b^2}}\left[\cos^{-1}\frac{b + a\cos x}{a + b\cos x}\right]_0^\pi \qquad \left|\because \; a^2 > b^2\right.$

$$= \frac{1}{\sqrt{a^2 - b^2}}\left[\cos^{-1}\frac{b - a}{a - b} - \cos^{-1}\frac{b + a}{a + b}\right]$$

$$= \frac{1}{\sqrt{a^2 - b^2}}[\cos^{-1}(-1) - \cos^{-1} 1] = \frac{1}{\sqrt{a^2 - b^2}}[\pi - 0] = \frac{1}{\sqrt{a^2 - b^2}} \quad \ldots(i)$$

Differentiating both sides of (i) w.r.t. a

$$\int_0^\pi \frac{-dx}{(a + b\cos x)^2} = \pi\cdot -\frac{1}{2}(a^2 - b^2)^{-3/2}\cdot 2a$$

Hence $\quad \displaystyle\int_0^\pi \frac{dx}{(a + b\cos x)^2} = \frac{\pi a}{(a^2 - b^2)^{3/2}} \quad \ldots(I)$

Differentiating both sides of (*i*) w.r.t. b

$$\int_0^\pi \frac{\partial}{\partial b}\left[\frac{1}{a+b\cos x}\right]dx = \frac{\partial}{\partial b}\left[\frac{\pi}{\sqrt{a^2-b^2}}\right]$$

$$\int_0^\pi \frac{-\cos x\,dx}{(a+b\cos x)^2} = \pi.-\frac{1}{2}(a^2-b^2)^{-3/2}.(-2b)$$

Hence $$\int_0^\pi \frac{\cos x\,dx}{(a+b\cos x)^2} = -\frac{\pi b}{(a^2-b^2)^{3/2}}. \quad \text{...(II)}$$

Example 20. *(a) From the value of* $\int_0^1 x^m\,dx$*, deduce the value of* $\int_0^1 x^m (\log x)^n\,dx$*, when* $m \ge 0$ *and n is a + ve integer.*

(b) Starting from $\int_0^\infty e^{-ax}\,dx = \frac{1}{a}$ *for* $a > 0$ *deduce that* $\int_0^\infty x^m e^{-ax}\,dx = \frac{m!}{a^{m+1}}$.

Sol. (*a*) $$\int_0^1 x^m\,dx = \left[\frac{x^{m+1}}{m+1}\right]_0^1 = \frac{1}{m+1} = (m+1)^{-1}$$

Differentiating w.r.t. m, we get

$$\int_0^1 x^m \log x\,dx = (-1)(m+1)^{-2}$$

Again differentiating w.r.t. m, we get

$$\int_0^1 x^m (\log x)^2\,dx = (-1)(-2)(m+1)^{-3}$$

Repeating differentiation w.r.t. m, n times, we get

$$\int_0^1 x^m (\log x)^n\,dx = (-1)(-2)(-3)\ldots(-n)(m+1)^{-(n+1)} = \frac{(-1)^n\,n!}{(m+1)^{n+1}}$$

(*b*) $$\int_0^\infty e^{-ax}\,dx = \frac{1}{a} = a^{-1}$$

Differentiating w.r.t. a, we get

$$\int_0^\infty e^{-ax}(-x)\,dx = (-1)\,a^{-2} \quad \Rightarrow \quad \int_0^\infty x\,e^{-ax}\,dx = 1.a^{-2}$$

Differentiating again w.r.t. a, we get

$$\int_0^\infty x\,e^{-ax}(-x)\,dx = 1.(-2)\,a^{-3} \quad \Rightarrow \quad \int_0^\infty x^2 e^{ax}\,dx = 1.2.a^{-3}$$

Repeating differentiation w.r.t. a, m times, we get

$$\int_0^\infty x^m e^{-ax}\,dx = 1.2.3.\ldots\ldots m.a^{(m+1)} = \frac{m!}{a^{m+1}}$$

Example 21. *From* $\int_0^\infty \frac{dx}{x^2+a^2} = \frac{\pi}{2a}$*, $a > 0$ deduce that*

$$\int_0^\infty \frac{dx}{(x^2+a^2)^n} = \frac{\pi}{2.a^{2n-1}}\,\frac{1.3.5\ldots(2n-3)}{2.4.6\ldots(2n-2)}$$ *where n is a positive integer.*

Hence by putting $x = a\tan\theta$ in the above result, show that

$$\int_0^{\pi/2} \cos^{2n}\theta\,d\theta = \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n}.\frac{\pi}{2}.$$

Sol. Since $$\int_0^\infty \frac{dx}{x^2+a^2} = \frac{\pi}{2a} \quad \text{...(1)}$$

$\therefore$ Differentiating both sides of (1) w.r.t. a, we get

$$\int_0^\infty \frac{\partial}{\partial a}\left[\frac{1}{x^2+a^2}\right]dx = \frac{\pi}{2}.\frac{-1}{a^2} \quad \text{or} \quad \int_0^\infty \frac{-2a}{(x^2+a^2)^2}\,dx = \frac{\pi}{2}.\frac{-1}{a^2}$$

Dividing both sides by $(-2a)$, we get $$\int_0^\infty \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{2}.\frac{1}{2}.\frac{1}{a^3}$$

Differentiating both sides w.r.t. a, we get

$$\int_0^\infty -2(x^2+a^2)^{-3}.\,2a\,dx = \frac{\pi}{2}.\frac{1}{2}.\frac{-3}{a^4} \quad \text{or} \quad \int_0^\infty \frac{-4a}{(x^2+a^2)^3}\,dx = \frac{\pi}{2}.\frac{1}{2}.\frac{-3}{a^4}$$

Dividing both sides by $(-4a)$, we get $$\int_0^\infty \frac{dx}{(x^2+a^2)^3} = \frac{\pi}{2}.\frac{1}{2}.\frac{3}{4}.\frac{1}{a^5}$$

Continuing like this till (1) is differentiated $(n-1)$ times w.r.t. a, we get

$$\int_0^\infty \frac{dx}{(x^2+a^2)^n} = \frac{\pi}{2}.\frac{1}{a^{2n-1}}.\frac{1.3\ldots(2n-3)}{2.4.(2n-2)} \quad \text{...(2)}$$

which is the required result. In (2) putting $x = a\tan\theta$, we have

$$\int_0^{\pi/2} \frac{a\sec^2\theta\,d\theta}{(a^2\tan^2\theta+a^2)^n} = \frac{\pi}{2}.\frac{1}{a^{n-1}}.\frac{1.3\ldots(2n-3)}{2.4\ldots(2n-2)}$$

$[\because \; dx = a\sec^2\theta\,d\theta$, when $x=\infty$, $\theta = \pi/2$, when $x = 0$, $\theta = 0]$

or $$\int_0^{\pi/2} \cos^{2n-2}\theta\,d\theta = \frac{\pi}{2}.\frac{1.3\ldots(2n-3)}{2.4\ldots(2n-2)}$$

Changing n to $(n+1)$, we get

$$\int_0^{\pi/2} \cos^{2n}\theta\,d\theta = \frac{\pi}{2}.\frac{1.3\ldots(2n-1)}{2.4\ldots(2n)}.$$

Example 22. *Evaluate* $\int_0^\alpha \frac{\log(1+\alpha x)}{1+x^2}\,dx$*, $\alpha \ge 0$ by applying differentiation under the integral sign.*

Sol. Let $$F(\alpha) = \int_0^\alpha \frac{\log(1+\alpha x)}{1+x^2}\,dx \quad \text{...(1)}$$

Here the upper limit of integration involves the parameter α.

Differentiating both sides w.r.t. a, we get

$$F'(a) = \int_0^a \frac{\partial}{\partial a}\left[\frac{\log(1+ax)}{1+x^2}\right]dx + \frac{d}{da}(a)\cdot\frac{\log(1+a\cdot a)}{1+a^2} - \frac{d}{da}(0)\cdot\frac{\log(1+a\cdot 0)}{1+a^2}$$

$$= \int_0^a \frac{1}{1+x^2}\cdot\frac{1}{1+ax}\cdot x\,dx + \frac{\log(1+a^2)}{1+a^2} \quad \ldots(2)$$

Let $\dfrac{x}{(1+ax)(1+x^2)} = \dfrac{A}{1+ax} + \dfrac{Bx+C}{1+x^2}$

Multiplying both sides by $(1+ax)(1+x^2)$, we get $x = A(1+x^2) + (Bx+C)(1+ax)$

Putting $x = -\dfrac{1}{a}$, $-\dfrac{1}{a} = A\left(1+\dfrac{1}{a^2}\right) \Rightarrow -\dfrac{1}{a} = \dfrac{1+a^2}{a^2}\cdot A \Rightarrow A = \dfrac{-a}{1+a^2}$

Comparing co-effs. of x^2, $0 = A + aB \Rightarrow B = -\dfrac{1}{a}A = \dfrac{1}{1+a^2}$

Comparing constants terms, $0 = A + C \Rightarrow C = -A = \dfrac{a}{1+a^2}$

$$\therefore \quad \frac{x}{(1+ax)(1+x^2)} = \frac{1}{1+a^2}\left[\frac{-a}{1+ax} + \frac{x+a}{1+x^2}\right]$$

$\therefore$ From (2), $$F'(a) = \int_0^a \frac{1}{1+a^2}\left[\frac{-a}{1+ax} + \frac{x+a}{1+x^2}\right]dx + \frac{\log(1+a^2)}{1+a^2}$$

$$= \frac{1}{1+a^2}\int_0^a\left[\frac{-a}{1+ax} + \frac{1}{2}\cdot\frac{2x}{1+x^2} + \frac{a}{1+x^2}\right]dx + \frac{\log(1+a^2)}{1+a^2}$$

$$= \frac{1}{1+a^2}\left[-a\cdot\frac{\log(1+ax)}{a} + \frac{1}{2}\log(1+x^2) + a\tan^{-1}x\right]_0^a + \frac{\log(1+a^2)}{1+a^2}$$

$$= \frac{1}{1+a^2}\left[-\log(1+a^2) + \frac{1}{2}\log(1+a^2) + a\tan^{-1}a\right] + \frac{\log(1+a^2)}{1+a^2}$$

$$= \frac{1}{1+a^2}\left[-\frac{1}{2}\log(1+a^2) + a\tan^{-1}a\right] + \frac{\log(1+a^2)}{1+a^2}$$

$$= \frac{1}{1+a^2}\left[\frac{1}{2}\log(1+a^2) + a\tan^{-1}a\right]$$

Integrating both sides w.r.t. a

$$F(a) = \frac{1}{2}\int \frac{\log(1+a^2)}{1+a^2}da + \int\frac{a\tan^{-1}a}{1+a^2}da + c$$

$$= \frac{1}{2}\int \log(1+a^2)\cdot\frac{1}{1+a^2}da + \int\frac{a\tan^{-1}a}{1+a^2}da + c$$

$$= \frac{1}{2}\left[\log(1+a^2)\cdot\tan^{-1}a - \int\frac{2a}{1+a^2}\cdot\tan^{-1}a\,da\right] + \int\frac{a\tan^{-1}a}{1+a^2}da + c$$

$$= \frac{1}{2}\log(1+a^2)\tan^{-1}a - \int\frac{a\tan^{-1}a}{1+a^2}da + \int\frac{a\tan^{-1}a}{1+a^2}da + c$$

$$= \frac{1}{2}\log(1+a^2)\tan^{-1}a + c \quad \ldots(3)$$

when $a = 0$, from (1), $F(0) = 0$

$\therefore$ from (3), $F(0) = 0 + c$ or $c = 0$

$\therefore$ $F(a) = \dfrac{1}{2}\log(1+a^2)\tan^{-1}a$

Hence $\displaystyle\int_0^a \frac{\log(1+ax)}{1+x^2}dx = \frac{1}{2}\log(1+a^2)\tan^{-1}a.$

Example 23. *Using the value of the integral* $\displaystyle\int_0^\infty \frac{dx}{x^2+a}$, *show that*

$$\int_0^\infty \frac{dx}{(x^2+a)^{n+1}} = \frac{\pi}{2}\cdot\frac{1.3.5\ldots\ldots(2n-1)}{2.4.6\ldots\ldots 2n}\cdot\frac{1}{a^{n+\frac{1}{2}}}.$$

Sol. *Please try yourself.*

Example 24. *Prove that* $\displaystyle\int_0^\infty \frac{1-\cos mx}{x}e^{-x}dx = \frac{1}{2}\log(1+m^2).$

Sol. Let $$F(m) = \int_0^\infty \frac{1-\cos mx}{x}e^{-x}dx \quad \ldots(i)$$

Differentiating both sides w.r.t. m, we get

$$F'(m) = \int_0^\infty \frac{\partial}{\partial m}\left(\frac{1-\cos mx}{x}\right)e^{-x}dx$$

$$= \int_0^\infty \frac{x\sin mx}{x}e^{-x}dx = \int_0^\infty e^{-x}\sin mx\,dx$$

$$= \left[\frac{e^{-x}}{1+m^2}(-1\sin mx - m\cos mx)\right]_0^\infty$$

$$\left[\because \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\right]$$

$$= \frac{m}{1+m^2}$$

Integrating both sides w.r.t. m, we get

$$F(m) = \frac{1}{2}\int\frac{2m}{1+m^2}dm + c = \frac{1}{2}\log(1+m^2) + c \quad \ldots(2)$$

When $m = 0$, from (1), $F(0) = 0$

From (2), $0 = c$

$$\therefore \quad F(m) = \frac{1}{2}\log(1+m^2)$$

Hence $\displaystyle\int_0^\infty \frac{1-\cos mx}{x} e^{-x}\, dx = \frac{1}{2}\log(1+m^2)$.

Example 25. *Prove that* $\displaystyle\int_0^\infty \frac{\tan^{-1}\alpha x \tan^{-1}\beta x}{x^2}\, dx = \frac{1}{2}\pi \log\left[\frac{(\alpha+\beta)^{\alpha+\beta}}{\alpha^\alpha \beta^\beta}\right]$, $\alpha > 0, \beta > 0$.

Sol. Let $\quad F(\alpha, \beta) = \displaystyle\int_0^\infty \frac{\tan^{-1}\alpha x \tan^{-1}\beta x}{x^2}\, dx$ ...(1)

Differentiating partially w.r.t. α, we have

$$\frac{\partial F}{\partial \alpha} = \int_0^\infty \frac{dx}{1+\alpha^2 x^2}\cdot\frac{\tan^{-1}\beta x}{x^2}\, dx \quad \text{or} \quad \frac{\partial F}{\partial \alpha} = \int_0^\infty \frac{\tan^{-1}\beta x}{x(1+\alpha^2x^2)}\, dx \quad ...(2)$$

Differentiating partially w.r.t. β, we have

$$\frac{\partial^2 F}{\partial\beta\,\partial\alpha} = \int_0^\infty \frac{x}{1+\beta^2x^2}\cdot\frac{dx}{x(1+\alpha^2x^2)} = \int_0^\infty \frac{dx}{(1+\alpha^2x^2)(1+\beta^2x^2)}$$

$$= \int_0^\infty \frac{1}{\alpha^2-\beta^2}\left[\frac{\alpha^2}{1+\alpha^2x^2} - \frac{\beta^2}{1+\beta^2x^2}\right]dx \qquad \text{(Partial Fractions)}$$

$$= \int_0^\infty \frac{1}{\alpha^2-\beta^2}\left[\frac{1}{\frac{1}{\alpha^2}+x^2} - \frac{1}{\frac{1}{\beta^2}+x^2}\right]dx$$

$$= \frac{1}{\alpha^2-\beta^2}\left[\alpha\tan^{-1}\alpha x - \beta\tan^{-1}\beta x\right]_0^\infty$$

$$= \frac{1}{\alpha^2-\beta^2}\left[\alpha\frac{\pi}{2} - \beta\frac{\pi}{2}\right] = \frac{\pi(\alpha-\beta)}{2(\alpha^2-\beta^2)} = \frac{\pi}{2(\alpha+\beta)}$$

Integrating w.r.t. β, we get $\quad \dfrac{\partial F}{\partial \alpha} = \dfrac{\pi}{2}\log(\alpha+\beta) + A$ ...(3)

When $\beta = 0$, from (1), $F(\alpha, 0) = 0$

From (2), $\quad \dfrac{\partial F}{\partial \alpha} = 0$

$\therefore$ From (3), $\quad 0 = \dfrac{\pi}{2}\log\alpha + A \quad$ or $\quad A = -\dfrac{\pi}{2}\log\alpha$

Hence (3) gives $\quad \dfrac{\partial F}{\partial \alpha} = \dfrac{\pi}{2}\log(\alpha+\beta) - \dfrac{\pi}{2}\log\alpha$

Now integrating w.r.t. α, we get

$$F(\alpha, \beta) = \frac{\pi}{2}\int \log(\alpha+\beta)\,.\,1\, d\alpha - \frac{\pi}{2}\int \log\alpha\,.\,1\, d\alpha + B$$

$$= \frac{\pi}{2}\left[\log(\alpha+\beta)\,.\,\alpha - \int\frac{1}{\alpha+\beta}\,.\,\alpha\, d\alpha\right] - \frac{\pi}{2}\left[\log\alpha\,.\,\alpha - \int\frac{1}{\alpha}\,.\,\alpha\, d\alpha\right] + B$$

$$= \frac{\pi}{2}\left[\alpha\log(\alpha+\beta) - \int\left(1-\frac{\beta}{\alpha+\beta}\right)d\alpha\right] - \frac{\pi}{2}[\alpha\log\alpha - \alpha] + B$$

$$= \frac{\pi}{2}[\alpha\log(\alpha+\beta) - \alpha + \beta\log(\alpha+\beta) - \alpha\log\alpha + \alpha] + B$$

$$= \frac{\pi}{2}[(\alpha+\beta)\log(\alpha+\beta) - \alpha\log\alpha] + B \quad ...(4)$$

When $\alpha = 0$, from (1) $F(0, \beta) = 0$

From (4), $\quad 0 = \dfrac{\pi}{2}\beta\log\beta + B \qquad \left[\because \lim\limits_{\alpha\to 0} \alpha\log\alpha = 0\right]$

or $\quad B = -\dfrac{\pi}{2}\beta\log\beta$

$\therefore$ From (4), $\quad F(\alpha,\beta) = \dfrac{\pi}{2}[(\alpha+\beta)\log(\alpha+\beta) - \alpha\log\alpha - \beta\log\beta]$

Hence $\quad \displaystyle\int_0^\infty \frac{\tan^{-1}\alpha x\tan^{-1}\beta x}{x^2}\, dx = \frac{\pi}{2}\log\left[\frac{(\alpha+\beta)^{\alpha+\beta}}{\alpha^\alpha\beta^\beta}\right]$.

Example 26. *Assuming the validity of differentiation under the integral sign, show that*

$$\int_0^\infty e^{-x^2}\cos\alpha x\, dx = \frac{\sqrt{\pi}}{2}e^{-\frac{\alpha^2}{4}}$$

Hence deduce that $\displaystyle\int_0^\infty x\, e^{-x^2}\sin\alpha x\, dx = \frac{\sqrt{\pi}}{4}\alpha\, e^{-\frac{\alpha^2}{4}}$.

Sol. Let $\quad F(\alpha) = \displaystyle\int_0^\infty e^{-x^2}\cos\alpha x\, dx$ ...(1)

Differentiating w.r.t α, we get

$$F'(\alpha) = \int_0^\infty \frac{\partial}{\partial\alpha}(e^{-x^2}\cos\alpha x)\, dx$$

$$= \int_0^\infty e^{-x^2}(-x\sin\alpha x)\, dx$$

$$= \frac{1}{2}\int_0^\infty \sin\alpha x\,(-2x\, e^{-x^2})\, dx$$

Integrating by parts $\quad = \dfrac{1}{2}\left[\left\{\sin\alpha x\,.\,e^{-x^2}\right\}_0^\infty - \displaystyle\int_0^\infty \alpha\cos\alpha x\,.\,e^{-x^2}\, dx\right]$

$$\left[\because \text{ putting } -x^2 = t \text{ so that } -2x\, dx = dt \int -2x\, e^{-x^2}\, dx = \int e^t\, dt = e^t = e^{-x^2}\right]$$

$$= \frac{1}{2}\left[0 - \alpha\int_0^\infty e^{-x^2}\cos\alpha x\, dx\right] = -\frac{\alpha}{2}\int_0^\infty e^{-x^2}\cos\alpha x\, dx = -\frac{\alpha}{2}F(\alpha)$$

$$\Rightarrow \qquad \frac{F'(\alpha)}{F(\alpha)} = -\frac{\alpha}{2}$$

Integrating w.r.t. α, we get $\log F(\alpha) = -\dfrac{\alpha^2}{4} + c$ ...(2)

When $\alpha = 0$, from (1), $F(0) = \int_0^\infty e^{-x^2}\, dx$

Putting $x^2 = z$ *i.e.*, $x = z^{1/2}$ so that $dx = \frac{1}{2} z^{-1/2}\, dz$

When $x = 0$, $z = 0$, when $x \to \infty$, $z \to \infty$

$$\int_0^\infty e^{-x^2}\, dx = \int_0^\infty e^{-z} \cdot \frac{1}{2} z^{-1/2}\, dz = \frac{1}{2}\int_0^\infty z^{-1/2} e^{-z}\, dz = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}$$

$$\therefore \qquad F(0) = \frac{1}{2}\sqrt{\pi}$$

When $\alpha = 0$, from (2), $\log F(0) = c \Rightarrow c = \log\dfrac{\sqrt{\pi}}{2}$

Putting this value of c in (2), we get

$$\log F(\alpha) = -\frac{\alpha^2}{4} + \log\frac{\sqrt{\pi}}{2}$$

$$\Rightarrow \qquad \log\left(\frac{F\alpha}{\frac{\sqrt{\pi}}{2}}\right) = -\frac{\alpha^2}{4} \Rightarrow \frac{F(\alpha)}{\frac{\sqrt{\pi}}{2}} = e^{-\frac{\alpha^2}{4}} \Rightarrow F(\alpha) = \frac{\sqrt{\pi}}{2} e^{-\frac{\alpha^2}{4}}$$

Hence $\displaystyle\int_0^\infty e^{-x^2} \cos \alpha x\, dx = \frac{\sqrt{\pi}}{2} e^{-\frac{\alpha^2}{4}}$.

Differentiating both sides w.r.t. α, we have

$$\int_0^\infty \frac{\partial}{\partial\alpha}(e^{-x^2}\cos\alpha x)\, dx = \frac{d}{d\alpha}\left(\frac{\sqrt{\pi}}{2} e^{-\frac{\alpha^2}{4}}\right)$$

$$\Rightarrow \qquad \int_0^\infty e^{-x^2}(-\sin\alpha x)\, x\, dx = \frac{\sqrt{\pi}}{2} e^{-\frac{\alpha^2}{4}} \cdot \left(-\frac{\alpha}{2}\right)$$

$$\Rightarrow \qquad \int_0^\infty x\, e^{-x^2} \sin\alpha x\, dx = \frac{\sqrt{\pi}}{4}\alpha\, e^{-\frac{\alpha^2}{4}}.$$

Example 27. *Evaluate* $\int_0^\infty e^{-ax^2} x^{2n}\, dx$ *by the method of differentiation under the integral sign.*

Sol. Let us first evaluate $\int_0^\infty e^{-ax^2}\, dx$.

Put $ax^2 = z$ *i.e.*, $x = \sqrt{\dfrac{z}{a}}$ so that $dx = \dfrac{1}{\sqrt{a}} \cdot \dfrac{1}{2} z^{-1/2}\, dz$

When $x = 0$, $z = 0$; when $x \to \infty$, $z \to \infty$

$$\therefore \qquad \int_0^\infty e^{-ax^2}\, dx = \int_0^\infty e^{-z} \cdot \frac{1}{2\sqrt{a}} z^{-1/2}\, dz$$

$$= \frac{1}{2\sqrt{a}}\int_0^\infty z^{-1/2} e^{-z}\, dz = \frac{1}{2\sqrt{a}}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2\sqrt{a}}$$

$$\Rightarrow \qquad \int_0^\infty e^{-ax^2}\, dx = \frac{\sqrt{\pi}}{2} \cdot a^{-1/2} \qquad \text{...(1)}$$

Differentiating both sides w.r.t. a, we get

$$\int_0^\infty e^{-ax^2} \cdot (-x^2)\, dx = \frac{\sqrt{\pi}}{2}\left(-\frac{1}{2}\right) a^{-3/2} \Rightarrow \int_0^\infty e^{-ax^2} \cdot x^2\, dx = \frac{\sqrt{\pi}}{2} \cdot \frac{1}{2} \cdot a^{-3/2}$$

Differentiating both sides w.r.t. a, we get

$$\int_0^\infty e^{-ax^2} \cdot (-x^2)\, x^2\, dx = \frac{\sqrt{\pi}}{2} \cdot \frac{1}{2}\left(-\frac{3}{2}\right) a^{-5/2}$$

$$\Rightarrow \qquad \int_0^\infty e^{-ax^2} \cdot (x^2)\, dx = \frac{\sqrt{\pi}}{2} \cdot \frac{1}{2}\frac{3}{2} \cdot a^{-5/2}$$

Continuing like this till (1) is differentiated n times w.r.t. a, we get

$$\int_0^\infty e^{-ax^2} \cdot (x^2)^n\, dx = \frac{\sqrt{\pi}}{2} \cdot \frac{1}{2} \cdot \frac{3}{2} \cdots\cdots \frac{2n-1}{2} \cdot a^{-\frac{1}{2}-n}$$

$$\Rightarrow \qquad \int_0^\infty e^{-ax^2} x^{2n}\, dx = \frac{\sqrt{\pi}}{2} \cdot \frac{1}{a^{n+\frac{1}{2}}} \cdot \frac{2n-1}{2} \cdot \frac{2n-3}{2} \cdots\cdots \frac{3}{2} \cdot \frac{1}{2}$$

$$= -\frac{1}{2a^{n+\frac{1}{2}}}\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\cdots\cdots\frac{3}{2} \cdot \frac{1}{2}\Gamma\left(\frac{1}{2}\right) \qquad \left[\because \sqrt{\pi} = \Gamma\left(\frac{1}{2}\right)\right]$$

$$= \frac{1}{2a^{n+\frac{1}{2}}}\Gamma\left(n+\frac{1}{2}\right) \qquad [\because (n-1)\,\Gamma(n-1) = \Gamma(n)]$$

Example 28. *Prove that* $\displaystyle\int_0^\infty \frac{e^{-x}}{x}(1-e^{-ax})\, dx = \log(1+a)$, $(a \geq 0)$.

Sol. Let $\displaystyle F(a) = \int_0^\infty \frac{e^{-x}}{x}(1-e^{-ax})\, dx$ ...(1)

Differentiating w.r.t. a, we have

$$F'(a) = \int_0^\infty \frac{\partial}{\partial a}\left[\frac{e^{-x}}{x}(1-e^{-ax})\right] dx$$

$$= \int_0^\infty \frac{e^{-x}}{x} \cdot (-e^{-ax})(-x)\, dx = \int_0^\infty e^{-(1+a)x}\, dx$$

$$\left[\frac{e^{-(1+a)x}}{-(1+a)}\right]_0^\infty = -\frac{1}{1+a}(0-1) = \frac{1}{1+a}$$

Integrating w.r.t. a, we get

$$F(a) = \log(1+a) + c \qquad \ldots(2)$$

When $a = 0$, from (1), $F(0) = 0$

Putting $a = 0$ in (2), $F(0) = \log 1 + c$ or $0 = 0 + c$

$\Rightarrow$ $c = 0$

$\therefore$ From (2), $F(a) = \log(1+a)$

or $$\int_0^\infty \frac{e^{-x}}{x}(1-e^{-ax})\,dx = \log(1+a).$$

Example 29. *Prove that* $\frac{d}{da}\left[\int_0^{a^2} \tan^{-1}\frac{x}{a}\,dx\right] = 2a\tan^{-1}a - \frac{1}{2}\log(a^2+1)$ *and verify by direct integration.*

Sol. $$\frac{d}{da}\left[\int_0^{a^2} \tan^{-1}\frac{x}{a}\,dx\right] = \int_0^{a^2} \frac{\partial}{\partial a}\left(\tan^{-1}\frac{x}{a}\right)dx + \frac{d}{da}(a^2)\cdot\tan^{-1}\frac{a^2}{a} - \frac{d}{da}(0)\cdot\tan^{-1}\frac{0}{a}$$

[$\because$ The upper limit involves the parameter a.]

$$= \int_0^{a^2} \frac{1}{1+\left(\frac{x}{a}\right)^2}\cdot\left(-\frac{x}{a^2}\right)dx + 2a\tan^{-1}a - 0$$

$$= -\int_0^{a^2} \frac{x}{a^2+x^2}\,dx + 2a\tan^{-1}a$$

$$= -\frac{1}{2}\int_0^{a^2} \frac{2x}{a^2+x^2}\,dx + 2a\tan^{-1}a$$

$$= -\frac{1}{2}\left[\log(a^2+x^2)\right]_0^{a^2} + 2a\tan^{-1}a$$

$$= -\frac{1}{2}\left[\log(a^2+a^4) - \log a^2\right] + 2a\tan^{-1}a$$

$$= 2a\tan^{-1}a - \frac{1}{2}\log\frac{a^2(1+a^2)}{a^2}$$

$$= 2a\tan^{-1}a - \frac{1}{2}\log(a^2+1) \qquad \ldots(1)$$

Now $$\int_0^{a^2} \tan^{-1}\frac{x}{a}\,dx = \int_0^{a^2} \tan^{-1}\frac{x}{a}\cdot 1\,dx$$

Integrating by parts, we have

$$= \left[\tan^{-1}\frac{x}{a}\cdot x\right]_0^{a^2} - \int_0^{a^2} \frac{1}{1+\left(\frac{x}{a}\right)^2}\cdot\frac{1}{a}\cdot x\,dx$$

$$= a^2\tan^{-1}\frac{a^2}{a} - 0 - a\int_0^{a^2} \frac{x}{a^2+x^2}\,dx$$

$$= a^2\tan^{-1}a - \frac{a}{2}\int_0^{a^2} \frac{2x}{a^2+x^2}\,dx$$

$$= a^2\tan^{-1}a - \frac{a}{2}\left[\log(a^2+x^2)\right]_0^{a^2}$$

$$= a^2\tan^{-1}a - \frac{a}{2}\left[\log(a^2+a^4) - \log a^2\right]$$

$$= a^2\tan^{-1}a - \frac{a}{2}\log\frac{a^2(1+a^2)}{a^2}$$

$$= a^2\tan^{-1}a - \frac{a}{2}\log(1+a^2)$$

$$\therefore \frac{d}{da}\left[\int_0^{a^2} \tan^{-1}\frac{x}{a}\,dx\right] = \frac{d}{da}\left[a^2\tan^{-1}a - \frac{a}{2}\log(1+a^2)\right]$$

$$= 2a\tan^{-1}a + a^2\cdot\frac{1}{1+a^2} - \frac{1}{2}\log(1+a^2) - \frac{a}{2}\cdot\frac{1}{1+a^2}\cdot 2a$$

$$= 2a\tan^{-1}a + \frac{a^2}{1+a^2} - \frac{1}{2}\log(1+a^2) - \frac{a^2}{1+a^2}$$

$$= 2a\tan^{-1}a - \frac{1}{2}\log(a^2+1)$$

which is the same as (1). Hence the verification.

Example 30. *Prove that* $\int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}} \sin\theta\cos^{-1}(\cos\alpha\,\mathrm{cosec}\,\theta)\,d\theta = \frac{\pi}{2}(1-\cos\alpha)$.

Sol. Let $$F(\alpha) = \int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}} \sin\theta\cos^{-1}(\cos\alpha\,\mathrm{cosec}\,\theta)\,d\theta \qquad \ldots(1)$$

Differentiating both sides w.r.t. α, we have

$$F'(\alpha) = \int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}} \frac{\partial}{\partial\alpha}\left[\sin\theta\cos^{-1}(\cos\alpha\,\mathrm{cosec}\,\theta)\right]d\theta + \frac{d}{d\alpha}\left(\frac{\pi}{2}\right)\cdot\sin\frac{\pi}{2}\cos^{-1}\left(\cos\alpha\,\mathrm{cosec}\,\frac{\pi}{2}\right)$$
$$- \frac{d}{d\alpha}\left(\frac{\pi}{2}-\alpha\right)\cdot\sin\left(\frac{\pi}{2}-\alpha\right)\cos^{-1}\left[\cos\alpha\,\mathrm{cosec}\left(\frac{\pi}{2}-\alpha\right)\right]$$

[$\because$ The lower limit involves the parameter α]

$$= \int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}} \sin\theta\cdot\frac{-1}{\sqrt{1-\cos^2\alpha\,\mathrm{cosec}^2\theta}}(-\sin\alpha\,\mathrm{cosec}\,\theta)\,d\theta + 0$$
$$-(-1)\cos\alpha\cos^{-1}(\cos\alpha\sec\alpha)$$

$$= \sin\alpha \int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-\frac{\cos^2\alpha}{\sin^2\theta}}} + \cos\alpha\cos^{-1}(1)$$

$$= \sin\alpha \int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}} \frac{\sin\theta\, d\theta}{\sqrt{\sin^2\theta-\cos^2\alpha}} + \cos\alpha\,.(0)$$

$$= \sin\alpha \int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}} \frac{\sin\theta\, d\theta}{\sqrt{1-\cos^2\theta-\cos^2\alpha}}$$

$$= \sin\alpha \int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}} \frac{\sin\theta\, d\theta}{\sqrt{\sin^2\alpha-\cos^2\theta}}$$

Put $\cos\theta = t$ so that $-\sin\theta\, d\theta = dt$ or ~~$\sin\theta\, d\theta = -dt$~~

when $\theta = \frac{\pi}{2}-\alpha$, $t = \sin\alpha$; when $\theta = \frac{\pi}{2}$, $t = \cos\frac{\pi}{2} = 0$

$\therefore$
$$F'(\alpha) = \sin\alpha \int_{\sin\alpha}^{0} \frac{-dt}{\sqrt{\sin^2\alpha-t^2}} = \sin\alpha \int_0^{\sin\alpha} \frac{dt}{\sqrt{\sin^2\alpha-t^2}}$$

$$= \sin\alpha\left[\sin^{-1}\frac{t}{\sin\alpha}\right]_0^{\sin\alpha} = \sin\alpha\,[\sin^{-1}1-\sin^{-1}0] = \frac{\pi}{2}\sin\alpha$$

Integrating w.r.t. α, we get

$$F(\alpha) = -\frac{\pi}{2}\cos\alpha + c \qquad \text{...(2)}$$

When $\alpha = 0$, from (1), $F(0) = \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\theta\cos^{-1}(\operatorname{cosec}\theta)\, d\theta = 0$ $\quad \left[\because \int_a^a f(x)\, dx = 0\right]$

Putting $\alpha = 0$ in (2), $F(0) = -\frac{\pi}{2}+c$ or $0 = -\frac{\pi}{2}+c$

$\Rightarrow \qquad c = \frac{\pi}{2}$

$\therefore$ From (2), $\qquad F(\alpha) = -\frac{\pi}{2}\cos\alpha + \frac{\pi}{2} = \frac{\pi}{2}(1-\cos\alpha)$

or
$$\int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}} \sin\theta\cos^{-1}(\cos\alpha\operatorname{cosec}\theta)\, d\theta = \frac{\pi}{2}(1-\cos\alpha)$$

Example 31. *If $y = \int_0^x f(t)\sin[k(x-t)]\, dt$, prove that y satisfies the differential equation*

$$\frac{d^2y}{dx^2} + k^2y = kf(x).$$

Sol. Given $\qquad y = \int_0^x f(t)\sin[k(x-t)]\, dt \qquad$ ...(1)

[Here the upper limit involves the parameter x.]

Differentiating w.r.t. x, we have

$$\frac{dy}{dx} = \int_0^x \frac{\partial}{\partial x}[f(t)\sin(k(x-t))]\, dt + \frac{d}{dx}(x)\,.\,f(x)\sin[k(x-x)] - \frac{d}{dx}(0)\,.\,f(0)\sin[k(x-0)]$$

$$= \int_0^x f(t)\cos(k(x-t))\,.\,k\, dt + 0 - 0$$

$$= \int_0^x kf(t)\cos(k(x-t)\, dt$$

Differentiating again w.r.t. x, we have

$$\frac{d^2y}{dx^2} = \int_0^x \frac{\partial}{\partial x}[kf(t)\cos(k(x-t)]\, dt + \frac{d}{dx}(x)\,.\,kf(x)\cos[k(x-x)] - \frac{d}{dx}(0)\,.\,kf(0)\cos[k(x-0)]$$

$$= \int_0^x kf(t)\,[-\sin k(x-t)]\,.\,k\, dt + kf(x)$$

$$= -k^2\int_0^x f(t)\sin[k(x-t)]\, dt + kf(x)$$

$$= -k^2y + kf(x) \qquad \text{[by (1)]}$$

$$\Rightarrow \qquad \frac{d^2y}{dx^2} + k^2y = kf(x).$$

14

Indeterminate Forms

14.1. If a function is such that for a certain assigned value of the variable involved, its value cannot be found by simply substituting that value of the variable, the function is said to take an indeterminate form.

The forms $\frac{0}{0}, \frac{\infty}{\infty}, 0 \times \infty, \infty - \infty, 0^0, 1^\infty, \infty^0$ are called indeterminate forms. The limiting value of an indeterminate form is called its **true value.**

We know that, in general, $\underset{x \to a}{\text{Lt}} \frac{f(x)}{g(x)} = \frac{\underset{x \to a}{\text{Lt}} f(x)}{\underset{x \to a}{\text{Lt}} g(x)}$

But, if $\underset{x \to a}{\text{Lt}} f(x) = 0 = \underset{x \to a}{\text{Lt}} g(x)$, then $\underset{x \to a}{\text{Lt}} \frac{f(x)}{g(x)}$ reduces to the indeterminate form $\frac{0}{0}$. This does not mean that $\underset{x \to a}{\text{Lt}} \frac{f(x)}{g(x)}$ does not exist. The only conclusion is that the method adopted is not suitable.

For example, if $f(x) = x^2 - 1$ and $g(x) = x - 1$ with $a = 1$

We have $\underset{x \to a}{\text{Lt}} \frac{f(x)}{g(x)} = \underset{x \to 1}{\text{Lt}} \frac{x^2 - 1}{x - 1}$ $\quad \left| \frac{0}{0} \text{ form} \right.$

$= \underset{x \to 1}{\text{Lt}} (x + 1) = 2$

The forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$ are regarded as fundamental forms and all other forms are dealt with by converting them to one of these two forms. The methods of effecting such conversions will be explained at their proper places.

14.2. L'HOSPITAL RULE

State and prove L'Hospital rule for determining the **true** value of the indeterminate form 0/0

Statement. *If $f(x)$ and $\phi(x)$ be two functions such that*

(i) they are continuous in the neighbourhood of a.

(ii) they are derivable in the deleted neighbourhood of a,

(iii) $f(a) = 0 = \phi(a)$, i.e., $\underset{x \to a}{Lt} f(x) = 0 = \underset{x \to a}{Lt} \phi(x)$,

and (iv) $\phi'(x) \neq 0$ in the deleted neighbourhood of a, then $\underset{x \to a}{Lt} \frac{f(x)}{\phi(x)} = \underset{x \to a}{Lt} \frac{f'(x)}{\phi'(x)} = \frac{f'(a)}{\phi'(a)}$

provided the latter limit exists, whether finite (zero or non-zero) or infinite.

Proof. Let $x = a + h$ be any point in the neighbourhood of $x = a$ whether h is positive or negative.

Now $\frac{f(x)}{\phi(x)} = \frac{f(a+h)}{\phi(a+h)} = \frac{f(a) + hf'(a+\theta_1 h)}{\phi(a) + h\phi'(a+\theta_2 h)}$ where $0 < \theta_1 < 1, 0 < \theta_2 < 1$

[on using Lagrange's mean-value theorem]

$\because$ $f(a) = 0 = \phi(a)$ given

$\therefore$ $\frac{f(x)}{\phi(x)} = \frac{hf'(a+\theta_1 h)}{h\phi'(a+\theta_2 h)} = \frac{f'(a+\theta_1 h)}{\phi'(a+\theta_2 h)}$

$\therefore$ $\underset{x \to a}{\text{Lt}} \frac{f(x)}{\phi(x)} = \underset{h \to 0}{\text{Lt}} \frac{f'(a+\theta_1 h)}{\phi'(a+\theta_2 h)}$ $\quad [\because x = a + h, \ \therefore \ x \to a, h \to 0]$

$= \frac{f'(a)}{\phi'(a)} = \underset{x \to 0}{\text{Lt}} \frac{f'(x)}{\phi'(x)}.$

This proves the result.

Note. Generalisation of L'Hospital Rule

If $f(x)$ and $\phi(x)$ vanish at $x = a$ and have their first $(n-1)$ derivatives all zero at $x = a$ while their nth derivatives are finite and non-zero, then

$\underset{x \to a}{\text{Lt}} \frac{f(x)}{\phi(x)} = \underset{x \to a}{\text{Lt}} \frac{f^n(x)}{\phi^n(x)} = \frac{f^n(a)}{\phi^n(a)}$, provided this limit exists finitely or infinitely.

Working Rule for finding the value of $\underset{x \to a}{\textbf{Lt}} \frac{f(x)}{\phi x}$ where $f(a) = 0 = \phi(a)$

1. Differentiate the numerator and denominator **separately.**
2. Put $x = a$ and remove the word limit.
3. If the indeterminate form $\frac{0}{0}$ still persists, repeat the above process.

Caution. *It should be carefully noted that $\frac{f(x)}{\phi(x)}$ should not be differentiated as a fraction. The numerator and denominator should be differentiated* **separately.**

Example 1. *Evaluate the following limits :*

(a) $\underset{x \to 0}{Lt} \frac{(1+x)^n - 1}{x}$ (b) $\underset{x \to 0}{Lt} \frac{xe^x - \log(1+x)}{x^2}$

(c) $\underset{x \to 0}{Lt} \frac{x - \tan x}{x^3}$ (d) $\underset{x \to 0}{Lt} \frac{x - \sin x}{x^3}$

(e) $\underset{x \to 0}{Lt} \frac{\log(1-x^2)}{\log \cos x}$ (f) $\underset{x \to 1}{Lt} \frac{x^x - x}{1 - x + \log x}$

Sol. (*a*) $\displaystyle \lim_{x\to 0} \frac{(1+x)^n - 1}{x}$ $\left[\text{form } \frac{0}{0}\right]$

[*Differentiate num. and denom separately*]

$$= \lim_{x\to 0} \frac{n(1+x)^{n-1}}{1} = n.$$

(*b*) $\displaystyle \lim_{x\to 0} \frac{xe^x - \log(1+x)}{x^2}$

When we put $x = 0$, the given expression takes the form $\frac{0}{0}$.

So, we differentiate the num. and denom. separately.

$$\therefore \quad \lim_{x\to 0} \frac{xe^x - \log(1+x)}{x^2} \qquad \left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{xe^x + e^x \cdot 1 - \frac{1}{1+x}}{2x} \qquad \left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{x \cdot e^x + e^x \cdot 1 + e^x + \frac{1}{(1+x)^2}}{2} = \frac{0+1+1+1}{2} = \frac{3}{2}.$$

(*c*) $\displaystyle \lim_{x\to 0} \frac{x - \tan x}{x^2}$ $\left[\text{form } \frac{0}{0}\right]$

$$= \lim_{x\to 0} \frac{1 - \sec^2 x}{3x^2} \qquad \left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{-2\sec x \cdot \sec x \tan x}{6x} = \lim_{x\to 0} \frac{-\sec^2 x \tan x}{3x} \qquad \left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{x\to 0} \left(-\frac{1}{3}\sec^2 x\right) \times \lim_{x\to 0} \frac{\tan x}{x} = -\frac{1}{3} \times 1 = -\frac{1}{3}.$$

(*d*) *Please try yourself.* **[Ans.** $\frac{1}{6}$**]**

(*e*) $\displaystyle \lim_{x\to 0} \frac{\log(1-x^2)}{\log \cos x}$ $\left[\text{form } \frac{0}{0}\right]$

$$= \lim_{x\to 0} \frac{\frac{-2x}{1-x^2}}{\frac{1}{\cos x} \cdot (-\sin x)} = \lim_{x\to 0} \frac{2x \cos x}{(1-x^2)\sin x} \qquad \left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{2[-x \sin x + \cos x \cdot 1]}{(1-x^2)\cos x - 2x \sin x} = \frac{2}{1} = 2.$$

(*f*) $\displaystyle \lim_{x\to 1} \frac{x^x - x}{1 - x + \log x}$ $\left[\text{form } \frac{0}{0}\right]$

$$= \lim_{x\to 1} \frac{x^x(1+\log x) - 1}{-1 + \frac{1}{x}} \qquad \left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{x\to 1} \frac{x^x\left(\frac{1}{x}\right) + (1+\log x) \cdot x^x(1+\log x)}{-\frac{1}{x^2}}$$

$= -2.$

$\because$ if $u = x^x$ then $\log u = x \log x$

Differentiating w.r.t. x,

$$\frac{1}{u}\frac{du}{dx} = x \cdot \frac{1}{x} + \log x \cdot 1$$

$$\therefore \quad \frac{du}{dx} = u(1+\log x) = x^x(1+\log x)$$

Example 2. *Evaluate the following limits :*

(*a*) $\displaystyle \lim_{x\to 0} \frac{e^x - 2\cos x + e^{-x}}{x \sin x}$ (*b*) $\displaystyle \lim_{x\to 0} \frac{\tan x - x}{x^2 \tan x}$ (*c*) $\displaystyle \lim_{x\to a} \frac{x^a - a^x}{x^x - a^a}$

(*d*) $\displaystyle \lim_{x\to 0} \frac{\cosh x - \cos x}{x \sin x}$ (*e*) $\displaystyle \lim_{x\to 0} \frac{x \cos x - \log(1+x)}{x^2}$ (*f*) $\displaystyle \lim_{x\to 0} \frac{e^x \sin x - x - x^2}{x^2 + x \log(1-x)}$

Sol. (*a*) $\displaystyle \lim_{x\to 0} \frac{e^x - 2\cos x + e^{-x}}{x \sin x}$ $\left[\text{form } \frac{0}{0}\right]$

[*Differentiate num. and denom. separately*]

$$= \lim_{x\to 0} \frac{e^x + 2\sin x - e^{-x}}{x \cos x + \sin x} \qquad \left[\text{form } \frac{0}{0}\right]$$

[*Differentiate the num. and denom. separately again*]

$$= \lim_{x\to 0} \frac{e^x + 2\cos x + e^{-x}}{-x \sin x + \cos x + \cos x} = \frac{1+2+1}{0+1+1} = \frac{4}{2} = 2.$$

(*b*) $\displaystyle \lim_{x\to 0} \frac{\tan x - x}{x^2 \tan x}$ $\left|\ \text{form } \frac{0}{0}\right.$

$$= \lim_{x\to 0} \frac{\tan x - x}{x^3} \cdot \frac{x}{\tan x} \qquad \text{| Note this step}$$

$$= \lim_{x\to 0} \frac{\tan x - x}{x^3} \cdot \lim_{x\to 0} \frac{x}{\tan x} = \lim_{x\to 0} \frac{\tan x - x}{x^3} \qquad \left|\ \because \lim_{x\to 0} \frac{x}{\tan x} = 1\right.$$

$$= \lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2} \qquad \left|\ \text{form } \frac{0}{0}\right.$$

$$= \lim_{x\to 0} \frac{2\sec x \cdot \sec x \tan x}{6x} = \lim_{x\to 0} \frac{1}{3}\sec^2 x \cdot \frac{\tan x}{x} = \frac{1}{3} \times 1 \times 1 = \frac{1}{3}.$$

(*c*) $\displaystyle \lim_{x\to a} \frac{x^a - a^x}{x^x - a^a}$ $\left|\ \text{form } \frac{0}{0}\right.$

$$= \underset{x \to 0}{\text{Lt}} \frac{ax^{a-1} - a^x \log a}{x^x(1+\log x) - 0}$$

$\because$ if $y = x^x$

$\log y = \log x^x = x \log x$

$\therefore \frac{1}{y} \cdot \frac{dy}{dx} = x \cdot \frac{1}{x} + \log x \cdot 1 = 1 + \log x$

or $\frac{dy}{dx} = y(1+\log x) = x^x(1+\log x)$

$$= \frac{a \cdot a^{a-1} - a^a \log a}{a^a(1+\log a)}$$

$$= \frac{a^a - a^a \log a}{a^a(1+\log a)} = \frac{a^a(1-\log a)}{a^a(1+\log a)}$$

$$= \frac{1-\log a}{1+\log a}$$

(*d*) *Please try yourself.* **[Ans. 1]**

(*e*) *Please try yourself.* $\left[\textbf{Ans.} \frac{1}{2}\right]$

(*f*) $$\underset{x \to 0}{\text{Lt}} \frac{e^x \sin x - x - x^2}{x^2 + x \log(1-x)} \qquad \left| \text{form} \frac{-0}{0} \right.$$

$$= \underset{x \to 0}{\text{Lt}} \frac{e^x \cos x + e^x \sin x - 1 - 2x}{2x + x\left(\frac{-1}{1-x}\right) + \log(1-x)} \qquad \left| \text{form} \frac{0}{0} \right.$$

$$= \underset{x \to 0}{\text{Lt}} \frac{-e^x \sin x + e^x \cos x + e^x \cos x + e^x \sin x - 2}{2 - \frac{(1-x) \cdot 1 - x(-1)}{(1-x)^2} - \frac{1}{1-x}}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{2e^x \cos x - 2}{2 - \frac{1}{(1-x)^2} - \frac{1}{1-x}} \qquad \left| \text{form} \frac{0}{0} \right.$$

$$= \underset{x \to 0}{\text{Lt}} \frac{-2e^x \sin x + 2e^x \cos x}{\frac{-2}{(1-x)^3} - \frac{1}{(1-x)^2}} = \frac{0+2}{-2-1} = -\frac{2}{3}.$$

Example 3. *Evaluate the following limits :*

(*a*) $\underset{x \to 0}{Lt} \frac{e^{ax} - e^{-ax}}{\log(1+bx)}$ (*b*) $\underset{x \to 0}{Lt} \frac{\sin^2 x - x^2}{x^2 \sin^2 x}$

(*c*) $\underset{x \to 0}{Lt} \frac{(1+x)^{1/x} - e}{x}$ (*d*) $\underset{x \to 0}{Lt} \frac{a^x - 1}{b^x - 1}, b \neq 0, a > 0, b > 0.$

Sol. (*a*) $$\underset{x \to 0}{\text{Lt}} \frac{e^{ax} - e^{-ax}}{\log(1+bx)} \qquad \left| \text{form} \frac{0}{0} \right.$$

$$= \underset{x \to 0}{\text{Lt}} \frac{ae^{ax} + ae^{-ax}}{\frac{1}{1+bx} \cdot b} = \frac{a+a}{b} = \frac{2a}{b}$$

(*b*) $$\underset{x \to 0}{\text{Lt}} \frac{\sin^2 x - x^2}{x^2 \sin^2 x} \qquad \left| \text{form} \frac{0}{0} \right.$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\sin^2 x - x^2}{x^4} \cdot \left(\frac{x}{\sin x}\right)^2 \qquad | \textbf{ Note carefully}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\sin^2 x - x^2}{x^4} \cdot \underset{x \to 0}{\text{Lt}} \left(\frac{x}{\sin x}\right)^2$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\sin^2 x - x^2}{x^4} \qquad \left[\because \underset{x \to 0}{\text{Lt}} \frac{x}{\sin x} = 1\right]$$

$$= \underset{x \to 0}{\text{Lt}} \frac{2 \sin x \cos x - 2x}{4x^3} = \underset{x \to 0}{\text{Lt}} \frac{\sin 2x - 2x}{4x^3} \qquad \left| \text{form} \frac{0}{0} \right.$$

$$= \underset{x \to 0}{\text{Lt}} \frac{2 \cos 2x - 2}{12x^2} \qquad \left| \text{form} \frac{0}{0} \right.$$

$$= \underset{x \to 0}{\text{Lt}} \frac{-4 \sin 2x}{24x} \qquad \left| \text{form} \frac{0}{0} \right.$$

$$= \underset{x \to 0}{\text{Lt}} \frac{-8 \cos 2x}{24} = \frac{-8}{24} = -\frac{1}{3}.$$

(*c*) $$\underset{x \to 0}{\text{Lt}} \frac{(1+x)^{1/x} - e}{x} \qquad \left[\text{from} \frac{0}{0} \because \underset{x \to 0}{\text{Lt}} (1+x)^{1/x} = e\right]$$

$$= \underset{x \to 0}{\text{Lt}} (1+x)^{1/x} \cdot \left[\frac{x - (1+x)\log(1+x)}{x^2(1+x)}\right] \Big/ 1$$

$$\left[\because \text{ if } u = (1+x)^{1/x}, \text{ then } \log u = \frac{1}{x} \log(1+x)\right.$$

$$\frac{1}{u} \cdot \frac{du}{dx} = -\frac{1}{x^2} \log(1+x) + \frac{1}{x(1+x)}$$

$$\left.\frac{du}{dx} = u\left\{-\frac{\log(1+x)}{x^2} + \frac{1}{x(1+x)}\right\} = (1+x)^{1/x} \cdot \frac{-(1+x)\log(1+x) + x}{x^2(1+x)}\right]$$

$$= \underset{x \to 0}{\text{Lt}} (1+x)^{1/x} \cdot \underset{x \to 0}{\text{Lt}} \frac{x - (1+x)\log(1+x)}{x^2(1+x)}$$

$$= e \underset{x \to 0}{\text{Lt}} \frac{x - (1+x)\log(1+x)}{x^2(1+x)} \qquad \left| \text{form} \frac{0}{0} \right.$$

$$= e \underset{x \to 0}{\text{Lt}} \frac{1 - 1 \cdot \log(1-x) - (1-x) \cdot \frac{1}{1+x}}{2x + 3x^2}$$

$$= e \underset{x \to 0}{\text{Lt}} \frac{-\log(1+x)}{2x + 3x^2} \qquad \left| \text{form} \frac{0}{0} \right.$$

$$= e \underset{x\to 0}{\text{Lt}} \frac{-\frac{1}{1+x}}{2+6x} = e\left(-\frac{1}{2}\right) = -\frac{e}{2}.$$

(d) *Please try yourself.* **[Ans. $\log_b a$]**

Example 4. (a) *What is wrong with the following application of L'Hospital's rule :*

$$\underset{x\to 1}{Lt} \frac{x^3+3x-4}{2x^2+x-3} = \underset{x\to 1}{Lt} \frac{3x^2+3}{4x+1} = \underset{x\to 1}{Lt} \frac{6x}{4} = \frac{3}{2}.$$

(b) *What is wrong with the following use of L'Hospital's rule :*

$$\underset{x\to 1}{Lt} \frac{x^4-4x^3+3}{3x^2-x-2} = \underset{x\to 1}{Lt} \frac{4x^3-12x^2}{6x-1} = \underset{x\to 1}{Lt} \frac{12x^2-24x}{6} = -2.$$

Sol. (a) $\underset{x\to 1}{\text{Lt}} \frac{x^3+3x-4}{2x^2+x-3}$ $\left| \text{form } \frac{0}{0} \right.$

$$= \underset{x\to 1}{\text{Lt}} \frac{3x^2+3}{4x+1}$$

Now the expression $\frac{3x^2+3}{4x+1}$ is not of the form $\frac{0}{0}$ as $x \to 1$. Therefore it is not correct to apply L'Hospital's rule to evaluate $\underset{x\to 1}{\text{Lt}} \frac{3x^2+3}{4x+1}$.

In fact, $\underset{x\to 1}{\text{Lt}} \frac{3x^2+3}{4x+1} = \frac{3+3}{4+1} = \frac{6}{5}$.

(b) *Please try yourself.* **[Ans.** Limit $= -\frac{8}{5}$**]**

Example 5. (a) *For what value of a does* $\frac{\sin 2x + a \sin x}{x^3}$ *tend to a finite limit l as* $x \to 0$? *When a has this value, what is the value of l ?*

(b) *Find the values of a and b in order that* $\underset{x\to 0}{Lt} \frac{x(1-a\cos x)+b\sin x}{x^3}$ *may be equal to* $\frac{1}{3}$.

(c) *Find the values of p and q for which* $\underset{x\to 0}{Lt} \frac{x(1+p\cos x)-q\sin x}{x^3}$ *exists and equals 1.*

Sol. (a) $\underset{x\to 0}{\text{Lt}} \frac{\sin 2x + a \sin x}{x^3}$ $\left| \text{form } \frac{0}{0} \right.$

$$= \underset{x\to 0}{\text{Lt}} \frac{2\cos 2x + a\cos x}{3x^2} \quad \text{...(1)}$$

The denominator of (1) $\to 0$ as $x \to 0$ but (1) $\to$ a finite limit l. (given)

$\therefore$ The numerator $(2\cos 2x + a\cos x)$ must $\to 0$ as $x \to 0$

$\therefore$ $2 + a = 0$ or $a = -2$

With this value of a,

(1) $= \underset{x\to 0}{\text{Lt}} \frac{2\cos 2x - 2\cos x}{3x^2}$ $\left| \text{form } \frac{0}{0} \right.$

$= \underset{x\to 0}{\text{Lt}} \frac{-4\sin 2x + 2\sin x}{6x}$ $\left| \text{form } \frac{0}{0} \right.$

$= \underset{x\to 0}{\text{Lt}} \frac{-8\cos 2x + 2\cos x}{6} = \frac{-8+2}{6} = -1$

$\therefore$ $l = -1$.

(b) $\underset{x\to 0}{\text{Lt}} \frac{x(1-a\cos x)+b\sin x}{x^3}$ $\left| \text{form } \frac{0}{0} \right.$

$= \underset{x\to 0}{\text{Lt}} \frac{1.(1-a\cos x)+x(a\sin x)+b\cos x}{3x^2}$ ...(1)

The denominator of (1) $\to 0$ as $x \to 0$ but (1) $\to$ a finite limit $\frac{1}{3}$. (given)

$\therefore$ The numerator $(1 - a\cos x + ax\sin x + b\cos x)$ must $\to 0$ as $x \to 0$.

$\therefore$ $1 - a + b = 0$ ...(2)

Also, if the relation (2) holds, then

(1) $= \underset{x\to 0}{\text{Lt}} \frac{1-a\cos x+ax\sin x+b\cos x}{3x^2}$ is of the form $\frac{0}{0}$

$= \underset{x\to 0}{\text{Lt}} \frac{a\sin x + a\sin x + ax\cos x - b\sin x}{6x}$ $\left| \text{form } \frac{0}{0} \right.$

$= \underset{x\to 0}{\text{Lt}} \frac{a\cos x + a\cos x + a\cos x - ax\sin x - b\cos x}{6}$

$= \frac{3a-b}{6} = \frac{1}{3}$ (given)

$\therefore$ $3a - b = 2$ ...(3)

From (2) and (3), $a = \frac{1}{2}, b = -\frac{1}{2}$.

(c) *Please try yourself.* **[Ans.** $p = -\frac{5}{2}, q = -\frac{3}{2}$**]**

14.3. TRUE VALUE OF THE INDETERMINATE FORM $\frac{\infty}{\infty}$

If f(x) and $\phi(x)$ be two functions such that

$$\underset{x\to a}{Lt} f(a) = \infty \text{ and } \underset{x\to a}{Lt} \phi(x) = \infty, \text{ then } \underset{x\to a}{Lt} \frac{f(x)}{\phi(x)} = \underset{x\to a}{Lt} \frac{f'(x)}{\phi'(x)}.$$

Proof. $\underset{x\to a}{\text{Lt}} \frac{f(x)}{\phi(x)} = \underset{x\to a}{\text{Lt}} \left[\frac{1}{\phi(x)}\right] \Big/ \left[\frac{1}{f(x)}\right]$ $\left| \text{form } \frac{0}{0} \right.$

$= \underset{x\to a}{\text{Lt}} -\frac{\phi'(x)}{[\phi(x)]^2} \Big/ -\frac{f'(x)}{[f(x)]^2}$ / L'Hospital Rule

$$= \underset{x \to a}{\text{Lt}} \frac{\phi'(x)}{f'(x)} \cdot \left[\underset{x \to a}{\text{Lt}} \frac{f(x)}{\phi(x)}\right]^2 \quad ...(1)$$

Three cases arise.

Case I. When $\underset{x \to a}{\text{Lt}} \frac{f(x)}{\phi(x)}$ is neither zero not infinity.

Then from (1), $1 = \underset{x \to a}{\text{Lt}} \frac{f(x)}{\phi(x)} \cdot \underset{x \to a}{\text{Lt}} \frac{\phi'(x)}{f'(x)}$

$$\therefore \quad \underset{x \to a}{\text{Lt}} \frac{f(x)}{\phi(x)} = \underset{x \to a}{\text{Lt}} \frac{f'(x)}{\phi'(x)}.$$

Case II When $\underset{x \to a}{\text{Lt}} \frac{f(x)}{\phi(x)} = 0$.

Now $\quad 1 + \underset{x \to a}{\text{Lt}} \frac{f(x)}{\phi(x)} = \underset{x \to a}{\text{Lt}} \frac{\phi(x)+f(x)}{\phi(x)} = \underset{x \to a}{\text{Lt}} \frac{\phi'(x)+f'(x)}{\phi'(x)}$

| *Using case I above,* $\because$ *the limit is now neither zero nor infinite.*

Subtracting 1 from each side, we have

$$\underset{x \to a}{\text{Lt}} \frac{f(x)}{\phi(x)} = \underset{x \to a}{\text{Lt}} \frac{f'(x)}{\phi'(x)}.$$

Case III. When $\underset{x \to a}{\text{Lt}} \frac{f(x)}{\phi(x)}$ infinite.

$$\underset{x \to a}{\text{Lt}} \frac{\phi(x)}{f(x)} = 0 = \underset{x \to a}{\text{Lt}} \frac{\phi'(x)}{f'(x)} \quad \text{| Using case II}$$

$$\therefore \quad \underset{x \to a}{\text{Lt}} \frac{f(x)}{\phi(x)} = \underset{x \to a}{\text{Lt}} \frac{f'(x)}{\phi'(x)} \quad \text{| On taking reciprocals}$$

$\therefore$ The result is true in all the cases.

Note 1. The above result is also true when $x \to \infty$.

• **Note 2.** In most of the problems of the form $\frac{\infty}{\infty}$, it is necessary to change it into the form $\frac{0}{0}$ at the proper stage, otherwise the process will never end. **[Remember]**

Example 6. *Evaluate the following limits :*

(*a*) $\underset{x \to 0}{Lt} \frac{\log x^2}{\cot x^2}$ (*b*) $\underset{x \to \pi/2}{Lt} \frac{\log(\theta - \pi/2)}{\tan \theta}$

(*c*) $\underset{x \to 0}{Lt} \frac{\text{cosec}\, x}{\log x}$ (*d*) $\underset{x \to \pi/2}{Lt} \frac{\tan 5x}{\tan x}$ (*e*) $\underset{x \to 0+}{Lt} \frac{\log \tan x}{\log x}$.

Sol. (*a*) $\underset{x \to 0}{\text{Lt}} \frac{\log x^2}{\cot x^2} = \underset{x \to 0}{\text{Lt}} \frac{2 \log x}{\cot x^2}$ | form $\frac{\infty}{\infty}$ $\because$ $\log 0 = -\infty, \cot 0 = \infty$

[Differentiate the num. and denom. separately]

$$= \underset{x \to 0}{\text{Lt}} \frac{2 \cdot \frac{1}{x}}{-\text{cosec}^2 x^2 \cdot 2x} = \underset{x \to 0}{\text{Lt}} -\frac{1}{x^2\, \text{cosec}^2 x^2}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{-\sin^2 x^2}{x^2} \quad \text{| form } \frac{0}{0} \text{ [Note this step]}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{-2 \sin x^2 \cdot \cos x^2 \cdot (2x)}{2x} = \underset{x \to 0}{\text{Lt}} -\sin(2x^2) = 0.$$

(*b*) $\underset{\theta \to \pi/2}{\text{Lt}} \frac{\log(\theta - \pi/2)}{\tan \theta}$ | form $\frac{\infty}{\infty}$ $\because$ $\log 0 = -\infty, \tan \frac{\pi}{2} = \infty$

$$= \underset{\theta \to \pi/2}{\text{Lt}} \frac{\frac{1}{\theta - \pi/2}}{\sec^2 \theta} \quad \text{| form } \frac{0}{0}$$

$$= \underset{\theta \to \pi/2}{\text{Lt}} \frac{\cos^2 \theta}{\theta - \pi/2} \quad \text{| form } \frac{0}{0} \text{ [Note this step]}$$

$$= \underset{\theta \to \pi/2}{\text{Lt}} \frac{-2\cos\theta \sin\theta}{1} = \underset{\theta \to \pi/2}{\text{Lt}} -\sin 2\theta = -\sin \pi = 0$$

(*c*) $\underset{x \to 0}{\text{Lt}} \frac{\text{cosec}\, x}{\log x}$ | form $\frac{\infty}{\infty}$ $\because$ $\text{cosec}\, 0 = -\infty, \log 0 = -\infty$

$$= \underset{x \to 0}{\text{Lt}} \frac{-\text{cosec}\, x \cot x}{\frac{1}{x}} \quad \text{| form } \frac{\infty}{\infty}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{-x \cos x}{\sin^2 x} \quad \text{| form } \frac{0}{0} \text{ [Note this step]}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{-\cos x + x \sin x}{2 \sin x \cos x} = \frac{-1}{0} = -\infty.$$

(*d*) *Please try yourself.* [Ans. $\frac{1}{5}$]

(*e*) $\underset{x \to 0+}{\text{Lt}} \frac{\log \tan x}{\log x}$ | form $\frac{\infty}{\infty}$

$$= \underset{x \to 0+}{\text{Lt}} \frac{\frac{1}{\tan x} \cdot \sec^2 x}{\frac{1}{x}} = \underset{x \to 0+}{\text{Lt}} \frac{x}{\sin x \cos x}$$

$$= \underset{x \to 0+}{\text{Lt}} \frac{2x}{\sin 2x} \quad \text{| form } \frac{\infty}{\infty}$$

$$= \underset{x \to 0+}{\text{Lt}} \frac{2}{2\cos 2x} = \frac{2}{2} = 1.$$

14.4. OTHER INDETERMINATE FORMS

The indeterminate forms $\infty - \infty$, $0 \times \infty$, 0^0, ∞^0 and 1^∞ can be reduced to any one of the two indeterminate forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by a suitable simplification and hence can be evaluated by L'Hospital's rule. This is illustrated by the following examples :

[Form $0 \times \infty$]

Example 1. *Evaluate the following limits :*

(a) $\underset{x \to 1}{Lt}\ (1-x) \cdot \tan \frac{\pi x}{2}$ (b) $\underset{x \to 0}{Lt}\ \sin x \cdot \log x$ (c) $\underset{x \to 0}{Lt}\ x \log x$

(d) $\underset{x \to 0}{Lt}\ x^m (\log x)^n$, *where m and n are + ve integers*

(e) $\underset{x \to \infty}{Lt}\ x^n e^{-x}$, *where n is a + ve integer.*

Sol. (a) $\underset{x \to 1}{\text{Lt}}\ (1-x) \cdot \tan \frac{\pi x}{2}$ | form $0 \times \infty$

$$= \underset{x \to 1}{\text{Lt}} \frac{1-x}{\cot \frac{\pi x}{2}} \qquad \left| \text{form } \frac{\infty}{\infty} \quad \because \cot \frac{\pi}{2} = 0 \right.$$

$$= \underset{x \to 1}{\text{Lt}} \frac{-1}{-\operatorname{cosec}^2 \frac{\pi x}{2} \cdot \frac{\pi}{2}} = \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi}.$$

(b) $\underset{x \to 0}{\text{Lt}}\ \sin x \log x$ | form $0 \times \infty$

$$= \underset{x \to 0}{\text{Lt}} \frac{\log x}{\operatorname{cosec} x}$$

| form $\frac{\infty}{\infty}$ **[Note this step,** *when one of the two factors in the form $0 \times \infty$ is* $\log x$, **do not take** *$\log x$ in the denominator.*]

$$= \underset{x \to 0}{\text{Lt}} \frac{\frac{1}{x}}{-\operatorname{cosec} x \cot x} \qquad \left| \text{form } \frac{\infty}{\infty} \right.$$

$$= \underset{x \to 0}{\text{Lt}} \frac{-\sin^2 x}{x \cos x} \qquad \left| \text{form } \frac{\infty}{\infty} \right. \textbf{[Note this step]}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{-2 \sin x \cos x}{\cos x \cdot 1 - x \sin x} = \frac{0}{1-0} = 0.$$

(c) *Please try yourself* as in (b) above. **[Ans. 0]**

(d) $\underset{x \to 0}{\text{Lt}}\ x^m (\log x)^n$ | form $0 \times \infty$

$$= \underset{x \to 0}{\text{Lt}} \frac{(\log x)^n}{x^{-m}} \qquad \left| \text{form } \frac{\infty}{\infty} \right.$$

$$= \underset{x \to 0}{\text{Lt}} \frac{n(\log x)^{n-1} \cdot \frac{1}{x}}{-m \cdot x^{-m-1}} = \underset{x \to 0}{\text{Lt}} \frac{n(\log x)^{n-1}}{(-m)\, x^{-m}} \qquad \left| \text{form } \frac{\infty}{\infty} \right.$$

$$= \underset{x \to 0}{\text{Lt}} \frac{n(n-1)(\log x)^{n-2} \cdot \frac{1}{x}}{(-m)^2\, x^{-m-1}} = \underset{x \to 0}{\text{Lt}} \frac{n(n-1) \cdot (\log x)^{n-2}}{(-m)^2 \cdot x^{-m}}$$

$$= \ldots\ldots = \underset{x \to 0}{\text{Lt}} \frac{n(n-1) \ldots 3.2.1}{(-m)^n \cdot x^{-m}} = \underset{x \to 0}{\text{Lt}} \frac{n!\, x^m}{(-m)^n} = 0$$

(e) $\underset{x \to \infty}{\text{Lt}}\ x^n e^{-x} = \underset{x \to \infty}{\text{Lt}} \frac{x^n}{e^x}$ $\left| \text{form } \frac{\infty}{\infty} \right.$

[*Differentiate the num. and denom. separately n times*]

$$= \underset{x \to \infty}{\text{Lt}} \frac{n!}{e^x} = \frac{n!}{\infty} = 0.$$

[Form $\infty - \infty$]

Example 2. *Evaluate the following limits :*

(a) $\underset{x \to 0}{Lt} \left(\frac{1}{x^2} - \frac{1}{x \tan x} \right)$ (b) $\underset{x \to 0}{Lt} \left(\frac{1}{x} - \cot x \right)$

(c) $\underset{x \to 0}{Lt} \left(\cot^2 x - \frac{1}{x^2} \right)$ (d) $\underset{x \to 0}{Lt} \left(\frac{1}{x^2} - \frac{1}{\sin^2 x} \right)$

(e) $\underset{x \to \pi/2}{Lt}\ (\sec x - \tan x)$ (f) $\underset{x \to 0}{Lt} \left[\frac{1}{x} - \frac{1}{x^2} \log(1+x) \right]$

(g) $\underset{x \to 0}{Lt} \left[\frac{\pi}{4x} - \frac{\pi}{2x(e^{\pi x}+1)} \right]$ (h) $\underset{x \to 1}{Lt} \left(\frac{2}{x^2-1} - \frac{1}{x-1} \right)$

(i) $\underset{x \to 4}{Lt} \left[\frac{1}{\log(x-3)} - \frac{1}{x-4} \right]$ (j) $\underset{x \to 0}{Lt} \left(\frac{1}{e^x - 1} - \frac{1}{x} \right)$.

Sol. (a) $\underset{x \to 0}{\text{Lt}} \left(\frac{1}{x^2} - \frac{1}{x \tan x} \right)$ | form $\infty - \infty$

$$= \underset{x \to 0}{\text{Lt}} \frac{\tan x - x}{x^2 \tan x} \qquad \left| \text{form } \frac{0}{0} \right.$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\tan x - x}{x^2} \cdot \frac{x}{\tan x} \qquad [\textit{Note carefully}]$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\tan x - x}{x^3} \times \underset{x \to 0}{\text{Lt}} \frac{x}{\tan x}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\tan x - x}{x^3} \quad \left[\because \underset{x \to 0}{\text{Lt}} \frac{x}{\tan x} = 1 \right] \qquad \left| \text{form } \frac{0}{0} \right.$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\sec^2 x - 1}{3x^2} \qquad \left| \text{form } \frac{0}{0} \right.$$

$$= \lim_{x\to 0} \frac{2\sec x.\sec x \tan x}{6x} = \lim_{x\to 0} \frac{\sec^2 x}{3} . \lim_{x\to 0} \frac{\tan x}{x} = \frac{1}{3}\times 1 = \frac{1}{3}.$$

(b) $\lim_{x\to 0}\left(\frac{1}{x} - \cot x\right)$ | *form* $\infty - \infty$

$$= \lim_{x\to 0}\left(\frac{1}{x} - \frac{\cos x}{\sin x}\right) = \lim_{x\to 0} \frac{\sin x - x\cos x}{x \sin x} \quad \Big| \text{ form } \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{\sin x - x\cos x}{x^2} . \frac{x}{\sin x} \quad \text{[Note carefully]}$$

$$= \lim_{x\to 0} \frac{\sin x - x\cos x}{x^2} . \lim_{x\to 0} \frac{x}{\sin x}$$

$$= \lim_{x\to 0} \frac{\sin x - x\cos x}{x^2} \quad \left[\because \lim_{x\to 0} \frac{x}{\sin x} = 1\right] \quad \Big| \text{ form } \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{\cos x - \cos x + x\sin x}{2x} = \lim_{x\to 0} \frac{\sin x}{2} = 0.$$

(c) $= \lim_{x\to 0}\left(\cot^2 x - \frac{1}{x^2}\right)$ | *form* $\infty - \infty$

$$= \lim_{x\to 0}\left(\frac{\cos^2 x}{\sin^2 x} - \frac{1}{x^2}\right)$$

$$= \lim_{x\to 0} \frac{x^2\cos^2 x - \sin^2 x}{x^2 \sin^2 x} \quad \Big| \text{ form } \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{x^2\cos^2 x - \sin^2 x}{x^4} . \frac{x^2}{\sin^2 x} \quad \text{[Note carefully]}$$

$$= \lim_{x\to 0} \frac{x^2\cos^2 x - \sin^2 x}{x^4} . \lim_{x\to 0}\left(\frac{x}{\sin x}\right)^2$$

$$= \lim_{x\to 0} \frac{x^2\left(\frac{1+\cos 2x}{2}\right) - \left(\frac{1-\cos 2x}{2}\right)}{x^4} \times 1$$

$$= \lim_{x\to 0} \frac{x^2(1+\cos 2x) - (1-\cos 2x)}{2x^4}$$

$$= \lim_{x\to 0} \frac{x^2 - 1 + (x^2+1)\cos 2x}{2x^4} \quad \Big| \text{ form } \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{2x + 2x\cos 2x - 2(x^2+1)\sin 2x}{8x^3} \quad \Big| \text{ form } \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{2 + 2\cos 2x - 4x\sin 2x - 4x\sin 2x - 4(x^2+1)\cos 2x}{24x^2}$$

$$= \lim_{x\to 0} \frac{2 - 8x\sin 2x - (4x^2+2)\cos 2x}{24x^2} \quad \Big| \text{ form } \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{-8\sin 2x - 16x\cos 2x - 8x\cos 2x + 2(4x^2+2)\sin 2x}{48x}$$

$$= \lim_{x\to 0} \frac{-24x\cos 2x + (8x^2-4)\sin 2x}{48x} \quad \Big| \text{ form } \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{-24\cos 2x + 48x\sin 2x + 16x\sin 2x + 2(8x^2-4)\cos 2x}{48}$$

$$= \frac{-24+0+0-8}{48} = \frac{-32}{48} = -\frac{2}{3}.$$

(d) *Please try yourself.* $\left[\textbf{Ans. } -\frac{1}{3}\right]$

(e) $\lim_{x\to \frac{\pi}{2}} (\sec x - \tan x)$ | *form* $\infty - \infty$

$$= \lim_{x\to\frac{\pi}{2}}\left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right) = \lim_{x\to\frac{\pi}{2}} \frac{1-\sin x}{\cos x} \quad \Big| \text{ form } \frac{0}{0}$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{-\cos x}{-\sin x} = \lim_{x\to\frac{\pi}{2}} \cot x = 0.$$

(f) $\lim_{x\to 0}\left[\frac{1}{x} - \frac{\log(1+x)}{x^2}\right] = \lim_{x\to 0} \frac{x - \log(1+x)}{x^2}$ | *form* $\frac{0}{0}$

$$= \lim_{x\to 0} \frac{1 - \frac{1}{1+x}}{2x} = \lim_{x\to 0} \frac{1}{2(1+x)} = \frac{1}{2}.$$

(g) $\lim_{x\to 0}\left[\frac{\pi}{4x} - \frac{\pi}{2x(e^{\pi x}+1)}\right] = \lim_{x\to 0} \pi\left[\frac{e^{\pi x}+1-2}{4x(e^{\pi x}+1)}\right]$ | *form* $\frac{0}{0}$

$$= \lim_{x\to 0} \pi\left[\frac{\pi e^{\pi x}}{4x(e^{\pi x}.\pi) + 4.(e^{\pi x}+1)}\right] = \frac{\pi^2}{0+4(1+1)} = \frac{\pi^2}{8}.$$

(h) *Please try yourself.* $\left[\textbf{Ans. } -\frac{1}{2}\right]$

(i) $\lim_{x\to 4}\left[\frac{1}{\log(x-3)} - \frac{1}{x-4}\right]$ | *form* $\infty - \infty$

$$= \lim_{x\to 4} \frac{(x-4) - \log(x-3)}{(x-4)\log(x-3)} \quad \Big| \text{ form } \frac{0}{0}$$

$$= \underset{x \to 4}{\text{Lt}} \frac{1 - \frac{1}{x-3}}{(x-4) \cdot \frac{1}{x-3} + \log(x-3)} = \underset{x \to 4}{\text{Lt}} \frac{1 - \frac{1}{x-3}}{1 - \frac{1}{x-3} + \log(x-3)} \qquad \Big| \text{ form } \frac{0}{0}$$

$$= \underset{x \to 4}{\text{Lt}} \frac{\frac{1}{(x-3)^2}}{\frac{1}{(x-3)^2} + \frac{1}{x-3}} = \frac{1}{1+1} = \frac{1}{2}.$$

(j) $$\underset{x \to 0}{\text{Lt}} \left(\frac{1}{e^x - 1} - \frac{1}{x} \right) \qquad | \text{ form } \infty - \infty$$

$$= \underset{x \to 0}{\text{Lt}} \frac{x - e^x + 1}{x(e^x - 1)} \qquad \Big| \text{ form } \frac{0}{0}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{1 - e^x}{(e^x - 1) + xe^x} \qquad \Big| \text{ form } \frac{0}{0}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{-e^x}{e^x + e^x + xe^x} = \frac{-1}{1+1+0} = -\frac{1}{2}.$$

Example 3. *Evaluate the following limits :*

(i) $\underset{x \to 0}{Lt} \left(\frac{1}{x} - \frac{1}{\sin x} \right)$ (ii) $\underset{x \to 0}{Lt} (\operatorname{cosec} x - \cot x)$.

Sol. *Please try yourself.* $\left[\textbf{Ans. } (i) -\frac{1}{2} \ (ii)\ 0 \right]$

[Forms 0°, 1^∞, ∞°]

Example 4. *Evaluate the following limits :*

(a) $\underset{x \to \infty}{Lt} \left(1 + \frac{a}{x} \right)^x$ (b) $\underset{x \to 1}{Lt} (x)^{\frac{1}{1-x}}$

(c) $\underset{x \to 0}{Lt} \left(\frac{\tan x}{x} \right)^{\frac{1}{x^2}}$ (d) $\underset{x \to 0}{Lt} \left(\frac{\sinh x}{x} \right)^{\frac{1}{x^2}}$

(e) $\underset{x \to 0}{Lt} \left(\frac{\sin x}{x} \right)^{\frac{1}{x}}$ (f) $\underset{x \to 0}{Lt} \left(\frac{\sin x}{x} \right)^{\frac{1}{x^2}}$

(g) $\underset{x \to 0}{Lt} \left(\frac{\tan x}{x} \right)^{\frac{1}{x}}$ (h) $\underset{x \to a}{Lt} \left(2 - \frac{x}{a} \right)^{\tan \frac{\pi x}{2a}}$

(i) $\underset{x \to \pi/2}{Lt} (\sin x)^{\tan x}$ (j) $\underset{x \to 0}{Lt} (\cos x)^{\cot^2 x}$.

Sol. (a) $$\underset{x \to \infty}{\text{Lt}} \left(1 + \frac{a}{x} \right)^x = l \text{ (say)} \qquad | \text{ form } 1^\infty$$

$\therefore$ $$\log l = \underset{x \to \infty}{\text{Lt}} \log \left(1 + \frac{a}{x} \right)^x \qquad | \text{ Taking logs on both sides}$$

$$= \underset{x \to \infty}{\text{Lt}} x \log \left(1 + \frac{a}{x} \right) \qquad | \text{ form } 0 \times \infty$$

$$= \underset{x \to \infty}{\text{Lt}} \frac{\log \left(1 + \frac{a}{x} \right)}{\frac{1}{x}} \qquad \Big| \text{ form } \frac{0}{0}$$

$$= \underset{x \to \infty}{\text{Lt}} \frac{\frac{1}{\left(1 + \frac{a}{x}\right)} \left(-\frac{a}{x^2} \right)}{-\frac{1}{x^2}} = \underset{x \to \infty}{\text{Lt}} \frac{a}{1 + \frac{a}{x}} = \frac{a}{1+0} = a$$

$\therefore$ $$l = e^a \quad \text{or} \quad \underset{x \to \infty}{\text{Lt}} \left(1 + \frac{a}{x} \right)^x = e^a.$$

(b) Let $$l = \underset{x \to 1}{\text{Lt}} (x)^{\frac{1}{1-x}} \qquad | \text{ form } 1^\infty$$

$\therefore$ $$\log l = \underset{x \to 1}{\text{Lt}} \log (x)^{\frac{1}{1-x}} = \underset{x \to 1}{\text{Lt}} \frac{1}{1-x} \log x = \underset{x \to 1}{\text{Lt}} \frac{\log x}{1-x} \qquad \Big| \text{ form } \frac{0}{0}$$

$$= \underset{x \to 1}{\text{Lt}} \frac{\frac{1}{x}}{-1} = \underset{x \to 1}{\text{Lt}} \frac{-1}{x} = -1$$

$\therefore$ $$l = e^{-1} \quad \text{or} \quad \underset{x \to 1}{\text{Lt}} (x)^{\frac{1}{1-x}} = e^{-1}.$$

(c) $$\underset{x \to 0}{\text{Lt}} \left(\frac{\tan x}{x} \right)^{\frac{1}{x^2}} = l \text{ (say)} \qquad | \text{ form } 1^\infty$$

$\therefore$ $$\log l = \underset{x \to 0}{\text{Lt}} \log \left(\frac{\tan x}{x} \right)^{\frac{1}{x^2}} = \underset{x \to 0}{\text{Lt}} \frac{\log \left(\frac{\tan x}{x} \right)}{x^2} \qquad \Big| \text{ form } \frac{0}{0}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\left[\frac{x}{\tan x} \cdot \frac{x \sec^2 x - \tan x}{x^2} \right]}{2x} \qquad | \text{ L'Hospital's Rule}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{x \sec^2 x - \tan x}{2x^2 \tan x} \qquad \left| \text{form } \frac{0}{0} \right.$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\sec^2 x + 2x \sec^2 x \tan x - \sec^2 x}{2x^2 \sec^2 x + 4x \tan x}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\sec^2 x \tan x}{x \sec^2 x + 2 \tan x} \qquad \left| \text{form } \frac{0}{0} \right.$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\sec^4 x + 2 \sec^2 x \tan^2 x}{2x \sec^2 x \tan x + \sec^2 x + 2 \sec^2 x} = \frac{1}{3}$$

$$\therefore \quad l = e^{\frac{1}{3}} \quad \text{or} \quad \underset{x \to 0}{\text{Lt}} \left(\frac{\tan x}{x} \right)^{\frac{1}{x^2}} = e^{\frac{1}{3}}.$$

(*d*) Let $\quad l = \underset{x \to 0}{\text{Lt}} \left(\frac{\sinh x}{x} \right)^{\frac{1}{x^2}}$ $\qquad$ | *form* 1^∞

$$\therefore \quad \log l = \underset{x \to 0}{\text{Lt}} \frac{1}{x^2} \log \left(\frac{\sinh x}{x} \right) = \underset{x \to 0}{\text{Lt}} \frac{\log \left(\frac{\sinh x}{x} \right)}{x^2} \qquad \left| \text{form } \frac{0}{0} \right.$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\left[\frac{x}{\sinh x} \cdot \frac{x \cosh x - \sinh x}{x^2} \right]}{2x} = \underset{x \to 0}{\text{Lt}} \frac{x \cosh x - \sinh x}{2x^2 \sinh x} \qquad \left| \text{form } \frac{0}{0} \right.$$

$$= \underset{x \to 0}{\text{Lt}} \frac{x \sinh x + \cosh x - \cosh x}{4x \sinh x + 2x^2 \cosh x} = \underset{x \to 0}{\text{Lt}} \frac{\sinh x}{4 \sinh x + 2x \cosh x} \qquad \left| \text{form } \frac{0}{0} \right.$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\cosh x}{4 \cosh x + 2x \sinh x + 2 \cosh x} = \frac{1}{6}$$

$$\therefore \quad l = e^{\frac{1}{6}}.$$

(*e*) *Please try yourself.* [Ans. 1]

(*f*) *Please try yourself.* [Ans. $e^{-1/6}$]

(*g*) *Please try yourself.* [Ans. 1]

(*h*) Let $\quad l = \underset{x \to a}{\text{Lt}} \left(2 - \frac{x}{a} \right)^{\tan \frac{\pi x}{2a}}$ $\qquad$ | *form* 1^∞

$$\therefore \quad \log l = \underset{x \to a}{\text{Lt}} \tan \frac{\pi x}{2a} \log \left(2 - \frac{x}{a} \right) \qquad | \text{ form } \infty \times 0$$

$$= \underset{x \to a}{\text{Lt}} \frac{\log \left(2 - \frac{x}{a} \right)}{\cot \frac{\pi x}{2a}} \qquad \left| \text{form } \frac{0}{0} \right.$$

$$= \underset{x \to a}{\text{Lt}} \frac{\frac{1}{2 - \frac{x}{a}} \cdot \frac{-1}{a}}{-\frac{\pi}{2a} \operatorname{cosec}^2 \frac{\pi x}{2a}} = \frac{2}{\pi}$$

$$\therefore \quad l = e^{\frac{2}{\pi}} \quad \text{or} \quad \underset{x \to a}{\text{Lt}} \left(2 - \frac{x}{a} \right)^{\tan \frac{\pi x}{2a}} = e^{\frac{2}{\pi}}.$$

(*i*) Let $\quad l = \underset{x \to \pi/2}{\text{Lt}} (\sin x)^{\tan x}$ $\qquad$ | *form* 1^∞

$$\therefore \quad \log l = \underset{x \to \pi/2}{\text{Lt}} \tan x \log \sin x = \underset{x \to \pi/2}{\text{Lt}} \frac{\log \sin x}{\cot x} \qquad \left| \text{form } \frac{0}{0} \right.$$

$$= \underset{x \to \pi/2}{\text{Lt}} \frac{\frac{1}{\sin x} \cdot \cos x}{-\operatorname{cosec}^2 x} = \frac{0}{-1} = 0$$

$$\therefore \quad l = e^0 = 1 \quad \text{or} \quad \underset{x \to \pi/2}{\text{Lt}} (\sin x)^{\tan x} = 1$$

(*j*) *Please try yourself.* $\left[\text{Ans. } e^{-\frac{1}{2}} \right]$

Example 5. *Evaluate the following limits :*

(*a*) $\underset{x \to \infty}{Lt} (1 + x)^{\frac{1}{x}}$ $\qquad$ (*b*) $\underset{x \to 0}{Lt} (\operatorname{cosec} x)^{\frac{1}{\log x}}$

(*c*) $\underset{x \to 0}{Lt} \left(\frac{1}{x} \right)^{\tan x}$ $\qquad$ (*d*) $\underset{x \to 0}{Lt} \left(\frac{1}{x} \right)^{2 \sin x}$

(*e*) $\underset{x \to 0+}{Lt} (\cos x)^x$ $\qquad$ (*f*) $\underset{x \to \pi/2}{Lt} (\tan x)^{\sin 2x}$

(*g*) $\underset{x \to 0}{Lt} \frac{\pi}{2} (\sin x)^{\tan^2 x}$.

Sol. (*a*) Let $\quad l = \underset{x \to \infty}{\text{Lt}} (1 + x)^{\frac{1}{x}}$ $\qquad$ | *form* ∞^0

$$\therefore \quad \log l = \underset{x \to \infty}{\text{Lt}} \frac{1}{x} \log (1 + x) = \underset{x \to \infty}{\text{Lt}} \frac{\log (1 + x)}{x} \qquad \left| \text{form } \frac{\infty}{\infty} \right.$$

$$= \underset{x \to \infty}{\text{Lt}} \frac{\frac{1}{1 + x}}{1} = \frac{1}{\infty} = 0.$$

$$\therefore \quad l = 0 = 1.$$

(*b*) Let $\quad l = \underset{x \to \infty}{\text{Lt}} (\operatorname{cosec} x)^{\frac{1}{\log x}}$ $\qquad$ | *form* ∞^0

$\therefore \quad \log l = \underset{x\to 0}{\text{Lt}} \dfrac{1}{\log x} . \log \text{cosec}\, x$

$= \underset{x\to 0}{\text{Lt}} \dfrac{\log \text{cosec}\, x}{\log x}$ | form $\dfrac{\infty}{\infty}$

$= \underset{x\to 0}{\text{Lt}} \dfrac{\dfrac{1}{\text{cosec}\, x}(-\text{cosec}\, x \cot x)}{\dfrac{1}{x}}$

$= \underset{x\to 0}{\text{Lt}} (-x \cot x) = \underset{x\to 0}{\text{Lt}} \dfrac{-x}{\tan x}$ | form $\dfrac{\infty}{\infty}$

$= \underset{x\to 0}{\text{Lt}} \dfrac{-1}{\sec^2 x} = -1$

$\therefore \quad l = e^{-1} = \dfrac{1}{e}.$

(c) Let $\quad l = \underset{x\to 0}{\text{Lt}} \left(\dfrac{1}{x}\right)^{\tan x}$ | form ∞^0

$\log l = \underset{x\to 0}{\text{Lt}} \tan x . \log \dfrac{1}{x} = \underset{x\to 0}{\text{Lt}} \dfrac{-\log x}{\cot x}$ | form $\dfrac{\infty}{\infty}$ **[Note this step]**

$= \underset{x\to 0}{\text{Lt}} \dfrac{-\dfrac{1}{x}}{\text{cosec}^2 x} = \underset{x\to 0}{\text{Lt}} \dfrac{\sin^2 x}{x}$ | form $\dfrac{0}{0}$

$= \underset{x\to 0}{\text{Lt}} \dfrac{2 \sin x \cos x}{1} = 0$

$\therefore \quad l = e^0 = 1.$ **[Ans. 1]**

(d) *Please try yourself.*

(e) Let $\quad l = \underset{x\to 0+}{\text{Lt}} (\cot x)^x$ | form ∞^0

$\therefore \quad \log l = \underset{x\to 0}{\text{Lt}}\, x \log (\cot x)$ | form $0 \times \infty$

$= \underset{x\to 0+}{\text{Lt}} \dfrac{\log (\cot x)}{\dfrac{1}{x}}$ | form $\dfrac{0}{0}$

$= \underset{x\to 0+}{\text{Lt}} \dfrac{\dfrac{1}{\cot x}.(-\text{cosec}^2 x)}{-\dfrac{1}{x^2}} = \underset{x\to 0+}{\text{Lt}} \dfrac{x^2 \tan x}{\sin^2 x} = \underset{x\to 0+}{\text{Lt}} \left(\dfrac{x}{\sin x}\right)^2 . \underset{x\to 0+}{\text{Lt}} \tan x$

$= (1)^2 \times 0 = 0$

$\therefore \quad l = e^0 = 1.$

(f) Let $\quad l = \underset{x\to \pi/2}{\text{Lt}} (\tan x)^{\sin 2x}$ | form ∞^0

$\therefore \quad \log l = \underset{x\to \pi/2}{\text{Lt}} \sin 2x \log (\tan x)$ | form $0 \times \infty$

$= \underset{x\to \pi/2}{\text{Lt}} \dfrac{\log (\tan x)}{\text{cosec}\, 2x}$ | form $\dfrac{\infty}{\infty}$

$= \underset{x\to \pi/2}{\text{Lt}} \dfrac{\dfrac{1}{\tan x}.\sec^2 x}{-2\,\text{cosec}\, 2x \cot 2x} = \underset{x\to \pi/2}{\text{Lt}} \dfrac{\dfrac{\cos x}{\sin x}.\dfrac{1}{\cos^2 x}}{-2 . \dfrac{1}{2 \sin x \cos x} . \cot 2x}$

$= \underset{x\to \pi/2}{\text{Lt}} - \tan 2x = 0$

$\therefore \quad l = e^0 = 1.$

(g) $\quad \underset{x\to 0}{\text{Lt}} \dfrac{\pi}{2} (\sin x)^{\tan^2 x} = \dfrac{\pi}{2} \underset{x\to 0}{\text{Lt}} (\sin x)^{\tan^2 x}$ ...(1)

Let $\quad l = \underset{x\to 0}{\text{Lt}} (\sin x)^{\tan^2 x}$ | form 0^0

$\therefore \quad \log l = \underset{x\to 0}{\text{Lt}} \tan^2 x \log (\sin x)$ | form $0 \times \infty$

$= \underset{x\to 0}{\text{Lt}} \dfrac{\log \sin x}{\cot^2 x}$ | form $\dfrac{\infty}{\infty}$

$= \underset{x\to 0}{\text{Lt}} \dfrac{\dfrac{1}{\sin x}.\cos x}{2 \cot x . (-\text{cosec}^2 x)} = \underset{x\to 0}{\text{Lt}} \dfrac{\cot x}{-2 \cot x\, \text{cosec}^2 x}$

$= \underset{x\to 0}{\text{Lt}} \left(-\dfrac{1}{2} \sin^2 x\right) = 0 \quad \Rightarrow \quad l = e^0 = 1$

$\therefore$ Form (1), $\quad \underset{x\to 0}{\text{Lt}} \dfrac{\pi}{2} (\sin x)^{\tan^2 x} = \dfrac{\pi}{2} \times 1 = \dfrac{\pi}{2}.$

Example 6. *Evaluate the following limits :*

(a) $\underset{x\to 0}{Lt}\, x^x$ (b) $\underset{x\to 1}{Lt}\, (1-x^2)^{\frac{1}{\log (1-x)}}$

(c) $\underset{x\to 0}{Lt}\, x^{2x}$ (d) $\underset{x\to 0+}{Lt}\, (\cot x)^{\sin x}$

Sol. (a) Let $\quad l = \underset{x\to 0}{\text{Lt}}\, x^x$

$\therefore \quad \log l = \cancel{\underset{x\to 0}{\text{Lt}}\, x \log x} = \underset{x\to 0}{\text{Lt}} \dfrac{\log}{\dfrac{1}{x}}$ | form $\dfrac{\infty}{\infty}$

$$= \underset{x\to 0}{\text{Lt}} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \underset{x\to 0}{\text{Lt}} (-x) = 0$$

$\therefore$ $l = e^0 = 1.$

(b) Let $l = \underset{x\to 1}{\text{Lt}} (1-x^2)^{\frac{1}{\log(1-x)}}$ | *form* 0°

$\therefore$ $\log l = \underset{x\to 1}{\text{Lt}} \frac{1}{\log(1-x)} . \log(1-x^2) = \underset{x\to 1}{\text{Lt}} \frac{\log(1-x^2)}{\log(1-x)}$ | *form* $\frac{\infty}{\infty}$

$$= \underset{x\to 1}{\text{Lt}} \frac{\frac{-2x}{1-x^2}}{\frac{-1}{1-x}} = \underset{x\to 1}{\text{Lt}} \frac{2x}{1+x} = \frac{2}{1+1} = 1$$

$\therefore$ $l = e^1 = e.$ **[Ans. 1]**

(c) *Please try yourself.*

(d) Let $l = \underset{x\to 0+}{\text{Lt}} (\cot x)^{\sin x}$ | *form* ∞°

$\log l = \underset{x\to 0+}{\text{Lt}} \sin x \log(\cot x)$ | *form* $0 \times \infty$

$$= \underset{x\to 0+}{\text{Lt}} \frac{\log(\cot x)}{\operatorname{cosec} x}$$ | *form* $\frac{\infty}{\infty}$

$$= \underset{x\to 0+}{\text{Lt}} \frac{\frac{1}{\cot x}.(-\operatorname{cosec}^2 x)}{-\operatorname{cosec} x \cot x} = \underset{x\to 0+}{\text{Lt}} \frac{\operatorname{cosec} x}{\cot^2 x}$$

$$= \underset{x\to 0+}{\text{Lt}} \frac{\tan^2 x}{\sin x}$$ | *form* $\frac{0}{0}$

$$= \underset{x\to 0+}{\text{Lt}} \frac{2\tan x . \sec^2 x}{\cos x} = \frac{2\times 0\times 1}{1} = 0$$

$\Rightarrow$ $l = e^0 = 1.$

15

Maxima and Minima

15.1. DEFINITIONS

Let $f(x)$ be a function defined on the interval $I = [a, b]$ and c be any interior point of I.

(i) **Maximum.** $f(x)$ is said to have a maximum value at $x = c$ if $\exists\ \delta > 0$ s.t. $\forall\ x \in (c-\delta, c+\delta), x \neq c$

$$f(x) < f(c)$$

(ii) **Minimum.** $f(x)$ is said to have a minimum value at $x = c$ if $\exists\ \delta > 0$ s.t. $\forall\ x \in (c-\delta, c+\delta), x \neq c$

$$f(x) > f(c)$$

(iii) **Extremum.** $f(x)$ is said to have an extreme value (or simply an extremum or a turning value) at $x = c$ if it has either a maximum or a minimum value at $x = c$.

[Extremum at $x = c$ is defined only in a nbd. of $x = c$].

☞15.2. THEOREM

1. **(Necessary Condition for the existence of extreme values).** *If $f(c)$ is an extreme value of a function f and $f(c)$ exists, then $f'(c) = 0$.*

Proof. $f'(c)$ = exists $\Rightarrow$ $Lf'(c)$ and $Rf'(c)$ both exist and $Lf'(c) = Rf'(c)$ ...(i)

$f(c)$ is an extreme value of $f(x)$.

$\Rightarrow$ $f(x)$ has either a maximum or a minimum value at $x = c$.

Let $f(x)$ have a maximum value at $x = c$ $\Rightarrow$ $\exists\ \delta > 0$ s.t.

$c - \delta < x < c \Rightarrow f(x) < f(c)$ ...(ii)

and $c < x < c + \delta \Rightarrow f(x) < f(c)$ ...(iii)

From (ii) $f(x) - f(c) < 0$ and $x - c < 0$

$\Rightarrow$ $\frac{f(x)-f(c)}{x-c} > 0$ when $c - \delta < x < c$

Also $Lf'(c)$ exists. [From (i)]

$\therefore$ $Lf'(c) \geq 0$ ...(iv)

From (ii), $f(x) - f(c) < 0$ and $x - c > 0$

$\Rightarrow$ $\frac{f(x)-f(c)}{x-c} < 0$ when $c < x < c + \delta$

Also $Rf'(c)$ exists. [From (i)]

$\therefore$ $Rf'(c) \leq 0$ ...(v)

But $\quad Lf'(c) = Rf'(c)$ [From (*i*)]

$\therefore$ From (*iv*) and (*v*), $\quad f'(c) = 0$

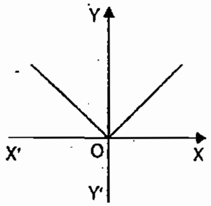
Again if $f(x)$ has a minimum at $x = c$, by similar arguments, we have $f'(c) = 0$.

Note 1. If $f'(c) = 0$, then $x = c$ is called a **stationary point** for f and $f(c)$ is called is **stationary value of** f.

Note 2. The above condition is a necessary but not a sufficient condition for $f(x)$ to have a maximum or minimum value at $x = c$.

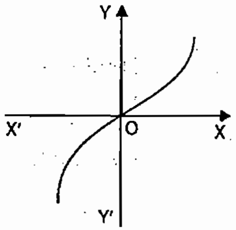
(*i*) A function may have an extremum at a point without having a derivative at that point.

For example, if $f(x) \mid x \mid x \in R$, then f is not derivable at $x = 0$ but it has minimum value at $x = 0$.



(*ii*) $f'(c) = 0$ may not imply that $f(x)$ has an extremum at $x = c$.

For example, if $f(x) = x^3, x \in R$, then $f'(0) = 0$ but $x = 0$ is not an extreme point.



Note 3. From Note 2. it is clear that if a function f has an extreme value at $x = c$, then either f is not derivable at $x = c$ or $f'(c) = 0$. Therefore, to investigate the maxima or minima of a function f, we first find the values of x for which either $f'(x)$ does not exist or if $f'(x)$ exists, then it vanishes. A point $x = c$ such that either $f'(c)$ does not exist of $f'(c) = 0$ is called a *critical point* and $f(c)$ is called a *critical value* of f.

(Sufficient Conditions for the existence of extreme values).

15.3. THEOREM 2. (First Derivative Test)

Let a function f be derivable on an open interval I and $f'(c) = 0, c \in I$. Then if $f'(x)$ changes sign

(*i*) *from positive to negative as x passes through c, then f has a maximum at $x = c$.*

(*ii*) *from negative to positive as x passes through c, then f has a minimum at $x = c$.*

Proof. Let $\quad I = (c - \delta, c + \delta)$.

f is derivable on I $\Rightarrow$ f is continuous on I.

(*i*) Let x_1 be any point of $(c - \delta, c)$. Then f satisfies the conditions of Lagrange's Mean Value Theorem in $[x_1, c]$.

$\therefore \quad f(c) - f(x_1) = (c - x_1) f'(d)$ for some $d \in (x_1, c)$

Since $\quad f'(x) > 0$ for $c - \delta < x < c$ and $c - \delta < x_1 < d < c$.

$\therefore \quad f'(d) > 0, c - x_1 > 0.$

Consequently $\quad (c - x_1) f'(d) > 0 \quad \therefore \quad f(c) - f(x_1) > 0$ or $f(x_1) < f(c)$

But x_1 is any point of $(c - \delta, c)$.

$\therefore \quad f(x) < f(c)$ for $c - \delta < x < c$ ...(1)

Now let x_2 be any point of $(c, c + \delta)$. Then f satisfies the conditions of Lagrange's Mean Value Theorem in $[c, x_2]$.

$\therefore \quad f(x_2) - f(c) = (x_2 - c) f'(e)$ for some $e \in (c, x_2)$

Since $f'(x) < 0$ for $c < x < c + \delta$ and $c < e < x_2 < c + \delta$

$\therefore \quad f'(e) < 0, x_2 - c > 0.$

Consequently $\quad (x_2 - c) f'(e) < 0 \quad \therefore \quad f(x_2) - f(c) < 0$ or $f(x_2) < f(c)$

But x_2 is any point of $(c, c + \delta)$.

$\therefore \quad f(x) < f(c)$ for $c < x < c + \delta$ ...(2)

From (1) and (2), $\quad f(x) < f(c)$ for $c - \delta < x < c + \delta, x \neq c$

Hence f has a maximum value at $x = c$.

(*ii*) *Please try yourself.*

15.4. THEOREM 3. (Generalised Test)

Let c be an interior point of the domain [a, b] of a function f. If $f^n(c)$ exists such that

(a) $f'(c = f'')(c) = f'''(c) = ... = f^{n-1}(c) = 0$. (b) $f^n(c) \neq 0$.

Then if (I) n is even (i) f(c) is minimum value of f, if $f^n(c) > 0$ and (ii) f(c) is maximum value of f, if $f^n(c) \leq 0$.

(II) n is odd, f(c) is not an extreme value of f and the point c is called the point of inflexion.

Proof. $f^n(c)$ exists $\Rightarrow$ $f', f'', f''' ...... f^{n-1}$ exist and are continuous in some nbd. $(c - \delta, c + \delta)$.

$\Rightarrow$ f satisfies all the conditions of Taylor's Theorem.

$$\therefore \quad f(c + h) = f(c) + hf'(c) + \frac{h^2}{2!} f''(c) + ... + \frac{h^{n-1}}{(n-1!)} f^{n-1}(c) + \frac{h^n}{n!} f^n(c + \theta h)$$

where $\quad 0 < \theta < 1, \mid h \mid < \delta$

Since $\quad f'(c) = f''(c) = f'''(c) = ... = f^{n-1}(c) = 0$

$$\therefore \quad f(c + h) - f(c) = \frac{h^n}{n!} f^n(c + \theta h) \qquad ...(1)$$

Also $f^n(c) \neq 0$, is continuous at c.

$\Rightarrow$ $f^n(x)$ and $f^n(c)$ have same sign if x is close to c

$\Rightarrow$ $f^n(c + \theta h)$ and $f^n(c)$ have same sign.

Case I. n is even

(*i*) **When $f^n(c) < 0$.**

$$\left.\begin{matrix} h < 0 \\ h > 0 \end{matrix}\right\} \Rightarrow h^n > 0. \text{ Also } f^n(c + \theta h) > 0$$

$$\therefore \quad \frac{h^n}{n!} f^n(c + \theta h) > 0 \Rightarrow f(c + h) > f(c) \qquad \text{[by (1)]}$$

$\Rightarrow$ $f(c)$ is minimum vlaue of $f(x)$.

(*ii*) **When $f^n(c) < 0$.**

$$\left.\begin{matrix} h < 0 \\ h > 0 \end{matrix}\right\} \Rightarrow h^n > 0. \text{ Also } f^n(c + \theta h) < 0$$

$$\therefore \quad \frac{h^n}{n!} f^n(c + \theta h) < 0 \Rightarrow f(c + h) < f(c) \qquad \text{[by (1)]}$$

$\Rightarrow$ $f(c)$ is maximum value of $f(x)$.

Case II. n is odd.

(*i*) **When $f^n(c) > 0$.**

$$h < 0 \Rightarrow h^n < 0. \text{ Also } f^n(c + \theta h) > 0$$

$$\therefore \quad \frac{h^n}{n!} f^n(c+\theta h) < 0 \Rightarrow f(c+h) < f(c) \qquad \text{[by (1)]} \quad ...(i)$$

$\Rightarrow$ $f(c)$ is maximum value of $f(x)$

Also $\quad h > 0 \Rightarrow h^n > 0$. Also $f^n(c + \theta h) > 0$

$$\therefore \quad \frac{h^n}{n!} f^n(c+\theta h) > 0 \Rightarrow f(c+h) > f(c) \qquad \text{[(by) (1)]} \quad ...(ii)$$

$\Rightarrow$ $f(c)$ is minimum value of $f(x)$.

From (*i*) and (*ii*) $f(c)$ is not an extreme value of $f(x)$.

Again in $f^n(c) < 0$, similar arguments show that $f(c)$ is not an extreme value of $f(x)$.

(Second Derivative Test)

Special Case. When n = 2

If a function f is such that $f'(c) = 0$ and $f''(c)$ exists and is non-zero, then

(*i*) $f(c)$ is a maximum if $f''(c) < 0$

(*ii*) $f(c)$ is a minimum if $f''(c) > 0$

☞ **Working Rule for finding maxima and minima.**

(First Derivative Test)

First Method

(*i*) Denote the given function by $f(x)$.

(*ii*) Find $f'(x)$ and equate it to zero. Let its roots be c_1, c_2, c_3 ......

(*iii*) Test these values in succession.

Consider $x = c_1$ (say).

Find the sings of $f'(x)$ for values of x slightly $< c_1$ and for values of x slightly $> c_1$.

If $f'(x)$ changes sign from + ve to – ve, $f(x)$ has a maximum at $x = c_1$.

If $f'(x)$ changes sign from – ve to + ve, $f(x)$ has a minimum at $x = c_1$.

If $f'(x)$ does not change sign, there is neither maximum nor minimum at $x = c_1$.

(*iv*) Similarly test all the value of x obtained in (*ii*).

(Second Derivative Test)

Second Method

(*i*) Denote the given function by $f(x)$.

(*ii*) Find $f'(x)$ and equate it to zero. Let its roots be c_1, c_2, c_3, ...

(*iii*) Find $f''(x)$. Put $x = c_1$.

If $f''(c_1) < 0$, $f(x)$ has a maximum at $x = c_1$

If $f''(c_1) > 0$, $f(x)$ has a minimum at $x = c_1$

*(*iv*) If $f''(c_1) = 0$, find $f'''(c_1)$.

If $f'''(c_1) \neq 0$, there is neither maximum nor minimum at $x = c_1$.

If $f'''(c_1) = 0$, find $f^{iv}(c_1)$.

If $f^{iv}(c_1) < 0$, $f(x)$ has a maximum at $x = c_1$

If $f^{iv}(c_1) > 0$, $f(x)$ has a minimum at $x = c_1$.

ILLUSTRATIVE EXAMPLES

Example 1. *Find the maximum and minimum values of $2x^3 - 9x^2 - 24x - 20$.*

Sol. Let
$$f(x) = 2x^3 - 9x^2 - 24x - 20$$
$$f'(x) = 6x^2 - 18x - 24$$
$$f''(x) = 12x - 18$$

For maximum or minimum, $f'(x) = 0$

$\Rightarrow \quad 6(x^2 - 3x - 4) = 0 \Rightarrow (x-4)(x+1) = 0 \Rightarrow x = 4$ or -1

At $x = 4$, $\quad f''(x) = 12(4) - 18 > 0$

$\Rightarrow$ f is minimum at $x = 4$

$\Rightarrow \quad f_{min.} = f(4) = 2(4)^3 - 9(4)^2 - 24(4) - 20 = -132$

At $x = -1$, $\quad f''(x) = -12 - 18 < 0$

$\Rightarrow$ f is maximum at $x = -1$

$\Rightarrow \quad f_{max.} = f(-1) = 2(-1)^3 - 9(-1)^2 - 24(-1) - 20 = -7.$

Example 2. *Find the maximum and ~~minimum points of~~ the function f given by*
$$f(x) = (x-1)(x-2)(x-3).$$

Sol.
$$f(x) = x^3 - 6x^2 + 11x - 6$$
$$f'(x) = 3x^2 - 12x + 11$$
$$f''(x) = 6x - 12$$

For maximum or minimum,
$$f'(x) = 0 \Rightarrow 3x^2 - 12x + 11 = 0$$

$$\Rightarrow \quad x = \frac{12 \pm \sqrt{144 + 132}}{6} = \frac{12 \pm 2\sqrt{3}}{6} = 2 \pm \tfrac{1}{3}\sqrt{3}$$

At $x = 2 + \frac{1}{3}\sqrt{3}$, $\quad f''(x) = 6(2 + \frac{1}{3}\sqrt{3}) - 12 = 2\sqrt{3} > 0$

$\Rightarrow$ f is minimum at $x = 2 + \frac{1}{3}\sqrt{3}$

At $x = 2 - \frac{1}{3}\sqrt{3}$, $\quad f''(x) = 6(2 - \frac{1}{3}\sqrt{3}) - 12 = -2\sqrt{3} < 0$

$\Rightarrow$ f is maximum at $x = 2 - \frac{1}{3}\sqrt{3}$.

Example 3. *Show that $x^5 - 5x^4 + 5x^3 - 1$ has a maximum when $x = 1$, a minimum when $x = 3$ and neither when $x = 0$.*

Sol. Let
$$f(x) = x^5 - 5x^4 + 5x^3 - 1$$
$$f'(x) = 5x^4 - 20x^3 + 15x^2$$
$$f''(x) = 20x^3 - 60x^2 + 30x$$

For maximum or minimum,
$$f'(x) = 0 \Rightarrow 5x^2(x^2 - 4x + 3) = 0$$

$\Rightarrow \quad x^2(x-1)(x-3) = 0 \Rightarrow x = 0, 1, 3$

At $x = 0$, $\quad f''(x) = 0$

Since $\quad f'''(x) = 60x^2 - 120x + 30, f'''(0) = 30 \neq 0$

$\Rightarrow$ f has neither a max. nor mini. when $x = 0$

At $x = 1$, $f''(x) = 20 - 60 + 30 = -10 < 0$

$\Rightarrow$ f has a maximum when $x = 1$

At $x = 3$, $f''(x) = 20(3)^3 - 60(3)^2 + 30(3) = 90 > 0$

$\Rightarrow$ f has a minimum when $x = 3$.

Example 4. *Examine the following function for extreme values : $(x-3)^5 (x+1)^4$.*

Sol. Let $f(x) = (x-3)^5 (x+1)^4$

then $f'(x) = (x-3)^5 \cdot 4(x+1)^3 + 5(x-3)^4 (x+1)^4$

$= (x-3)^4 (x+1)^3 [4(x-3) + 5(x+1)] = (x-3)^4 (x+1)^3 (9x-7)$

For maximum or minimum,

$f'(x) = 0 \Rightarrow x = 3, -1, 7/9$

Let us test values one by one.

(*i*) For x slightly < 3, $f'(x) = (+)(+)(+) = +$ ve

For x slightly > 3, $f'(x) = (+)(+)(+) = +$ ve

Since $f'(x)$ does not changes sign as x passes through 3, f is neither maximum nor minimum at $x = 3$.

(*ii*) For x slightly < -1, $f'(x) = (+)(-)(-) = +$ ve

For x slightly > -1, $f'(x) = (+)(+)(-) = -$ ve

Since $f'(x)$ changes sign from +ve to –ve as x passes through -1, f is maximum at $x = -1$

$$f_{max.} = f(-1) = 0$$

(*iii*) For x slightly $< \frac{7}{9}$, $f'(x) = (+)(+)(-) = -$ve

For x slightly $> \frac{7}{9}$, $f'(x) = (+)(+)(+) = +$ve

Since $f'(x)$ changes sign from – ve to + ve as x passes through $\frac{7}{9}$, f is minimum at $x = \frac{7}{9}$.

$$f_{min.} = f\left(\frac{7}{9}\right) = \left(\frac{7}{9} - 3\right)^5 \left(\frac{7}{9} + 1\right)^4 = \frac{4^{13} \cdot 5^5}{3^{18}}.$$

Example 5. *Examine for maxima and minima the function f defined by $f(x) = x^2(1-x)^3$.*

Sol. *Please try yourself.* [**Ans.** Min. 0 at $x = 0$, max. 108 at $x = -2$
Neither max. nor min. at $x = 1$]

Example 6. *Find the maximum and minimum values, if any, of the function $(1-x)^2 e^x$.*

Sol. Let $f(x) = (1-x)^2 e^x$

then $f'(x) = (1-x)^2 \cdot e^x - 2(1-x) e^x$

$= (1-x)(1-x-2) e^x = (1-x)(-1-x) e^x$

$= (x+1)(x-1) e^x = (x^2-1) e^x$

$f''(x) = (x^2-1) \cdot e^x + 2x \cdot e^x = (x^2 + 2x - 1) e^x$

For maximum or minimum, $f'(x) = 0$

$\Rightarrow$ $(x^2-1) e^x = 0 \Rightarrow x^2 - 1 = 0$ [$\because$ $e^x \neq 0$ for any $x \in$ R]

$\therefore$ $x = \pm 1$

When x = 1, $f''(x) = 2e > 0 \Rightarrow f$ is minimum at $x = 1$

$\therefore$ $f_{min.} = f(1) = 0$

When x = – 1, $f''(x) = -2e^{-1} < 0 \Rightarrow f$ is maximum at $x = -1$

$f_{max.} = f(-1) = 4e^{-1} = 4/e$.

Example 7. *Find the maximum value of $\frac{\log x}{x}$, $0 < x < \infty$.*

Sol. Let $f(x) = \frac{\log x}{x}$

then
$$f'(x) = \frac{x \cdot \frac{1}{x} - \log x \cdot 1}{x^2} = \frac{1 - \log x}{x^2}$$

$$f''(x) = \frac{x^2\left(-\frac{1}{x}\right) - (1 - \log x) \cdot 2x}{x^4} = \frac{2x \log x - 3x}{x^4} = \frac{2 \log x - 3}{x^3}$$

For maximum or min., $f'(x) = 0 \Rightarrow 1 - \log x = 0$

$\Rightarrow$ $\log x = 1 = \log e \Rightarrow x = e$

When $x = e$, $f''(x) = \frac{2 \log e - 3}{e^3} = \frac{2-3}{e^3} = -\frac{1}{e^3} < 0$

$\Rightarrow$ f is maximum at $x = e$.

$$f_{max.} = f(e) \frac{\log e}{e} = \frac{1}{e}.$$

Example 8. *Prove that the function $\left(\frac{1}{x}\right)^x$, $x > 0$ has a maximum at $x = \frac{1}{e}$.*

Sol. Let $f(x) = \left(\frac{1}{x}\right)^x$ then $\log f(x) = x \log \frac{1}{x} = -x \log x$

$\therefore$ $\frac{1}{f(x)} \cdot f'(x) = -\left[x \cdot \frac{1}{x} + 1 \cdot \log x\right] \Rightarrow f'(x) = -f(x) [1 + \log x]$

$\therefore$ $f''(x) = -f(x) \cdot \frac{1}{x} - f'(x) \cdot [1 + \log x]$

For max. (or min.) $f'(x) = 0 \Rightarrow -f(x)[1 + \log x] = 0$.

$\Rightarrow$ $1 + \log x = 0$ [$\because$ $f(x) \neq 0$]

$\Rightarrow$ $\log x = -1 \Rightarrow x = e^{-1}$

Also $f''(e^{-1}) = -f(e^{-1}) \cdot e - f'(e^{-1})[1 + \log e^{-1}] = -(e)^{1/e} \cdot e - 0 < 0$

[$\because$ $f'(e^{-1}) = 0$]

$\Rightarrow$ f is maximum at $x = e^{-1} = \frac{1}{e}$.

Example 9. *Prove that the function x^x, $x > 0$ has a minimum at $x = \frac{1}{e}$.*

Sol. *Please try yourself.*

Example 10. *Prove that $x^{1/x}$ has a maximum value at $x = e$.*

Sol. *Please try yourself.*

Example 11. *Show that $\sin x(1+\cos x)$ is a maximum when $x = \frac{\pi}{3}$.*

Sol. Let $f(x) = \sin x(1+\cos x)$

then
$$f'(x) = \sin x(-\sin x) + \cos x(1+\cos x)$$
$$= \cos^2 x - \sin^2 x + \cos x = \cos 2x + \cos x$$
$$f''(x) = -2\sin 2x - \sin x$$

For max. or min., $f'(x) = 0$

$\Rightarrow \quad \cos 2x + \cos x = 0 \quad \Rightarrow \quad 2\cos\frac{3x}{2}\cos\frac{x}{2} = 0$

$\Rightarrow \quad$ either $\frac{3x}{2} = \frac{\pi}{2}$ or $\frac{x}{2} = \frac{\pi}{2} \quad \Rightarrow \quad x = \frac{\pi}{3}$ or $x = \pi$.

Here we have to consider only the point $x = \frac{\pi}{3}$

$$f''\left(\frac{\pi}{3}\right) = -2\sin\frac{2\pi}{3} - \sin\frac{\pi}{3} = -2\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = -\frac{3\sqrt{3}}{2} < 0$$

$\Rightarrow$ $f(x)$ has a maximum at $x = \frac{\pi}{3}$.

Example 12. *Find the maximum and minimum values of the function*

$$\sin x + \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x,\ 0 \le x \le \pi.$$

Sol. Let $f(x) = \sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x$

then
$$f'(x) = \cos x + \cos 2x + \cos 3x$$
$$f''(x) = -[\sin x + 2\sin 2x + 3\sin 3x]$$

For max. or min., $f'(x) = 0$

$\Rightarrow \quad \cos x + \cos 2x + \cos 3(x) = 0 \quad \Rightarrow \quad (\cos 3x + \cos x) + \cos 2x = 0$

$\Rightarrow \quad 2\cos 2x\cos x + \cos 2x = 0 \quad \Rightarrow \quad \cos 2x(2\cos x + 1) = 0$

$\Rightarrow \quad$ either $\cos 2x = 0$ or $\cos x = -\frac{1}{2}$

$\Rightarrow \quad$ either $2x = \frac{\pi}{2}, \frac{3\pi}{2}$ or $x = \frac{2\pi}{3} \quad \Rightarrow \quad x = \frac{\pi}{4}, \frac{2\pi}{3}, \frac{3\pi}{4}$

When $x = \frac{\pi}{4}$, $f''(x) = -\left[\sin\frac{\pi}{4} + 2\sin\frac{\pi}{2} + 3\sin\frac{3\pi}{4}\right] = -\left[\frac{1}{\sqrt{2}} + 2 + \frac{3}{\sqrt{2}}\right] = -(2+2\sqrt{2}) < 0$

$\Rightarrow$ f is maximum at $x = \frac{\pi}{4}$

$$f_{max} = f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} + \frac{1}{2}\sin\frac{\pi}{2} + \frac{1}{3}\sin\frac{3\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{2} + \frac{1}{3}\cdot\frac{1}{\sqrt{2}} = \frac{4\sqrt{2}+3}{6}$$

When $x = \frac{2\pi}{3}$, $f''(x) = -\left[\sin\frac{2\pi}{3} + 2\sin\frac{4\pi}{3} + 3\sin 2\pi\right] = -\left[\frac{\sqrt{3}}{2} - 2\cdot\frac{\sqrt{3}}{2} + 0\right] = \frac{\sqrt{3}}{2} > 0$

$\Rightarrow$ f is minimum at $x = \frac{2\pi}{3}$

$$f_{min} = f\left(\frac{2\pi}{3}\right) = \sin\frac{2\pi}{3} + \frac{1}{2}\sin\frac{4\pi}{3} + \frac{1}{3}\sin 2\pi = \frac{\sqrt{3}}{2} + \frac{1}{2}\left(-\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{4}$$

When $x = \frac{3\pi}{4}$, $f''(x) = -\left[\sin\frac{3\pi}{4} + 2\sin\frac{3\pi}{2} + 3\sin\frac{9\pi}{4}\right]$
$$= -\left[\frac{1}{\sqrt{2}} - 2 + 3\cdot\frac{1}{\sqrt{2}}\right] = -[2\sqrt{2} - 2] < 0$$

$\Rightarrow$ f is maximum at $x = \frac{3\pi}{4}$

$$f_{max} = f\left(\frac{3\pi}{4}\right) = \sin\frac{3\pi}{4} + \frac{1}{2}\sin\frac{3\pi}{4} + \frac{1}{3}\sin\frac{9\pi}{4}$$
$$= \frac{1}{\sqrt{2}} - \frac{1}{2} + \frac{1}{3}\cdot\frac{1}{\sqrt{2}} = \frac{4\sqrt{2}-3}{6}.$$

Example 13. *Prove that the function $f(\theta) = \sin^p\theta\cos^q\theta$ has a maximum at*

$$\theta = \tan^{-1}\left(\sqrt{\frac{p}{q}}\right).$$

Sol.
$$f(\theta) = \sin^p\theta\cos^q\theta$$
$$f'(\theta) = \sin^p\theta\,.\,q\cos^{q-1}\theta(-\sin\theta) + p\sin^{p-1}\theta(\cos\theta)\cos^q\theta$$
$$= -q\sin^{p-1}\theta\cos^{q-1}\theta + p\sin^{p-1}\theta\cos^{q+1}\theta$$
$$= \sin^{p-1}\theta\cos^{q-1}\theta(p\cos^2\theta - q\sin^2\theta)$$

For maxima or minima, $f'(\theta) = 0$

$\Rightarrow \quad \sin\theta = 0$ or $\cos\theta = 0$ or $p\cos^2\theta = q\sin^2\theta$

$\Rightarrow \quad \theta = 0$ or $\theta = \frac{\pi}{2}$ or $\tan\theta = \sqrt{\frac{p}{q}}$

Also $\quad f'(\theta) = \sin^{p-1}\theta\cos^{q-1}\theta(p\cos^2\theta - q\sin^2\theta)$

$$= \frac{\sin^p\theta\cos^q\theta}{\sin\theta\cos\theta}(p\cos^2\theta - q\sin^2\theta) = f(\theta)[p\cot\theta - q\tan\theta]$$

$\therefore \quad f''(\theta) = f(\theta)[-p\operatorname{cosec}^2\theta - q\sec^2\theta] + f'(\theta)(p\cot\theta - q\tan\theta)$

At $\theta = \tan^{-1}\sqrt{\frac{p}{q}}$, $f'(\theta) = 0$

$\therefore \quad f''(\theta) = -f(\theta)[p\operatorname{cosec}^2\theta + q\sec^2\theta]$ at $\theta = \tan^{-1}\sqrt{\frac{p}{q}}$ which is negative.

Hence $f(\theta)$ is maximum at $\theta = \tan^{-1}\left(\sqrt{\frac{p}{q}}\right)$.

Example 14. *Investigative for maximum and minimum values the function $f(x) = \sin x + \cos 2x$.*

Sol. *Please try yourself.* $\quad$ $\left[\textbf{Ans.}\ \text{min.} -1 \text{ at } x = \frac{\pi}{2},\ \text{max.}\ \frac{9}{8} \text{ at } x = \sin^{-1}\left(\frac{1}{4}\right)\right]$